Vaughan–Lee's loop is finitely based

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Mathematics
UNIVERSITY OF COLORADO **BOULDER**

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Examples: (expansions of) groups, loops with $m(x, y, z) = (x/y)z$

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For groups: nilpotent = supernilpotent

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Open

Is 'super' necessary in 1, 2?

Nilpotent $\neq$ supernilpotent

Example (Vaughan–Lee 1983)

Let $\mathbf{L} := (\mathbb{Z}_{12}, x + y + t(x, y))$ with

$$t(x,y) := \begin{cases} 4 & \text{if } (x,y) \equiv_4 (1,3), (3,1), \\ 0 & \text{else.} \end{cases}$$

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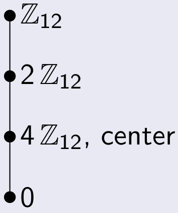
$\mathbf{L}$ is a loop with normal subloops



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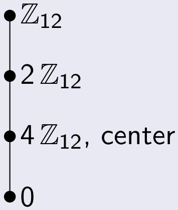

$\mathbf{L}$ is nilpotent but not supernilpotent.

Main results

(A, T) is a (polynomial) reduct of $\mathbf{A} = (A, F)$ if T is a set of (polynomial) term functions of $\mathbf{A}$.

Theorem (Kompatscher, M, Wynne 2022)

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$x * y := m(x, 0, y)$ is a loop multiplication for $\mathbf{A}$ nilpotent, $0 \in A$.

Loops

Main Lemma

Let p be a prime and $\mathbf{A}$ a finite loop with central subloop C of p-power order and $\mathbf{A}/C$ supernilpotent.
Then $\mathbf{A}$ has a normal subloop P with $|P|$ a p-power, $|\mathbf{A}/P|$ coprime to p, and a supernilpotent reduct isomorphic to $\mathbf{P} \times \mathbf{A}/P$.

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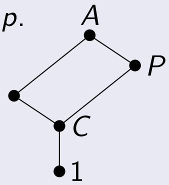
2. Let $P \trianglelefteq \mathbf{A}$ such that $\mathbf{A}/P = \mathbf{V}$.
   $|P| = |C||U|$ is the maximal p-power dividing $|A|$.



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3. Since **P** and **A**$/P$ are coprime, there exists $n \geq 1$ such that

$$x^n := \underbrace{(.(xx)x\dots)x}_{n \text{ times}} = x \text{ for all } x \in P, \quad x^n \in P \text{ for all } x \in A.$$

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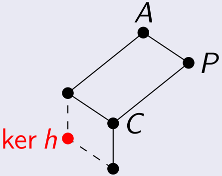

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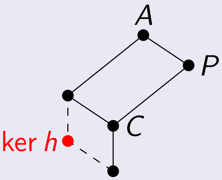

□

Example

Vaughan–Lee's loop **L** is an extension of $P = C \cong (\mathbb{Z}_3 +)$ by $(\mathbb{Z}_4, +)$, hence has a reduct isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$, in fact $* = +$ on $\mathbb{Z}_{12}$.

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Proof.

Use the Main Lemma inductively down a central series of $\mathbf{A}$ with factors of prime power order. □

Normal forms for term functions of Vaughan–Lee's loop **L**

$\mathrm{Clo}(\mathbf{A})$ …clone of term functions of an algebra **A**

Lemma (M, 2022)

$\mathrm{Clo}_k(\mathbf{L}) = \mathrm{Clo}_k(\mathbb{Z}_{12}, +) \oplus W_k$ for

$$W_k := \{w\colon\ \mathbb{Z}_{12}^k \to 4\,\mathbb{Z}_{12} \mid \begin{array}{l} w(2\,\mathbb{Z}_{12}^k) = 0, \\ w(x + 4\,\mathbb{Z}_{12}^k) = w(x), \\ w(x) = w(-x) \text{ for all } x \in \mathbb{Z}_{12}^k \end{array} \}.$$

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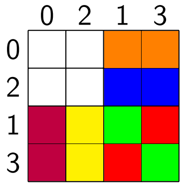
$W_2 =$ functions that are constant with values $\{0, 4, 8\}$ on any colored line segment, 0 else



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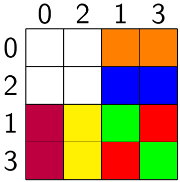

Note

$W := \bigcup_{k\in\mathbb{N}} W_k$ is a set of finitary functions from $\mathbb{Z}_{12}$ to $4\,\mathbb{Z}_{12}$ that is closed under $+$ on domain and codomain.
W is no clone but a **clonoid** ($\rightarrow$ P. Wynne's talk).

Subpower membership for Vaughan–Lee's loop **L**

SMP(A)

Input: $a_1, \ldots, a_k, b \in A^n$

Question: Is $b \in \langle a_1, \ldots, a_k \rangle \leq \mathbf{A}^n$?

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2. The additive normal form of term functions of **L** yields a polytime reduction of $\mathrm{SMP}(\mathbf{L})$ to $\mathrm{SMP}(\mathbb{Z}_{12}, +)$. □

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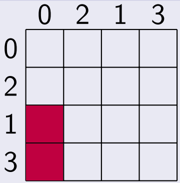

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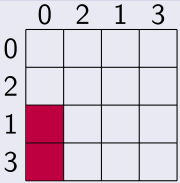

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   3. $f(0,y) \approx 0$
   4. $f(x+y,z) \approx$
      $f(x, 2x+2y+2z) + f(x, 2y+z) - f(x, 2y) + f(x, 2z) + f(y,z)$

Proof, continued

3. Every term s of $\mathbf{A}$ can be transformed into a unique normal form s_0 using the identities of $(\mathbb{Z}_{12}, +)$ and
   1. $f(x+2u, y+4v) \approx f(x,y)$
   2. $3f(x,y) \approx 0$
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   4. $f(x+y,z) \approx$ $f(x,2x+2y+2z) + f(x,2y+z) - f(x,2y) + f(x,2z) + f(y,z)$
   5. $f(x,2x+y) \approx f(x,2y) - f(x,y)$

Proof, continued

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   5. $f(x, 2x+y) \approx f(x,2y) - f(x,y)$
   6. $f(x, 2y+z) \approx f(x,z) - f(y,z) + f(y, 2x+z)$

Proof, continued

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4. $\mathbf{A}$ satisfies $s \approx t$ iff $s_0 = t_0$.

Proof, continued

3. Every term s of $\mathbf{A}$ can be transformed into a unique normal form s_0 using the identities of $(\mathbb{Z}_{12}, +)$ and
   1. $f(x+2u, y+4v) \approx f(x,y)$
   2. $3f(x,y) \approx 0$
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   4. $f(x+y,z) \approx$ $f(x, 2x+2y+2z) + f(x, 2y+z) - f(x,2y) + f(x,2z) + f(y,z)$
   5. $f(x, 2x+y) \approx f(x,2y) - f(x,y)$
   6. $f(x, 2y+z) \approx f(x,z) - f(y,z) + f(y, 2x+z)$
4. $\mathbf{A}$ satisfies $s \approx t$ iff $s_0 = t_0$.
5. Hence the identities above are a finite basis for $\mathbf{A}$. □