The Stone Space of an Orthomodular Lattice

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Introduction

1. Goldblatt (1975) proved that every ortholattice A is isomorphic to the clopen $\perp$-stable subsets of a Stone space X_A endowed with an orthogonality relation $\perp \subseteq X_A^2$ that is irreflexive and symmetric.
2. Bimbó (2007) added a topology to the general class of orthoframes and extended Goldblatt's representation to that of a full duality.
3. By Goldblatt (1984), orthomodularity cannot be characterized by any first-order condition on $\perp$.
4. Hartonas (2016) introduced orthomodular frames (which are first-order definable) and proved orthomodular quantum logic to be sound and complete with respect to this class of structures.
5. This talk is based on our paper (2022) in which we topologize orthomodular frames along the lines of Bimbó (2007) and develop a full (i.e., functorial) duality for orthomodular lattices.

Orthomodular Lattices

Definition

An *orthomodular lattice* is an algebra $A = \langle A; \wedge, \vee, -, 0, 1\rangle$ of similarity type $\langle 2, 2, 1, 0, 0\rangle$ satisfying the following conditions:

1. $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice
2. $-: A \to A$ is an orthocomplementation operator, i.e.,
    1. $a \wedge -a = 0$, $a \vee -a = 1$
    2. $a \leq b \Rightarrow -b \leq -a$
    3. $a = --a$
3. $a \leq b \Rightarrow b = a \vee (b \wedge -a)$

Example

Let H be an infinite-dimensional separable Hilbert space over the complex numbers. Then the lattice $\mathbb{C}(H)$ of closed linear subspaces of H is an orthomodular lattice ordered by subspace inclusion.

Orthomodular Frames

Definition

An *orthomodular frame* is a structure $\langle X; \preceq, \perp, \Omega \rangle$ with $\preceq, \perp \subseteq X \times X$ and a distinguished subset $\Omega \subseteq X$ satisfying the following conditions:

1. $\langle X; \preceq \rangle$ is a meet-semilattice
2. $\langle X; \preceq \rangle$ has a least upper bound ω such that $\omega \in \Omega$
3. For each point $x \in \Omega$, the set $\{y : x \perp y\}$ is generated as the upper closure of a single point $z \in \Omega$. Moreover, the set $\{v : z \perp v\}$ generated by z is in the upper closure of x
4. $\forall x \forall y (x \in \Omega \wedge y \in \Omega \Rightarrow (y \leq x \wedge \forall z (y \preceq z \wedge z \perp x \Rightarrow z = \omega) \Rightarrow x = y))$
5. $\forall x (x \perp x \Rightarrow x = \omega)$
6. $\forall x \forall y (x \in \Omega \wedge y \in \Omega \Rightarrow x \cap y \in \Omega)$ where $x \cap y := \inf\{x, y\}$
7. $\forall x (x \perp \omega)$
8. $\forall x \forall y (x \perp y \Rightarrow y \perp x)$
9. $\forall x \forall y \forall z (x \perp y \wedge x \preceq z \Rightarrow z \perp y)$.

Orthomodular Spaces

Definition

Let $X = \langle X; \preceq, \perp, \Omega \rangle$ be an orthomodular frame. The $\perp$-*stable* sets are $(X)^{\dagger} = \{U \in \wp(X) : U = U^{\perp\perp}\}$ where $U^{\perp} = \{x \in X : \forall y(y \in U \Rightarrow x \perp y)\}$. Also, let $\Gamma = \lambda\rho : \wp(X) \to \wp(X)$ be a closure operator induced by a Galois connection on $\langle X; \preceq \rangle$. Then $\Gamma x = \uparrow x = \{y \in X : x \preceq y\}$ where $\Gamma x := \Gamma(\{x\})$.

Definition

An *orthomodular space* is an ordered relational topological space $X = \langle X; \preceq, \perp, \Omega, \mathcal{T}_X \rangle$ satisfying the following conditions:

1. $\langle X; \preceq, \perp, \Omega \rangle$ is an orthomodular frame
2. X is a compact space
3. $\mathrm{CO}(X)^{\dagger}$ is closed under $\cap$ and $^{\perp}$
4. each $U \in \mathrm{CO}(X)^{\dagger}$ is of the form Γz for some $z \in \Omega$
5. $x \not\preceq y$ implies there exists $U \in \mathrm{CO}(X)^{\dagger}$ such that $x \in U$ and $y \notin U$
6. $x \perp y$ implies there exists $U \in \mathrm{CO}(X)^{\dagger}$ such that $x \in U$ and $y \in U^{\perp}$

The Dual Space of an Orthomodular Lattice

Lemma

Let $A = \langle A; \wedge, -, 0\rangle$ be an orthomodular lattice, let $\mathfrak{F}(A)$ be the set of all filters on A, let $\mathfrak{P}(A)$ be the set of all principal filters on A, and let $h\colon A \to \wp(\mathfrak{F}(A))$ be a function defined by $h(a) = \{x \in \mathfrak{F}(A) : a \in x\}$. Moreover, let $\mathscr{T}(S)$ be a topology on $\mathfrak{F}(A)$ generated by the subbasis $S = \{h(a) : a \in A\} \cup \{-h(a) : a \in A\}$. Lastly, define $x \perp_A y$ iff $\exists a \in A$ such that $a \in x$ and $-a \in y$. Then the space $\mathscr{S}_0(A) = \langle \mathfrak{F}(A); \subseteq, \perp_A, \mathfrak{P}(A), \mathscr{T}(S)\rangle$ is an orthomodular space. Hence, $\mathscr{S}_0(A)$ denotes an orthomodular space whenever A is an orthomodular lattice.

Corollary

If A is an orthomodular lattice, then $\mathscr{S}_0(A)$ is a Stone space. Moreover, $\perp_A$ is an orthogonality relation on $\mathfrak{F}(A) \setminus \{\omega\}$.

The Dual Lattice of an Orthomodular Space

Lemma

If $X = \langle X; \preceq, \perp, \Omega, \mathscr{T}_X \rangle$ is an orthomodular space, then the algebra $\mathscr{A}_0(X) = \langle \mathrm{CO}(X)^{\dagger}; \cap, {}^{\perp}, \{\omega\} \rangle$ is an orthomodular lattice. Hence, $\mathscr{A}_0(X)$ is an orthomodular lattice whenever X is an orthomodular space.

1. ${}^{\perp}$ is clearly an involution on $\mathrm{CO}(X)^{\dagger}$ since by definition,

$$(X)^{\dagger} = \{U \in \wp(X) : U = U^{\perp\perp}\}$$

2. ${}^{\perp}$ is order reversing on $\mathrm{CO}(X)^{\dagger}$ by the symmetry of $\perp$ and the definition of ${}^{\perp}$.
3. $U \cap U^{\perp} = \{\omega\}$ follows from the fact that for each point $x \in X$, we have that $x \perp \omega$ and $x \perp x \Rightarrow x = \omega$.
4. The orthomodularity condition falls out of condition 4 of orthomodular frames and condition 4 of orthomodular spaces.

Object Duality

Theorem (Topological representation of orthomodular lattices)

For every orthomodular lattice $A = \langle A; \wedge, -, 0\rangle$, there exists an isomorphism $A \to \mathscr{A}_0(\mathscr{S}_0(A))$.

We show that the canonical map $h: A \to \wp(\mathscr{S}_0(A))$ defined by

$$h(a) = \{x \in \mathfrak{F}(A) : a \in x\}$$

exhibits the desired isomorphism from A to $\mathscr{A}_0(\mathscr{S}_0(A))$ where $h(0) = \{\omega\}$.

Theorem (Algebraic realization of orthomodular spaces)

For any orthomodular space $X = \langle X; \preceq, \perp, \Omega, \mathscr{T}_X\rangle$, there exists a relational homeomorphism $X \to \mathscr{S}_0(\mathscr{A}_0(X))$.

We show that the function $f: X \to \wp(\mathscr{A}_0(X))$ defined by:

$$f(x) = \{U \in \mathrm{CO}(X)^{\dagger} : x \in U\}$$

exhibits the desired relaitonal homeomorphism from X to $\mathscr{S}_0(\mathscr{A}_0(X))$.

Homomorphisms and Continuous Weak p-Morphisms

Definition

The category $\mathbb{OML}$ has as objects, orthomodular lattices, and has as morphisms, *orthomodular lattice homomorphisms*, i.e., homomorphisms $\phi : A \to A'$ which preserve the orthomodular lattice operations:

1. $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$
2. $\phi(-a) = -\phi(a)$
3. $\phi(0) = 0$

Definition

The category $\mathbb{OMS}$ has as objects, orthomodular spaces and has as morphisms, *continuous weak p-morphisms*, i.e., continuous functions $\psi : X \to X'$ satisfying the following conditions:

1. $x \not\perp_X y \Rightarrow \psi(x) \not\perp_{X'} \psi(y)$
2. $z \not\perp_{X'} \psi(y) \Rightarrow \exists x(x \not\perp_X y \wedge z \preceq_{X'} \psi(x))$

Matching Morphisms

Lemma (The contravariant functor $\mathscr{S}_F$)

Let A and A' be orthomodular lattices and $\phi : A \to A'$ an orthomodular lattice homomorphism. Then the map $\mathscr{S}_1(\phi) \colon \mathscr{S}_0(A') \to \mathscr{S}_0(A)$ with $\mathscr{S}_1(\phi) := \phi^{-1}$ and $\mathrm{dom}(\mathscr{S}_1(\phi)) = \mathfrak{F}(A')$ is a continuous weak p-morphism. Hence $\mathscr{S}_1(\phi)$ denotes a continuous weak p-morphism whenever ϕ is an orthomodular lattice homomorphism. Hence, $\mathscr{S}_F := \langle \mathscr{S}_0, \mathscr{S}_1 \rangle \colon \mathbb{OML} \to \mathbb{OMS}$ is a contravariant functor.

Lemma (The contravariant functor $\mathscr{A}_F$)

Let X and X' be orthomodular spaces and $\psi : X \to X'$ a continuous weak p-morphism. Then the corresponding map $\mathscr{A}_1(\psi) \colon \mathscr{A}_0(X') \to \mathscr{A}_0(X)$ with $\mathscr{A}_1(\psi) := \psi^{-1}$ and $\mathrm{dom}(\mathscr{A}_1(\psi)) = \mathrm{CO}(X')^\dagger$ is an orthomodular lattice homomorphism. Hence $\mathscr{A}_1(\psi)$ denotes an orthomodular lattice homomorphism whenever ψ is a continuous weak p-morphism. Moreover, $\mathscr{A}_F := \langle \mathscr{A}_0, \mathscr{A}_1 \rangle \colon \mathbb{OMS} \to \mathbb{OML}$ is a contravariant functor.

The Main Result

Theorem (Functorial duality)

The contravariant functors $\mathscr{A}_F : \mathbb{OMS} \to \mathbb{OML}$ *and* $\mathscr{S}_F : \mathbb{OML} \to \mathbb{OMS}$ *constitute a dual equivalence of categories.*

It remains to demonstrate for each $a \in A$ that

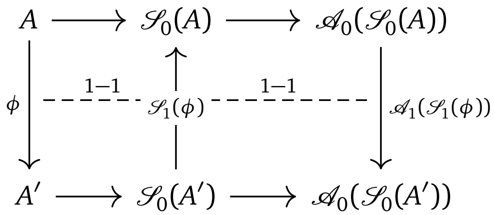
$$h(\phi(a)) = \mathscr{A}_1(\mathscr{S}_1(\phi))$$

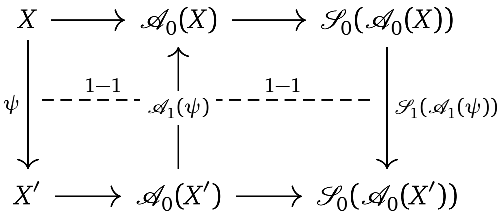
where $\mathscr{A}_1(\mathscr{S}_1(\phi)) = \phi^{-1-1}[h(a)]$ and for each $x \in X$ that,

$$f(\psi(x)) = \mathscr{S}_1(\mathscr{A}_1(\psi))$$

where $\mathscr{S}_1(\mathscr{A}_1(\psi)) = \psi^{-1-1}[f(x)]$

In summary, we have proven that the following diagrams commute:





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