The definable content of (co)homological invariants. Part 1: background, motivation, and main results.

Aristotelis Panagiotopoulos

Carnegie Mellon University

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These series of lectures are about an ongoing research program that is in joint work with Jeff Bergfalk and Martino Lupini.

[1] The definable content of homological invariants I: $\mathrm{Ext}\ \&\ \lim^1$, (2020), arXiv:2008.08782.

[2] The definable content of homological invariants II: cohomology and homotopy classication, (2022), in preparation.

The big picture

Goal. Enriching various classical invariants of **homological algebra** and **algebraic topology** with additional structures pertinent to "definability".

This results into much **finer** invariants for:

(1) classifying spaces up to homotopy equivalence

Steenrod homology $\rightsquigarrow$ **definable homology**

Čech cohomology $\rightsquigarrow$ **definable cohomology**

(2) classifying various algebraic structures up to isomorphism

$\mathrm{Ext}(-,-)$-bifunctor $\rightsquigarrow$ **definable** $\mathbf{Ext}(-,-)$

$\lim^1(-)$-functor $\rightsquigarrow$ **definable** $\mathbf{lim}^1(-)$

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- Each $C^n(X)$ is an abelian group recording the pertinent n-dim data.
- Coboundary maps $\delta^n \colon C^{n-1}(X) \to C^n(X)$ are group homomorphism.

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Example. Steenrod homology of the $\mathbf{2}$-solenoid

The $\mathbf{2}$-solenoid $\Sigma_{\mathbf{2}}$ is the inverse limit of the inverse sequence:

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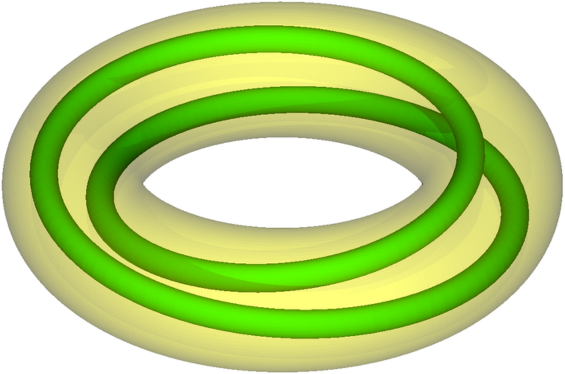

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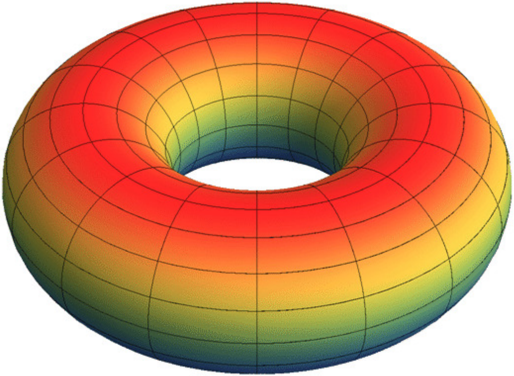

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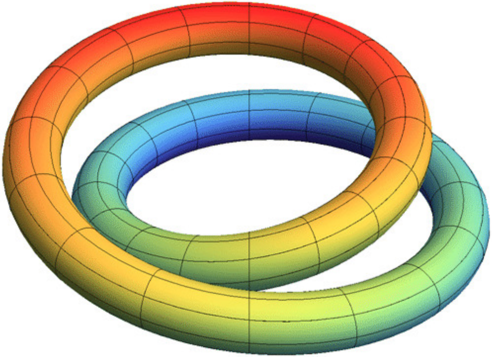

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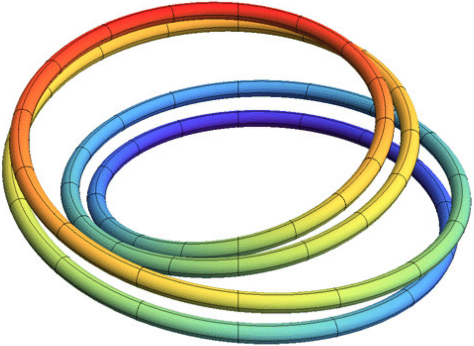

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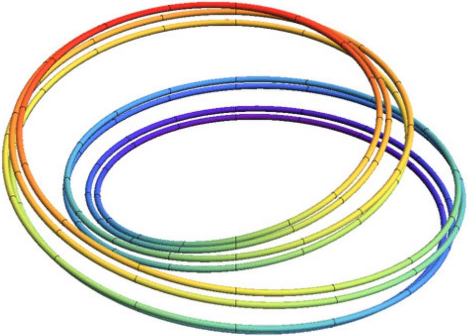

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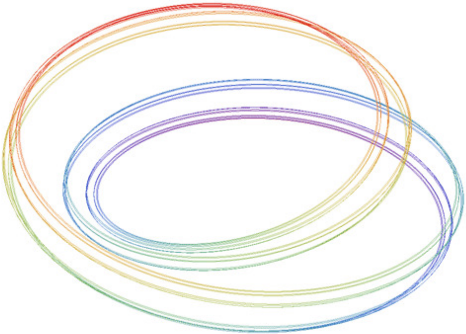

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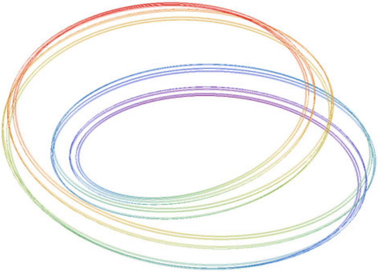

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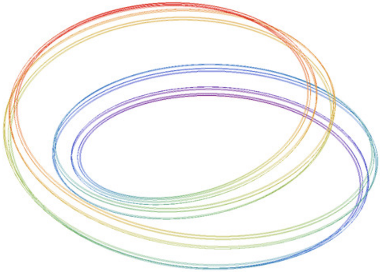

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Notice that $\mathbb{Z}$ is a **dense** subgroup of $\mathbb{Z}_{\mathbf{2}}$.

Hence, the quotient topology on $\mathbb{Z}_{\mathbf{2}}/\mathbb{Z}$ is **trivial**, i.e., equal to $\{\emptyset, \mathbb{Z}_{\mathbf{2}}/\mathbb{Z}\}$.

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Two parallel stories (1950s)

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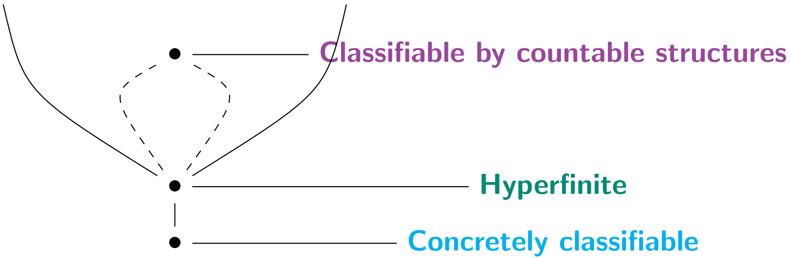
Let (X, E) and (Y, F) be two classification problems. A **Borel reduction** from E to F is any Borel map $f\colon X \to Y$ with $xEx' \iff f(x)Ff(x')$.

We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

The Borel Reduction Hierarchy



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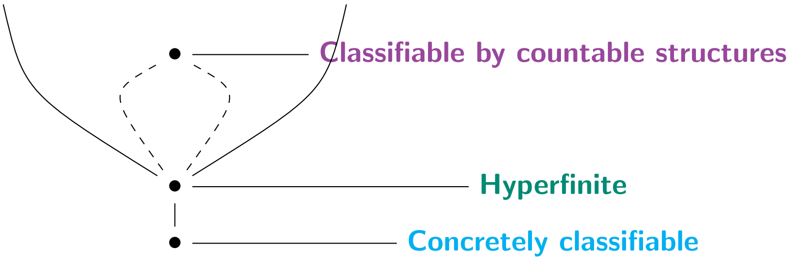

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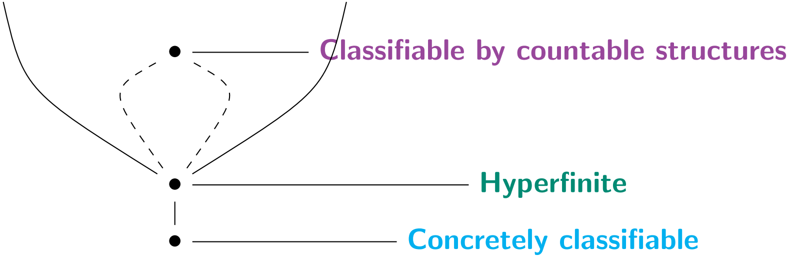

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(X, E) is **hyperfinite** if $(X, E) \leq_B (2^{\mathbb{N}}, E_0)$, where E_0 denotes eventual equality: $(k_0, k_1, \ldots)E_0(\ell_0, \ell_1, \ldots)$ if there is $n \in \mathbb{N}$ so that $k_m = \ell_m$ for all $m \geq n$.

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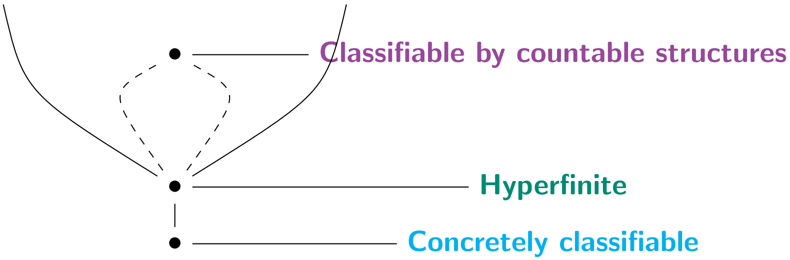

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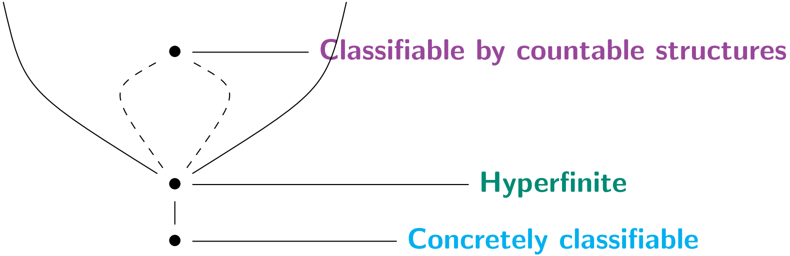

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Heuristic. (X, E) belongs high up in the Borel reduction hierarchy whenever the E-classes are "highly dense" and "wildly entangled" with one another.

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Homework. Consider the group with a Polish cover $\mathbb{R}/\mathbb{Q}$.
(1) How many abstract isomorphisms $\mathbb{R}/\mathbb{Q} \to \mathbb{R}/\mathbb{Q}$ can you find?
(2) How many definable isomorphisms $\mathbb{R}/\mathbb{Q} \to \mathbb{R}/\mathbb{Q}$ can you find?

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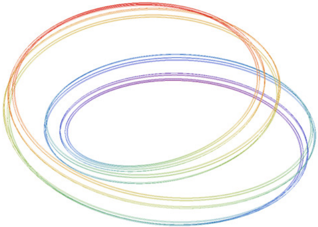

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Similarly, our **definable cohomology** completely classifies solenoid complements $S^3 \setminus K$ while classical Čech cohomology fails to do so.

Rigidity in topology...

Definition

A class $\mathcal{C}$ of topological spaces is **topologically rigid** if whenever $X, Y \in \mathcal{C}$ are homotopy equivalent, then they are homeomorphic.

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Example

The class of compact 2-manifolds is topologically rigid.

Example (Mostow)

The class $\mathcal{C}_M := \{$all finite volume, hyperbolic n-manifolds$, n > 2\}$ is topologically rigid.

Conjecture (Armand Borel)

The class of compact aspherical manifolds is topologically rigid.

Rigidity in descriptive set theory...

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Theorem (Veličković 1986)

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Theorem (Kanovei-Reeken 2000)

Let Λ_1, Λ_2 be countable dense subgroups of $(\mathbb{R}, +)$. Every group homomorphism $f\colon \mathbb{R}/\Lambda_1 \to \mathbb{R}/\Lambda_2$ which admits a Borel lift $\mathbb{R} \to \mathbb{R}$ is **trivial** *in a certain concrete sense.*

In the same paper, Kanovei-Reeken ask the following question:

Question. Do quotients of p-adic groups satisfy similar rigidity phenomena?

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The key ingredient our proof of topological rigidity...

Corollary

The class $\mathcal{S}$ of solenoids is **topologically rigid***:*
if $K, L \in \mathcal{S}$ are homotopy equivalent then they are homeomorphic.

...is a descriptive set-theoretic rigidity result.

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Let Λ_1, Λ_2 be countable dense subgroups of the non-archimedean abelian Polish groups G_1, G_2. Every homomorphism $f\colon G_1/\Lambda_1 \to G_2/\Lambda_2$ which admits a Borel lift is **trivial** *in a certain concrete sense.*

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The latter answers Kanovei-Reeken's question in a much greater generality.

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The latter answers Kanovei-Reeken's question in a much greater generality. It also allows us to characterize when two 0-dim **definable homology** groups $\mathbb{Z}_{\boldsymbol{n}}/\mathbb{Z}$ and $\mathbb{Z}_{\boldsymbol{m}}/\mathbb{Z}$ are (definably) isomorphic