On the Number of Clonoids Between Finite Modules

Patrick Wynne
Joint work with Peter Mayr

University of Colorado Boulder

BLAST 2022, Chapman University
8 August 2022

Clonoids

Clonoids

Clonoid

For algebras $\mathbb{A}$ and $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a **clonoid** from $\mathbb{A}$ to $\mathbb{B}$ if

$$C\,\mathrm{Clo}(\mathbb{A}) \subseteq C \qquad \& \qquad \mathrm{Clo}(\mathbb{B})C \subseteq C$$

Clonoids

Clonoid

For algebras $\mathbb{A}$ and $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a **clonoid** from $\mathbb{A}$ to $\mathbb{B}$ if

$$C\,\mathrm{Clo}(\mathbb{A}) \subseteq C \qquad \& \qquad \mathrm{Clo}(\mathbb{B})C \subseteq C$$

- C is closed under precomposition with term operations of $\mathbb{A}$, and

Clonoids

Clonoid

For algebras $\mathbb{A}$ and $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a **clonoid** from $\mathbb{A}$ to $\mathbb{B}$ if

$$C\,\mathrm{Clo}(\mathbb{A}) \subseteq C \qquad \& \qquad \mathrm{Clo}(\mathbb{B})C \subseteq C$$

- C is closed under precomposition with term operations of $\mathbb{A}$, and
- C is closed under postcomposition with term operations of $\mathbb{B}$;

Clonoids

Clonoid

For algebras $\mathbb{A}$ and $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a **clonoid** from $\mathbb{A}$ to $\mathbb{B}$ if

$$C\,\mathrm{Clo}(\mathbb{A}) \subseteq C \qquad \& \qquad \mathrm{Clo}(\mathbb{B})C \subseteq C$$

- C is closed under precomposition with term operations of $\mathbb{A}$, and
- C is closed under postcomposition with term operations of $\mathbb{B}$;

Example: Let $\mathbb{A}$ be an abelian group and let $\mathbb{B}$ be a lattice. Let C be a clonoid from $\mathbb{A}$ to $\mathbb{B}$ with $f, g \in C$. Then

$$h(x_1, x_2, x_3) := f(x_1 + x_2) \wedge g(x_1 + x_2, x_1 + x_3) \in C.$$

Polymorphisms and Clonoids

Polymorphisms and Clonoids

Definition

For $R \subseteq A^n, S \subseteq B^n$, let

$$\mathrm{Pol}(R,S) = \bigcup_{k \in \mathbb{N}} \{f \colon A^k \to B \mid f(R,\dots,R) \subseteq S\}$$

denote the set of **polymorphisms** of the relational pair (R, S).

Polymorphisms and Clonoids

Definition

For $R \subseteq A^n, S \subseteq B^n$, let

$$\mathrm{Pol}(R, S) = \bigcup_{k \in \mathbb{N}} \{f \colon A^k \to B \mid f(R, \ldots, R) \subseteq S\}$$

denote the set of **polymorphisms** of the relational pair (R, S).

Theorem (Couceiro, Foldes 2009)

For $R \leq \mathbb{A}^n$ and $S \leq \mathbb{B}^n$ subalgebras, $\mathrm{Pol}(R, S)$ is a clonoid from $\mathbb{A}$ to $\mathbb{B}$.

Polymorphisms and Clonoids

Definition

For $R \subseteq A^n, S \subseteq B^n$, let

$$\mathrm{Pol}(R,S) = \bigcup_{k\in\mathbb{N}} \{f\colon A^k \to B \mid f(R,\dots,R) \subseteq S\}$$

denote the set of **polymorphisms** of the relational pair (R,S).

Theorem (Couceiro, Foldes 2009)

For $R \leq \mathbb{A}^n$ and $S \leq \mathbb{B}^n$ subalgebras, $\mathrm{Pol}(R,S)$ is a clonoid from $\mathbb{A}$ to $\mathbb{B}$.

Example:

- $\mathbb{A} = (\mathbb{Z}_{p^2}, +), \qquad \mathbb{B} = (\mathbb{Z}_q, +)$

Polymorphisms and Clonoids

Definition

For $R \subseteq A^n, S \subseteq B^n$, let

$$\mathrm{Pol}(R,S) = \bigcup_{k \in \mathbb{N}} \{f \colon A^k \to B \mid f(R,\dots,R) \subseteq S\}$$

denote the set of **polymorphisms** of the relational pair (R,S).

Theorem (Couceiro, Foldes 2009)

For $R \leq \mathbb{A}^n$ and $S \leq \mathbb{B}^n$ subalgebras, $\mathrm{Pol}(R,S)$ is a clonoid from $\mathbb{A}$ to $\mathbb{B}$.

Example:

- $\mathbb{A} = (\mathbb{Z}_{p^2}, +), \qquad \mathbb{B} = (\mathbb{Z}_q, +)$
- $\mathrm{Pol}(\equiv_p, =)$ is the clonoid of functions from $\mathbb{A}$ to $\mathbb{B}$ that are constant on classes modulo p.

How Many Clonoids?

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

Motivation: By Freese and McKenzie, clonoids arise in the study of nilpotent algebras.

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

Motivation: By Freese and McKenzie, clonoids arise in the study of nilpotent algebras.

To understand finite 2-nilpotent algebras, we study clonoids between finite modules.

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

Motivation: By Freese and McKenzie, clonoids arise in the study of nilpotent algebras.
To understand finite 2-nilpotent algebras, we study clonoids between finite modules.

Problem

Given two finite modules $\mathbb{A}$ and $\mathbb{B}$, classify the clonoids from $\mathbb{A}$ to $\mathbb{B}$.

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

Motivation: By Freese and McKenzie, clonoids arise in the study of nilpotent algebras.
To understand finite 2-nilpotent algebras, we study clonoids between finite modules.

Problem

Given two finite modules $\mathbb{A}$ and $\mathbb{B}$, classify the clonoids from $\mathbb{A}$ to $\mathbb{B}$.

- Kreinecker (2019): For p an odd prime, the number of clonoids from the group $\mathbb{Z}_p$ to $\mathbb{Z}_p$ is countably infinite.

How Many Clonoids?

Theorem (Aichinger, Mayr 2018)

If $\mathbb{A}$ is a finite algebra and $\mathbb{B}$ is a finite Mal'cev algebra then clonoids from $\mathbb{A}$ to $\mathbb{B}$ are finitely related.

Motivation: By Freese and McKenzie, clonoids arise in the study of nilpotent algebras.
To understand finite 2-nilpotent algebras, we study clonoids between finite modules.

Problem

Given two finite modules $\mathbb{A}$ and $\mathbb{B}$, classify the clonoids from $\mathbb{A}$ to $\mathbb{B}$.

- Kreinecker (2019): For p an odd prime, the number of clonoids from the group $\mathbb{Z}_p$ to $\mathbb{Z}_p$ is countably infinite.
- We generalize to modules of non-coprime order as follows:

Not coprime implies many clonoids

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Proof idea:

- **Reduction:** $\mathbb{A} = (\mathbb{Z}_p^n, +, M_n(\mathbb{Z}_p))$ and $\mathbb{B} = (\mathbb{Z}_p^m, +, M_m(\mathbb{Z}_p))$.

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Proof idea:

- **Reduction:** $\mathbb{A} = (\mathbb{Z}_p^n, +, M_n(\mathbb{Z}_p))$ and $\mathbb{B} = (\mathbb{Z}_p^m, +, M_m(\mathbb{Z}_p))$.
- $h\colon A \to B,\ (x_1, \dots, x_n) \mapsto \begin{cases} (x_1, \dots, x_m) & \text{if } m \leq n, \\ (x_1, \dots, x_n, 0, \dots, 0) & \text{else.} \end{cases}$

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Proof idea:

- **Reduction:** $\mathbb{A} = (\mathbb{Z}_p^n, +, M_n(\mathbb{Z}_p))$ and $\mathbb{B} = (\mathbb{Z}_p^m, +, M_m(\mathbb{Z}_p))$.
- $h\colon A \to B,\ (x_1, \dots, x_n) \mapsto \begin{cases} (x_1, \dots, x_m) & \text{if } m \leq n, \\ (x_1, \dots, x_n, 0, \dots, 0) & \text{else.} \end{cases}$
- $\cdot\ : A^2 \to A,\ (\bar{x}, \bar{y}) \mapsto (x_1 y_1, \dots, x_n y_n)$

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Proof idea:

- **Reduction:** $\mathbb{A} = (\mathbb{Z}_p^n, +, M_n(\mathbb{Z}_p))$ and $\mathbb{B} = (\mathbb{Z}_p^m, +, M_m(\mathbb{Z}_p))$.
- $h\colon A \to B,\ (x_1, \ldots, x_n) \mapsto \begin{cases} (x_1, \ldots, x_m) & \text{if } m \leq n, \\ (x_1, \ldots, x_n, 0, \ldots, 0) & \text{else.} \end{cases}$
- $\cdot\ : A^2 \to A,\ (\bar{x}, \bar{y}) \mapsto (x_1 y_1, \ldots, x_n y_n)$
- $f_k\colon A^k \to B,\ (\bar{x}_1, \ldots, \bar{x}_k) \mapsto h(\bar{x}_1 \cdot \ldots \cdot \bar{x}_k)$

Not coprime implies many clonoids

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite modules whose orders have a common prime factor, p. The number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is countably infinite.

Proof idea:

- **Reduction:** $\mathbb{A} = (\mathbb{Z}_p^n, +, M_n(\mathbb{Z}_p))$ and $\mathbb{B} = (\mathbb{Z}_p^m, +, M_m(\mathbb{Z}_p))$.
- $h\colon A \to B,\ (x_1, \dots, x_n) \mapsto \begin{cases} (x_1, \dots, x_m) & \text{if } m \leq n, \\ (x_1, \dots, x_n, 0, \dots, 0) & \text{else.} \end{cases}$
- $\cdot\ : A^2 \to A,\ (\bar{x}, \bar{y}) \mapsto (x_1 y_1, \dots, x_n y_n)$
- $f_k\colon A^k \to B,\ (\bar{x}_1, \dots, \bar{x}_k) \mapsto h(\bar{x}_1 \cdot \dots \cdot \bar{x}_k)$
- **Claim:** $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

Not coprime implies many clonoids

Not coprime implies many clonoids

Claim: $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

Not coprime implies many clonoids

Claim: $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

- **Idea:** Notice that f_k is zero-absorbing.

Not coprime implies many clonoids

Claim: $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

- **Idea:** Notice that f_k is zero-absorbing.
- i.e. $f_k(\bar{x}_1, \ldots, \bar{x}_k) = 0$ whenever $\bar{x}_i = 0$.

Not coprime implies many clonoids

Claim: $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

- **Idea:** Notice that f_k is zero-absorbing.
- i.e. $f_k(\bar{x}_1, \ldots, \bar{x}_k) = 0$ whenever $\bar{x}_i = 0$.
- The only zero-absorbing function in $\langle f_1, \ldots, f_k \rangle$ of arity $k+1$ is the zero function.

Not coprime implies many clonoids

Claim: $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$

- **Idea:** Notice that f_k is zero-absorbing.
- i.e. $f_k(\bar{x}_1, \ldots, \bar{x}_k) = 0$ whenever $\bar{x}_i = 0$.
- The only zero-absorbing function in $\langle f_1, \ldots, f_k \rangle$ of arity $k+1$ is the zero function.
- But f_{k+1} is zero-absorbing. ■

Modules of Coprime Order

Modules of Coprime Order

Theorem (Fioravanti, 2020)

For p and q distinct primes, clonoids from the group $\mathbb{Z}_p$ to the group $\mathbb{Z}_q$ are uniquely determined by their unary functions.

Modules of Coprime Order

Theorem (Fioravanti, 2020)

For p and q distinct primes, clonoids from the group $\mathbb{Z}_p$ to the group $\mathbb{Z}_q$ are uniquely determined by their unary functions.

- $f : \mathbb{Z}_p^k \to \mathbb{Z}_q$ can be interpolated on lines in $\mathbb{Z}_p^k$ passing through the origin, i.e.

Modules of Coprime Order

Theorem (Fioravanti, 2020)

For p and q distinct primes, clonoids from the group $\mathbb{Z}_p$ to the group $\mathbb{Z}_q$ are uniquely determined by their unary functions.

- $f : \mathbb{Z}_p^k \to \mathbb{Z}_q$ can be interpolated on lines in $\mathbb{Z}_p^k$ passing through the origin, i.e.
- unary functions in $\langle f \rangle$ generate $f_v : \mathbb{Z}_p^k \to \mathbb{Z}_q$,

$$f_v(x) = \begin{cases} f(x) & \text{if } x \in \text{span}(v) \\ 0 & \text{otherwise} \end{cases}.$$

Modules of Coprime Order

Theorem (Fioravanti, 2020)

For p and q distinct primes, clonoids from the group $\mathbb{Z}_p$ to the group $\mathbb{Z}_q$ are uniquely determined by their unary functions.

- $f : \mathbb{Z}_p^k \to \mathbb{Z}_q$ can be interpolated on lines in $\mathbb{Z}_p^k$ passing through the origin, i.e.
- unary functions in $\langle f \rangle$ generate $f_v : \mathbb{Z}_p^k \to \mathbb{Z}_q$,

$$f_v(x) = \begin{cases} f(x) & \text{if } x \in \text{span}(v) \\ 0 & \text{otherwise} \end{cases}.$$

- So for f such that $f(0, \ldots, 0) = 0$, $f = \sum f_v$ for v's generating distinct lines.

Unary Generation

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.
(1) $\implies$ (2):

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.

(1) $\implies$ (2):

- If $\mathbb{A}$ is k-generated then there are clonoids not generated by their $< k$-ary functions.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.

(1) $\implies$ (2):

- If $\mathbb{A}$ is k-generated then there are clonoids not generated by their $< k$-ary functions.
- So $\mathbb{A}$ is a cyclic R-module.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.

(1) $\implies$ (2):

- If $\mathbb{A}$ is k-generated then there are clonoids not generated by their $< k$-ary functions.
- So $\mathbb{A}$ is a cyclic R-module.
- R commutative implies $\mathbb{A} \cong R/I$.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.
(1) $\implies$ (2):

- If $\mathbb{A}$ is k-generated then there are clonoids not generated by their $< k$-ary functions.
- So $\mathbb{A}$ is a cyclic R-module.
- R commutative implies $\mathbb{A} \cong R/I$.
- $\mathrm{Rad}(R) > 0 \implies$ some clonoids are not generated by unary functions.

Unary Generation

Theorem (Mayr, Wynne 2022)

Let R be a finite commutative ring and let $\mathbb{A}$ be a finite faithful R-module. Let $\mathbb{B}$ be an S-module. The following are equivalent:

1. Every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its unary functions.
2. $\gcd(|A|, |B|) = 1$, R is a direct product of finite fields and $\mathbb{A} =_R R$.

(2) $\implies$ (1): Interpolate f on cyclic submodules of $\mathbb{A}^k$.
(1) $\implies$ (2):

- If $\mathbb{A}$ is k-generated then there are clonoids not generated by their $< k$-ary functions.
- So $\mathbb{A}$ is a cyclic R-module.
- R commutative implies $\mathbb{A} \cong R/I$.
- $\text{Rad}(R) > 0 \implies$ some clonoids are not generated by unary functions.
- So R is semisimple and commutative, i.e. a product of finite fields.

Unary functions are not always enough

Unary functions are not always enough

Example

For p a prime, let $\mathbb{A} = (\mathbb{Z}_{p^2}, +)$ and let $\mathbb{B}$ be a nontrivial group.

Unary functions are not always enough

Example

For p a prime, let $\mathbb{A} = (\mathbb{Z}_{p^2}, +)$ and let $\mathbb{B}$ be a nontrivial group.
There exist distinct clonoids from $\mathbb{A}$ to $\mathbb{B}$ with the same unary functions.

Unary functions are not always enough

Example

For p a prime, let $\mathbb{A} = (\mathbb{Z}_{p^2}, +)$ and let $\mathbb{B}$ be a nontrivial group.
There exist distinct clonoids from $\mathbb{A}$ to $\mathbb{B}$ with the same unary functions.

Proof:

- $C :=$ clonoid from $\mathbb{A}$ to $\mathbb{B}$ of all functions that are constant on line segments $\{\bar{x} + rp\bar{x} \ : \ 0 \leq r \leq p-1\}$ and zero on pA^k.

Unary functions are not always enough

Example

For p a prime, let $\mathbb{A} = (\mathbb{Z}_{p^2}, +)$ and let $\mathbb{B}$ be a nontrivial group.
There exist distinct clonoids from $\mathbb{A}$ to $\mathbb{B}$ with the same unary functions.

Proof:

- $C :=$ clonoid from $\mathbb{A}$ to $\mathbb{B}$ of all functions that are constant on line segments $\{\bar{x} + rp\bar{x} \ : \ 0 \leq r \leq p-1\}$ and zero on pA^k.
- $D :=$ clonoid from $\mathbb{A}$ to $\mathbb{B}$ of all functions that are constant modulo p (i.e. satisfying $f(\bar{x} + pA^k) = f(\bar{x})$), and are zero on pA^k.

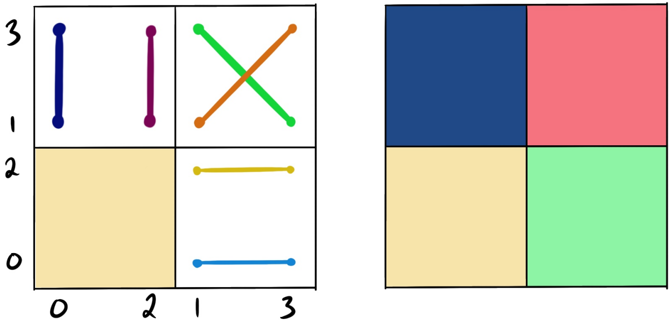
Unary functions are not always enough

Unary functions are not always enough

- Such line segments in $A^k \setminus pA^k$ are equal to $\equiv_p$-blocks iff $k = 1$.

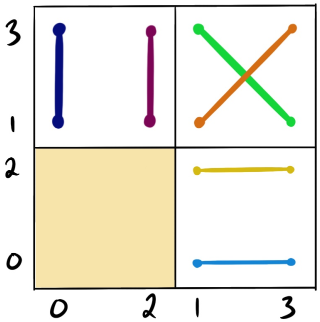
Unary functions are not always enough

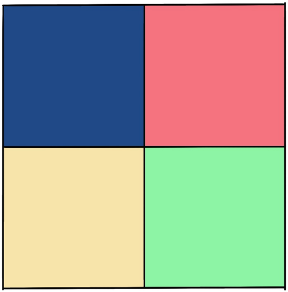
- Such line segments in $A^k \setminus pA^k$ are equal to $\equiv_p$-blocks iff $k = 1$.



Unary functions are not always enough

- Such line segments in $A^k \setminus pA^k$ are equal to $\equiv_p$-blocks iff $k = 1$.

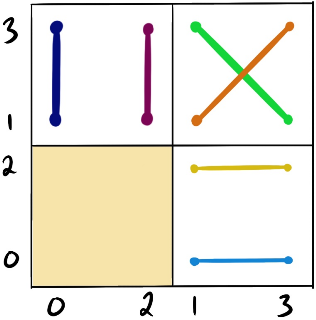


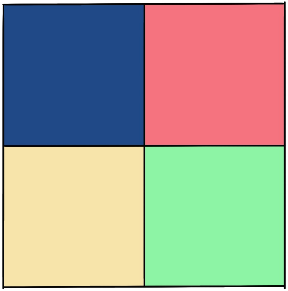


- So the unary parts of C and D are equal but the binary parts are not.

Unary functions are not always enough

- Such line segments in $A^k \setminus pA^k$ are equal to $\equiv_p$-blocks iff $k = 1$.





- So the unary parts of C and D are equal but the binary parts are not.

Theorem (Mayr, Wynne 2022)

Let $\mathbb{A} = (\mathbb{Z}_{p^2}, +)$ and let $\mathbb{B}$ be a module of coprime order. Then every clonoid from $\mathbb{A}$ to $\mathbb{B}$ is generated by its binary functions.

Thank You!