Weak pseudoradiality of CSC spaces

Joint work with Alan Dow

Hector Barriga-Acosta

hbarriga@uncc.edu
Postdoc at University of North Carolina at Charlotte

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Figure: Random tree.

Figure: Architecture.

Figure: Spiritual.

Figure: Intellectual.

Figure: Physical.

Figure: Social.

Figure: Laguna beach sunset.

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Cardinal invariant of the continuum

An almost dijoint family on ω is a collection $\mathcal{A} \subset \mathcal{P}(\omega)$ such that $\forall a, b \in \mathcal{A}$, $a \cap b$ is finite.

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{P}(\omega) \text{ is a maximal almost disjoint family}\}$$

$$\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$$

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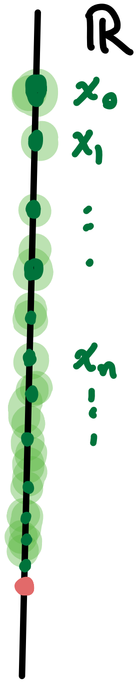
$$\omega_1 \leq \mathfrak{a} \leq \mathfrak{c}$$

- $\mathfrak{a}$ almost disjoint number
- $\mathfrak{b}$ bounding number
- $\mathfrak{c}$ continuum
- $\mathfrak{d}$ dominating number
- $\mathfrak{e}$ evasion number
- $\mathfrak{f}$
- $\mathfrak{g}$ groupwise density number
- $\mathfrak{h}$ distributivity number
- $\mathfrak{i}$ independence number
- $\mathfrak{j}$
- $\mathfrak{k}$
- $\mathfrak{l}$
- $\mathfrak{m}$ Martin's number
- $\mathfrak{n}$
- $\mathfrak{o}$
- $\mathfrak{p}$ pseudointersection number
- $\mathfrak{q}$
- $\mathfrak{r}$ reaping number
- $\mathfrak{s}$ splitting number
- $\mathfrak{t}$ tower number
- $\mathfrak{u}$ ultrafilter number
- $\mathfrak{v}$
- $\mathfrak{w}$
- $\mathfrak{x}$
- $\mathfrak{y}$
- $\mathfrak{z}$

Sequences

- Convergent sequence in $\mathbb{R}$ (Frechet)



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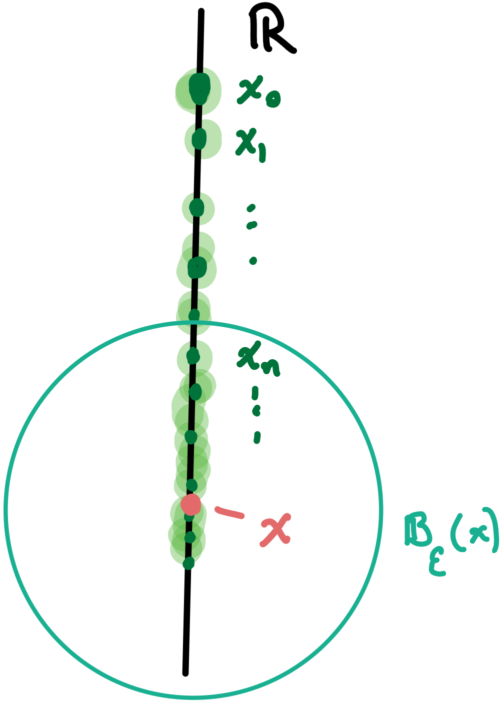

Sequences

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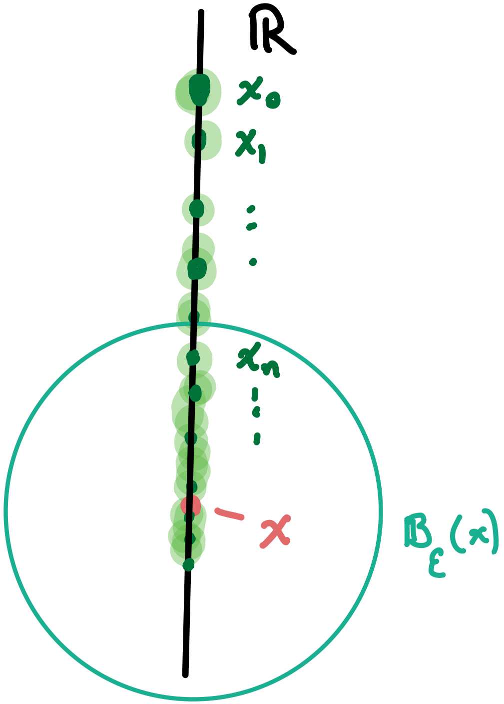

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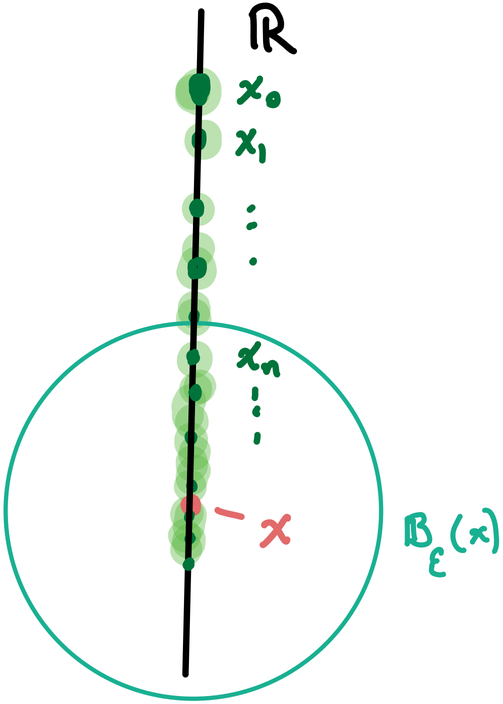


Sequences

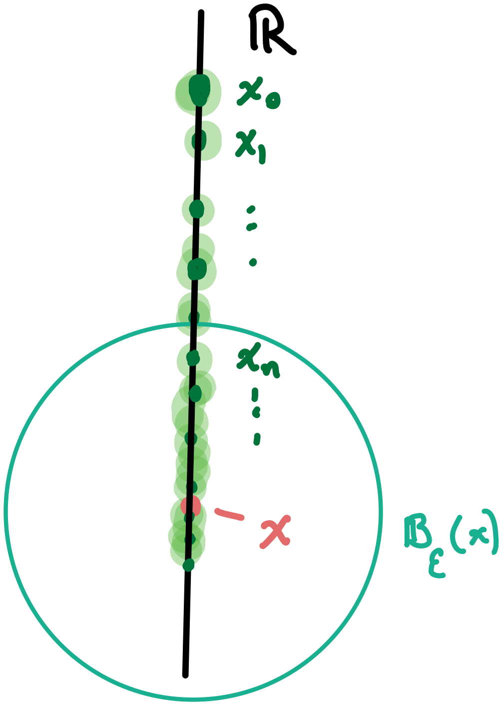
- Convergent sequence in $\mathbb{R}$ (Frechet)
- Cauchy sequence
- Free sequence
- Convergent filter

Definitions involving sequences

- Sequentially compact
- Sequential
- Fréchet-Urysohn
- Radiality
- Pseudoradiality

Convergence under the presence of

- Covering properties
- Numerability axioms
- Linear orders
- Metric

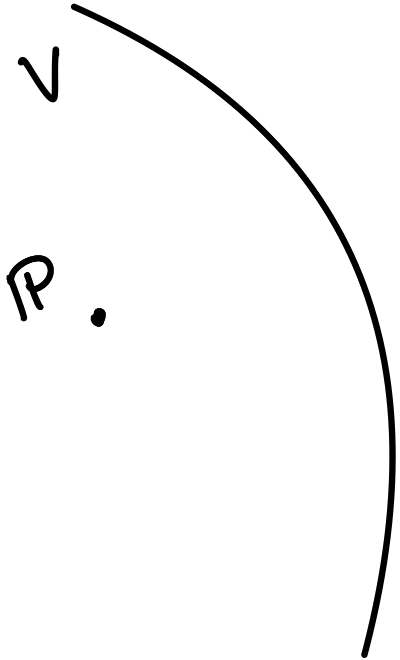


BLAST with Forcing flavor

$V \models ZFC$

Paul Cohen
(1937 - 2007)

BLAST with Forcing flavor



BLAST with Forcing flavor

V

$V[G]$

$\mathbb{P}$

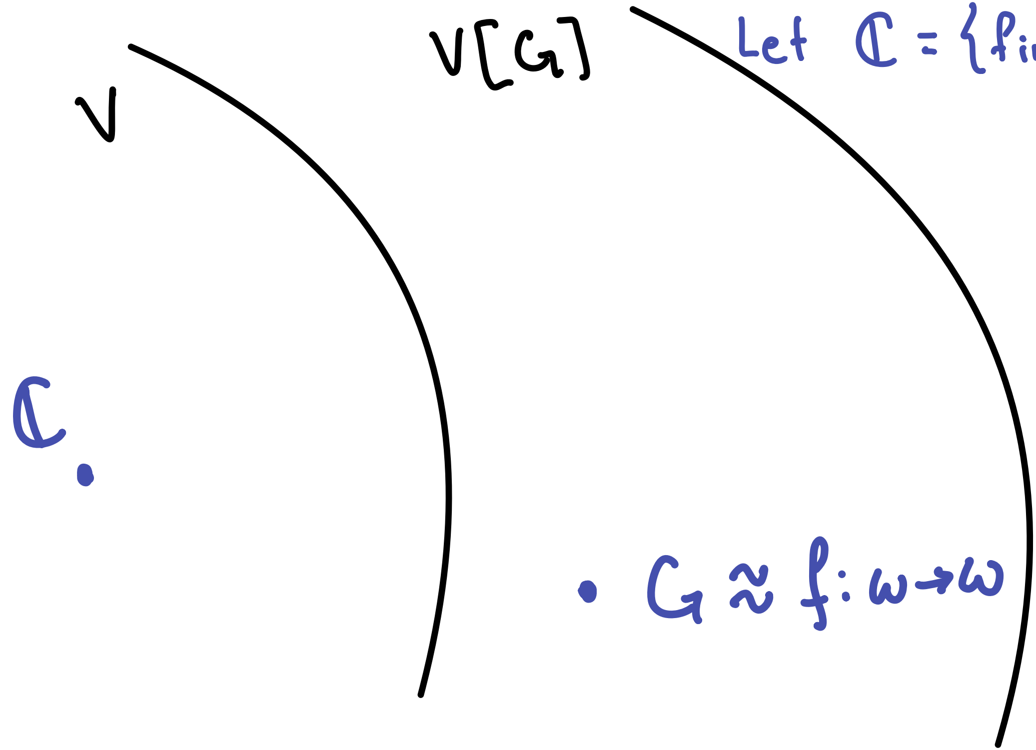
elements are promises

$p \in \mathbb{P} \to p \Vdash$ " "

G

BLAST with Forcing flavor

Let $\mathbb{C} = \{$finite partial functions from $\omega \to \omega\}$



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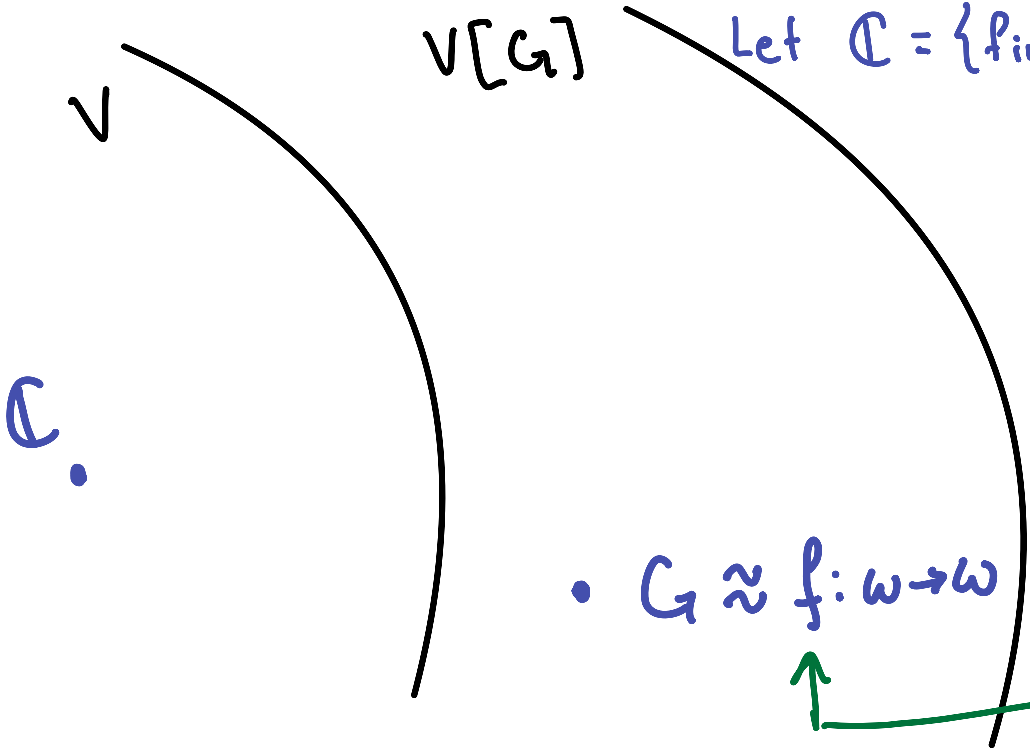
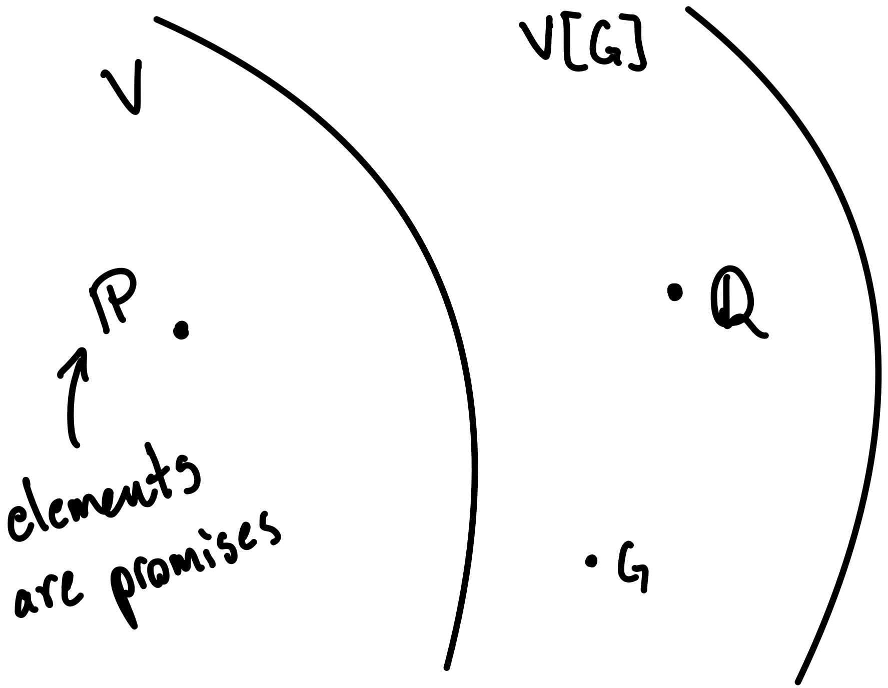


Cohen real

BLAST with Forcing flavor



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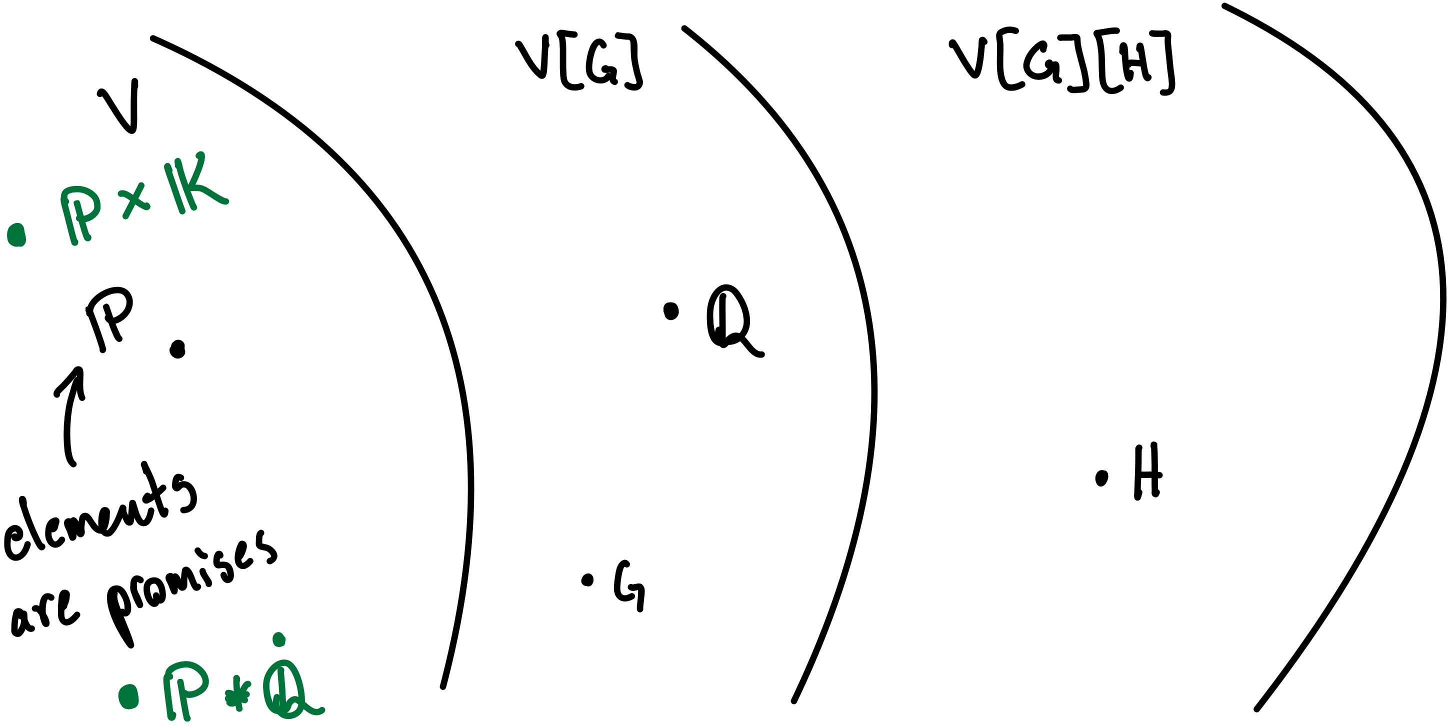

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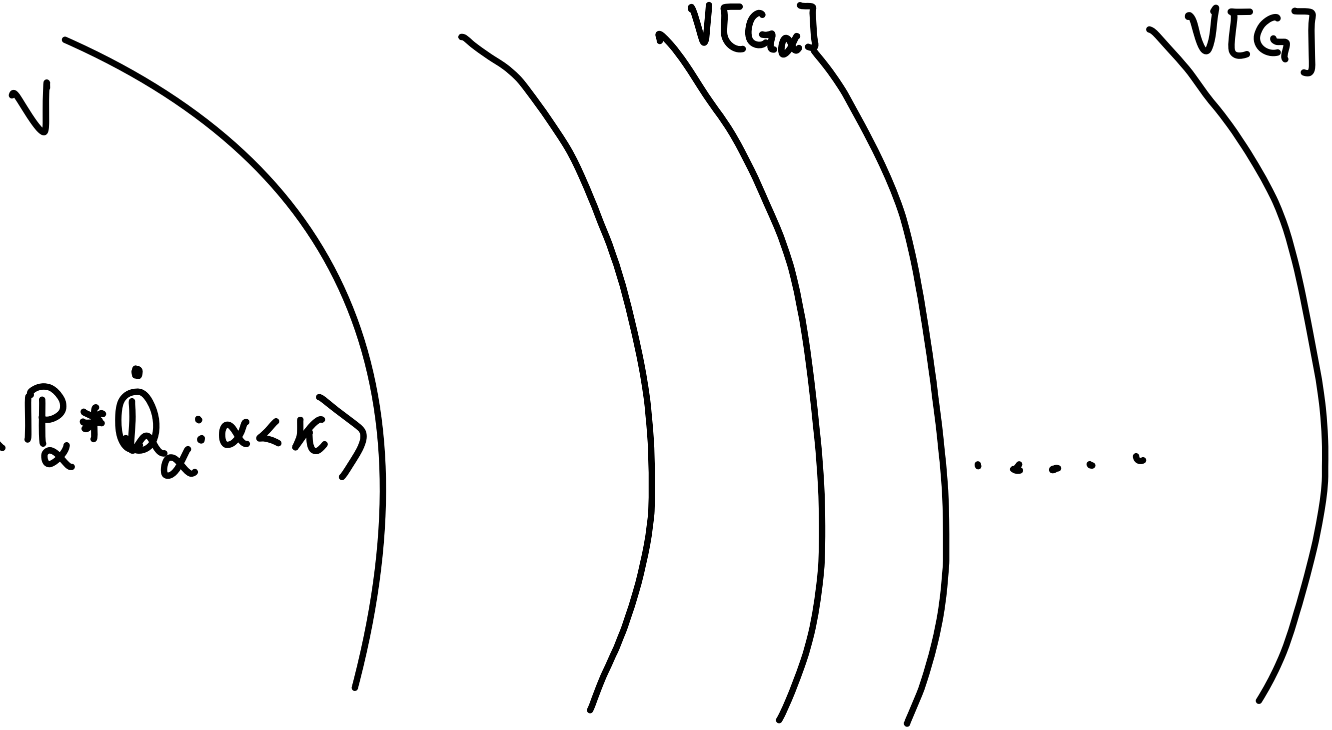

- $\mathbb{P} = \langle \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$

Pseudoradiality and known results

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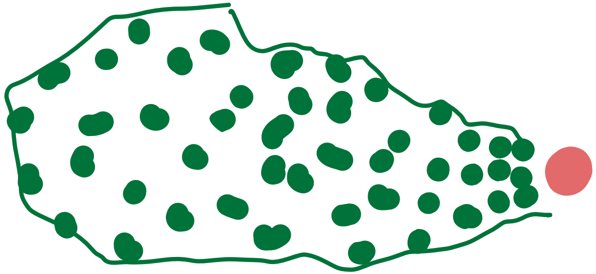
The following spaces are pseudoradial

- First countable
- Ordinals

Remark

If X is compact, then X is sequentially compact if and only if for every countable non-closed $A \subset X$, there is a sequence in A converging out of A (pseudoradial to countable sets).

$\Rightarrow)$



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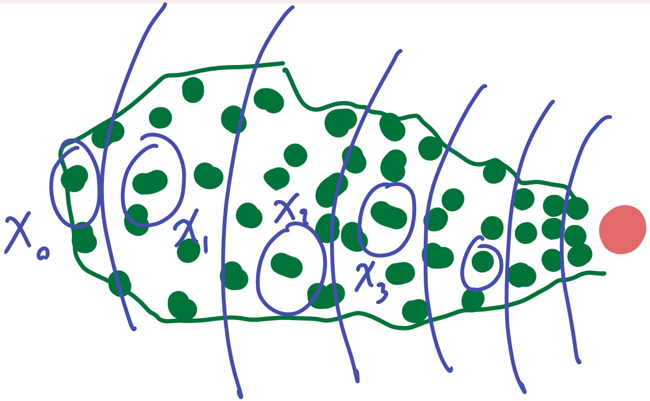

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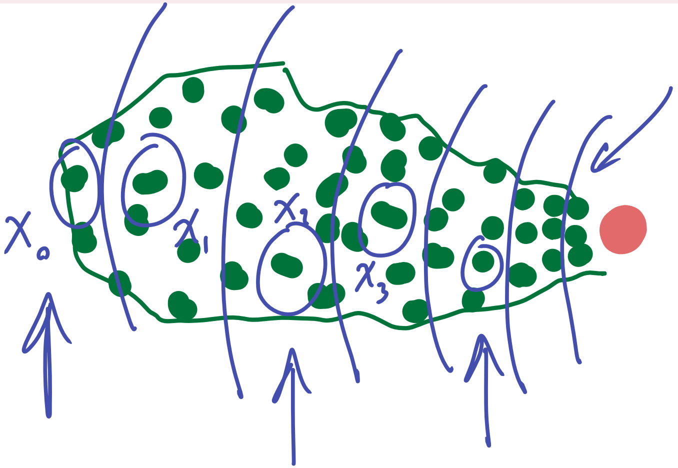

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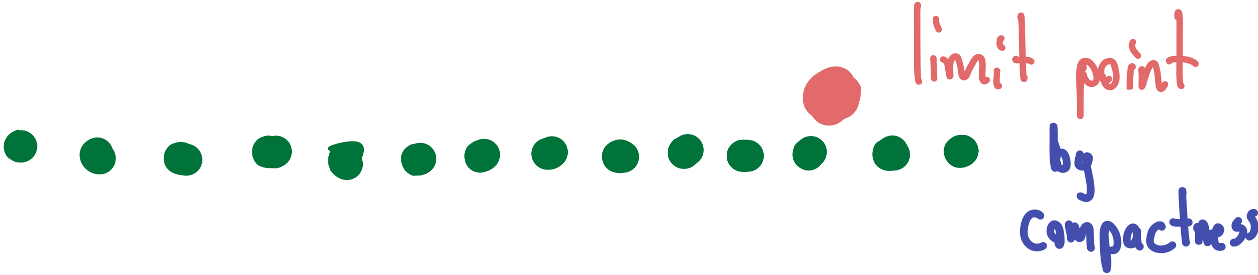

Theorem (Šapirovskiĭ, 1990)

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1. If $\mathfrak{c} \leq \aleph_2$, every CSC space is pseudoradial.
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   - Does $\mathfrak{c} = \aleph_3$ imply there is a non-pseudoradial CSC space?

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b In the forcing extension by adding any number of Cohen reals over a model of CH, every CSC space is pseudoradial.

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Separable subspaces

Suppose X is compact and non-pseudoradial.

Theorem (Bella, Dow, Tironi, 2001)

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Is it true in ZFC that 2^{ω_2} is not pseudoradial?
Independent from the axioms of ZFC

- 2^{ω_2} is not pseudoradial : Juhász, Szentmiklóssy
- 2^{ω_2} is pseudoradial : Alan Dow

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Remark

$$(1)\ X \text{ is pseudoradial}$$
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(4) $\Rightarrow$ (1): Under CH (Šapirovskiĭ)

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$$\begin{gathered}\text{(1) } X \text{ is pseudoradial}\\ \Downarrow \\ \text{(2) 'closed separable subspaces are pseudoradial'} \\ \Downarrow \\ \text{(3) 'radial closures of countable sets are closed'} == \textit{weakly pseudoradial} \\ \Downarrow \\ \text{(4) } X \text{ is sequentially compact}\end{gathered}$$

(4) $\Rightarrow$ (1): Under CH (Šapirovskiĭ)
(4) $\Rightarrow$ (3): Under $\mathfrak{p} = \mathfrak{c}$

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($\mathfrak{s} = \min\{\kappa : 2^\kappa$ is not sequentially compact$\}$)
(2) $\nRightarrow$ (1): ??? (For this we need model in which 2^{ω_2} is pseudoradial)

Elementary submodels

Given two models $M \subseteq V$, M is an elementary submodel of N $(M \prec V)$ if any statement, with parameters in M, true in V is also true in M.

Nice features:

1) If M is countable, $M \cap \omega_1$ is a countable ordinal.

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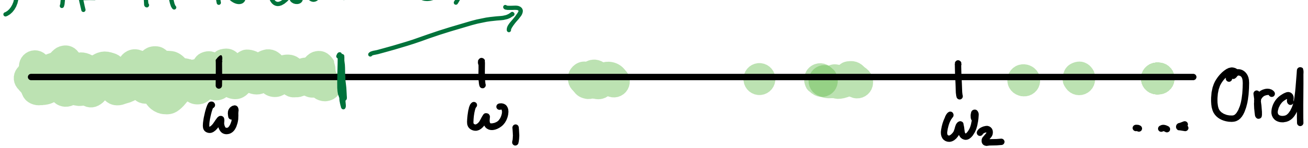

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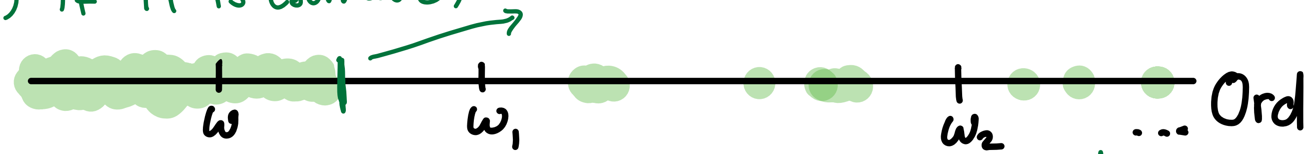

2) If $A \in M$, M countable and there is no bijection $\varphi : \omega \to A$ in M, then A is uncountable.

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Lemma: If $\{A_\alpha : \alpha < \omega_1\}$ is a family of finite sets, then it has an uncoutable Δ-system.

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Proof:

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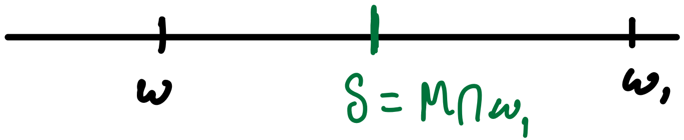

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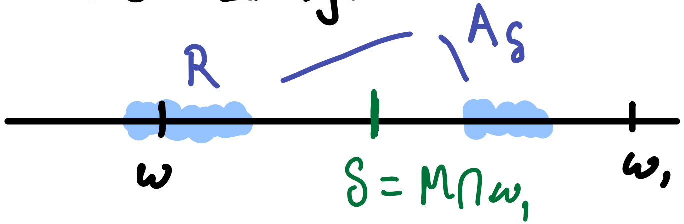

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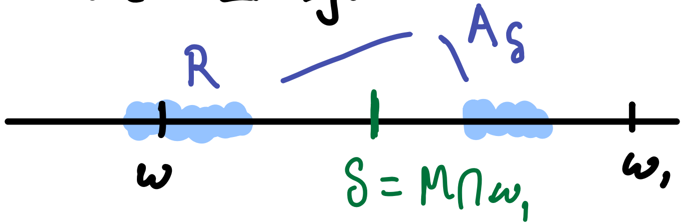

2) Let $R = M \cap A_\delta$
3) Consider $B \subseteq \omega_1$ s.t. $\{A_\alpha : \alpha \in B\}$ is a maximal Δ-system with root R.

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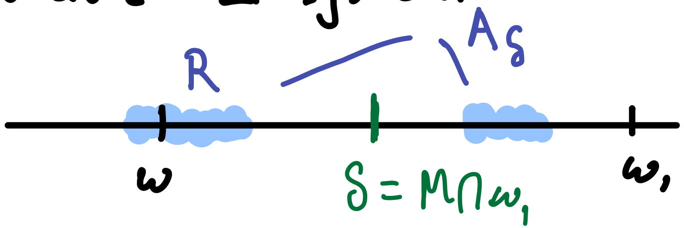

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3) Consider $B \subseteq \omega_1$ s.t. $\{A_\alpha : \alpha \in B\}$ is a maximal Δ-system with root R.

4) If there is a bijection $\varphi : \omega \to A$ in M, then $\delta \in M$!

Forcing, sequences and elementarity

Definition

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- $M[G] \cap [\omega]^\omega = M[G_M] \cap [\omega]^\omega = V[G_M] \cap [\omega]^\omega$

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Recall a space is **weakly pseudoradial** if the radial closure of any countable sets is closed.

Theorem (Dow, Juhász, Soukup, Szentmiklóssy, 1996)

In the forcing extension by adding any number of Cohen reals over a model of CH, every CSC space is pseudoradial.

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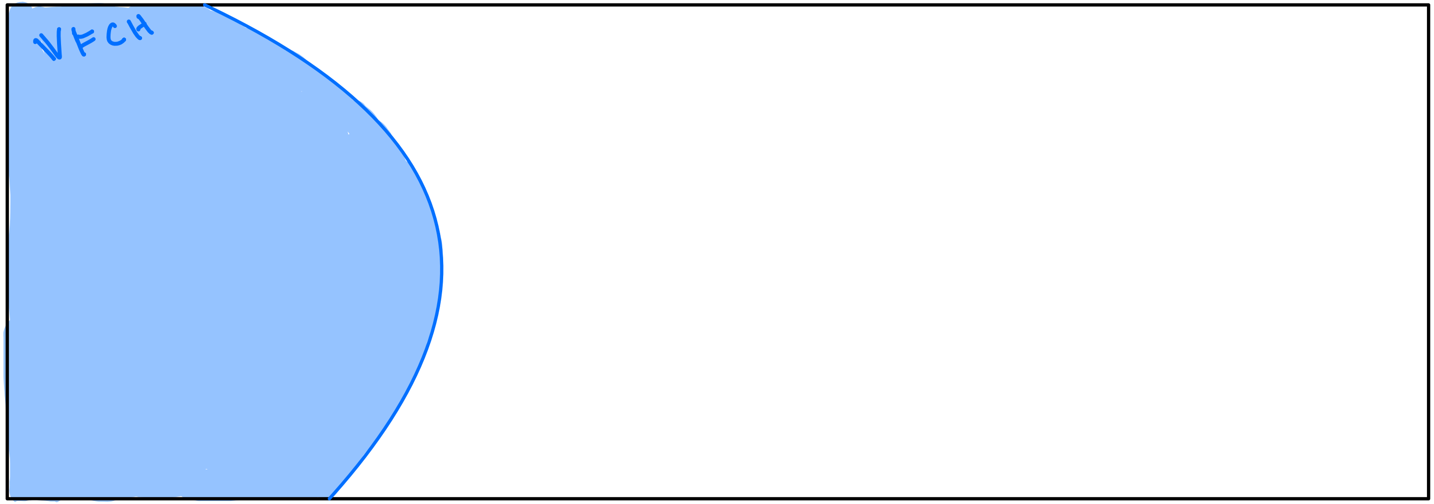
Theorem (Dow, B. , 2022)

Let $V \models \square$ +GCH, and let G be a generic filter for a poset with finally property K. Then, in $V[G]$ every CSC space is weakly pseudoradial.



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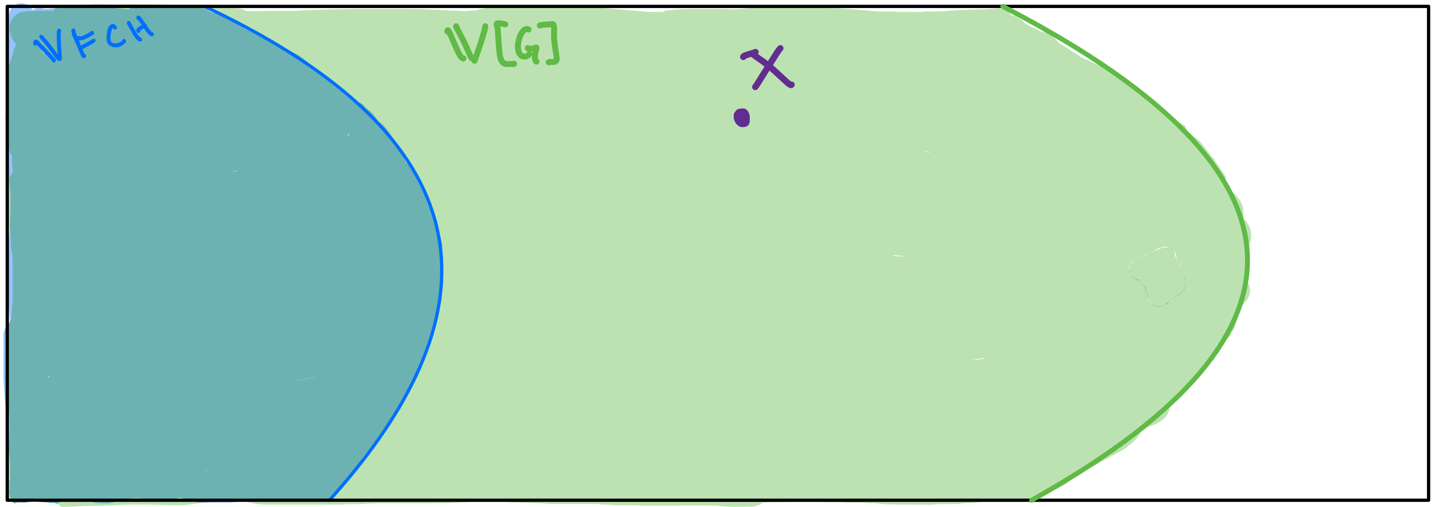

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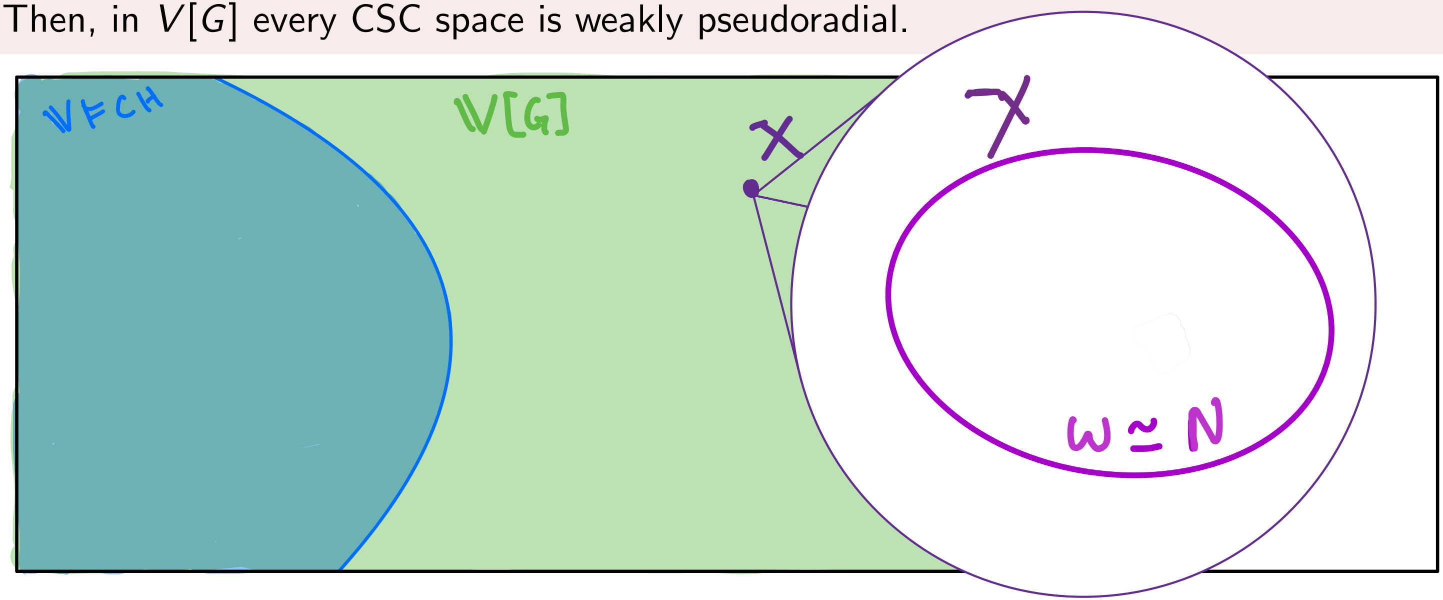

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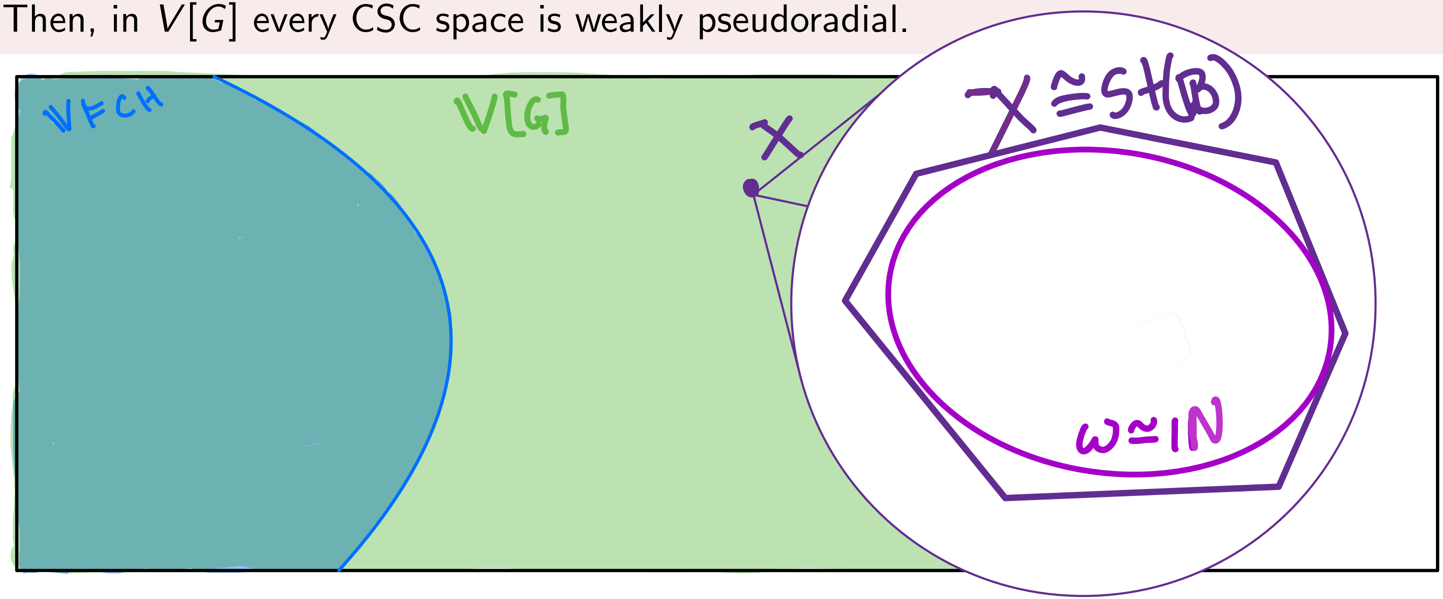

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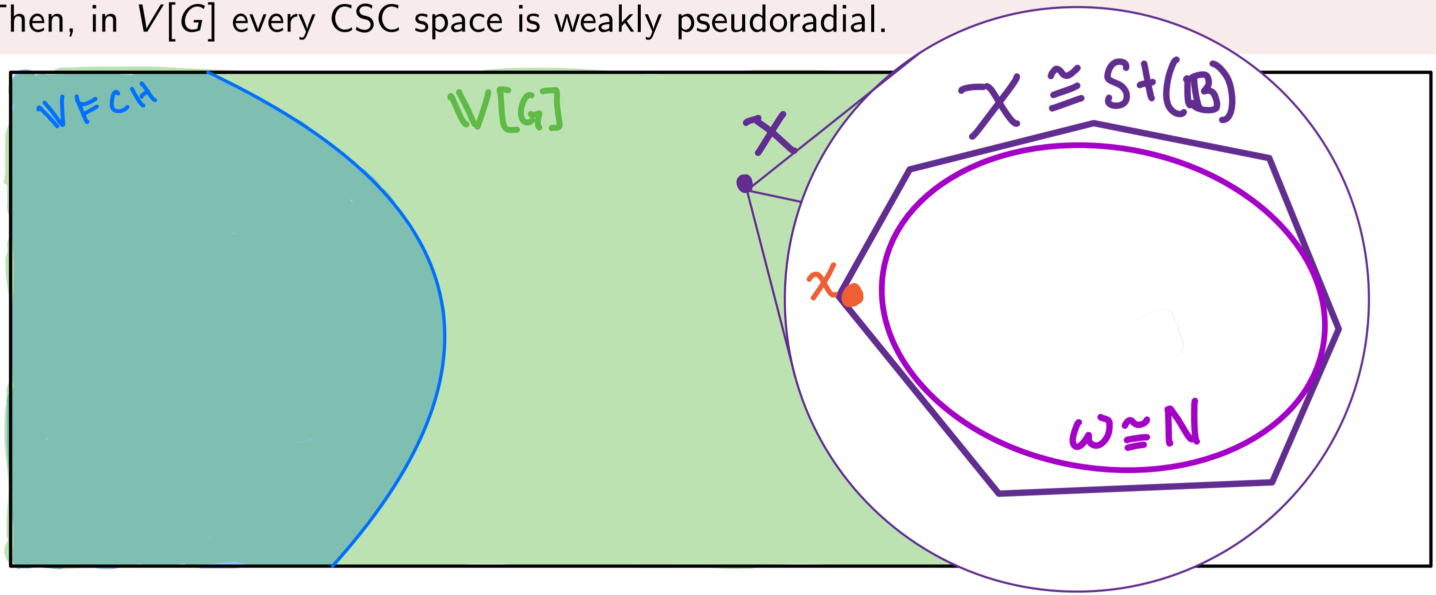

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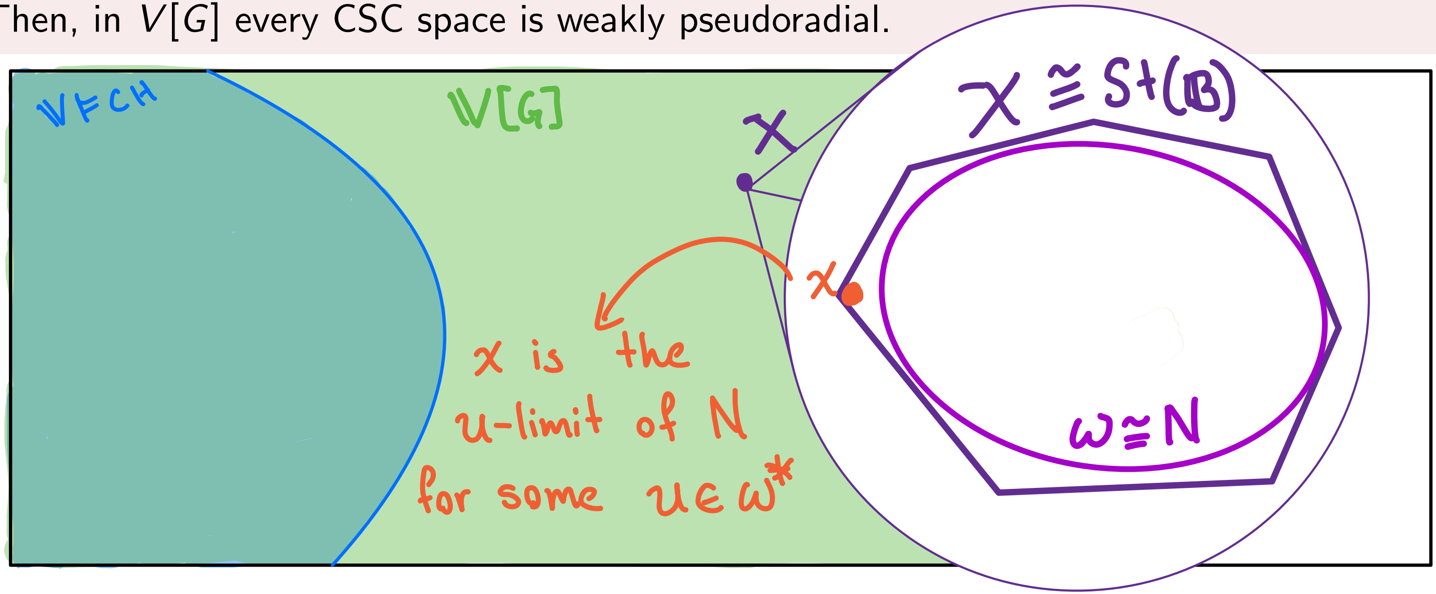

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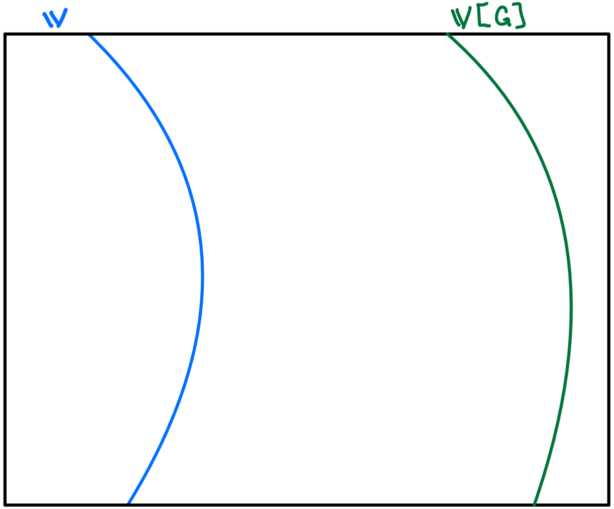

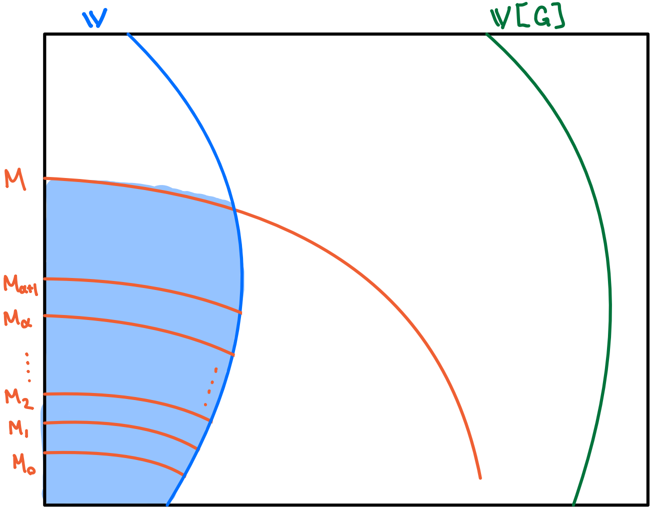

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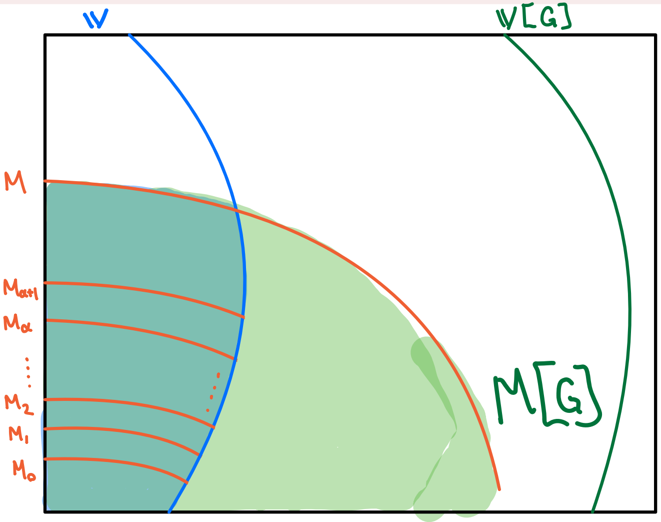

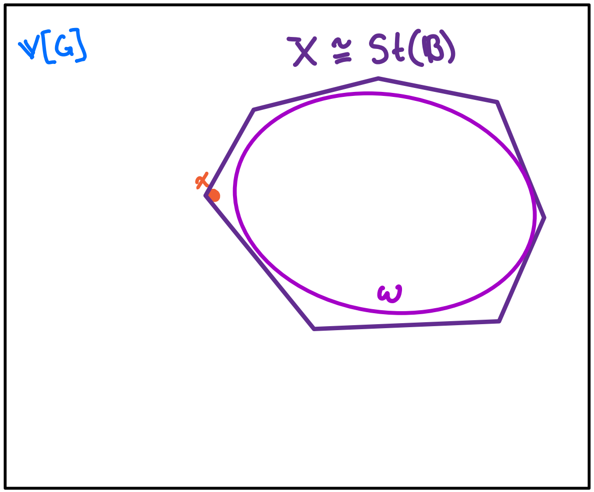

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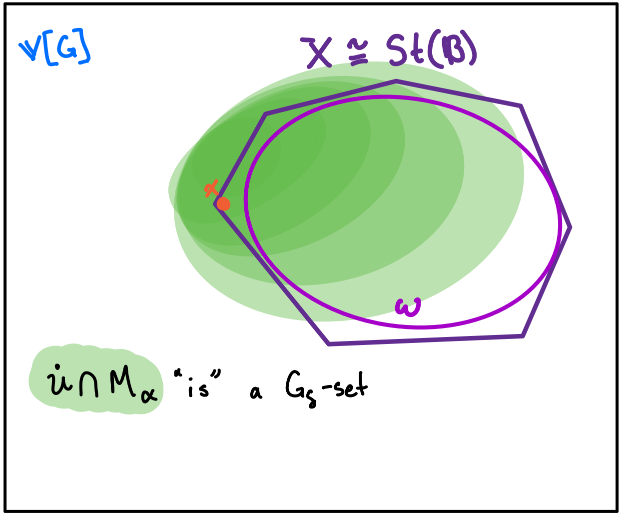

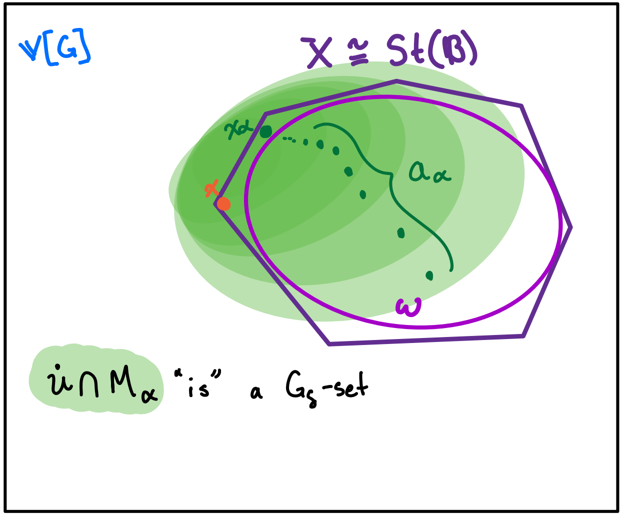

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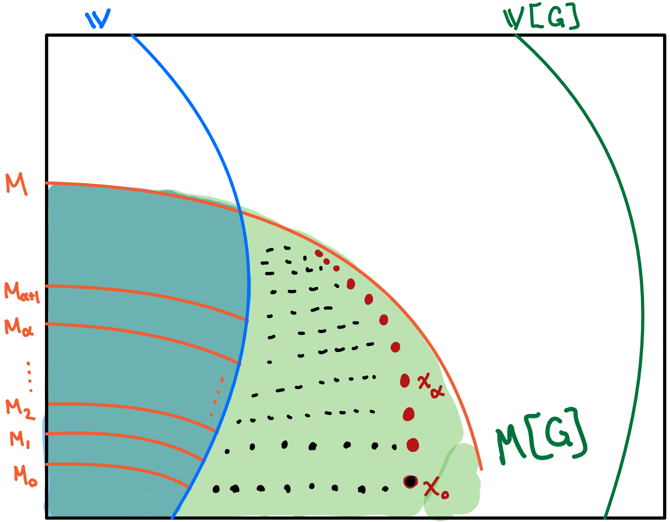
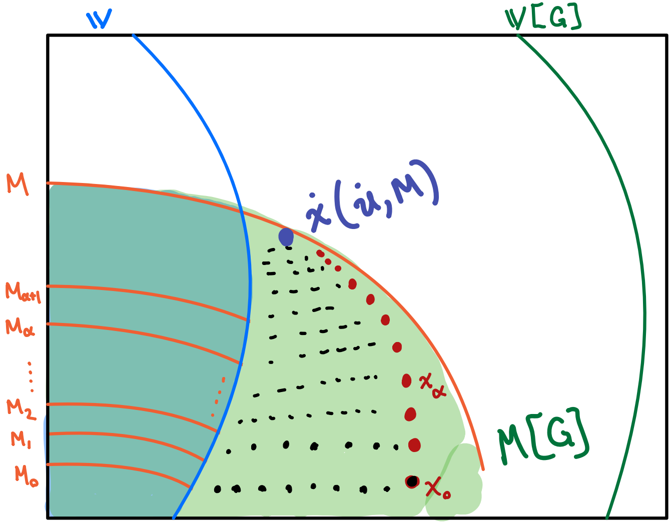
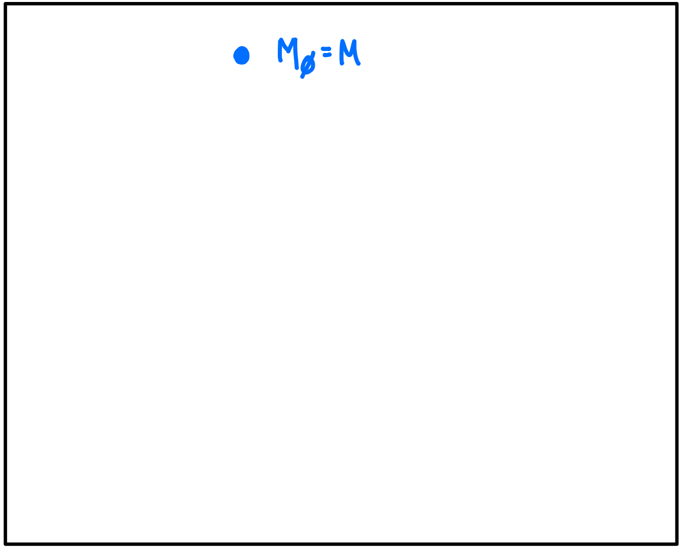
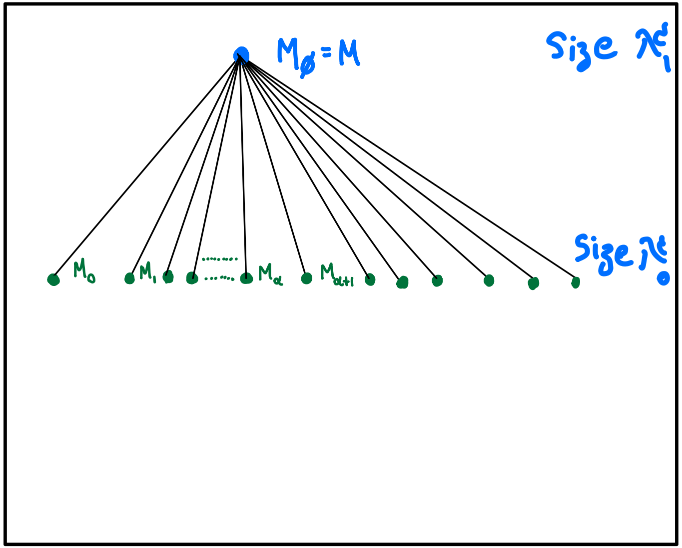


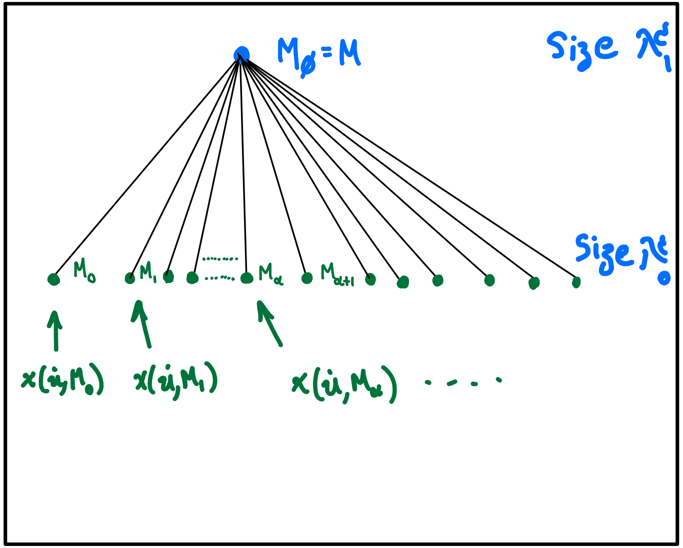




$M_\emptyset = M$



$M_\emptyset = M$
Size $\aleph_1$
Size $\aleph_0$
M_0
M_1
M_α
$M_{\alpha+1}$



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Size $\aleph_1$
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M_0
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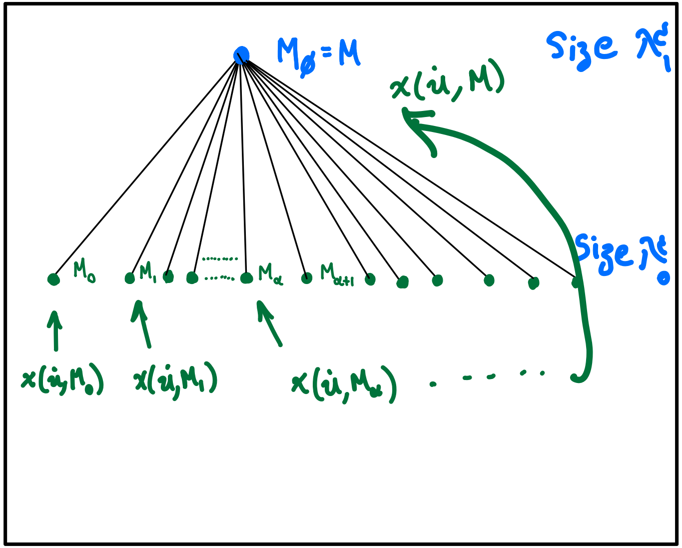
$x(u, M_0)$
$x(u, M_1)$
$x(u, M_\alpha)$



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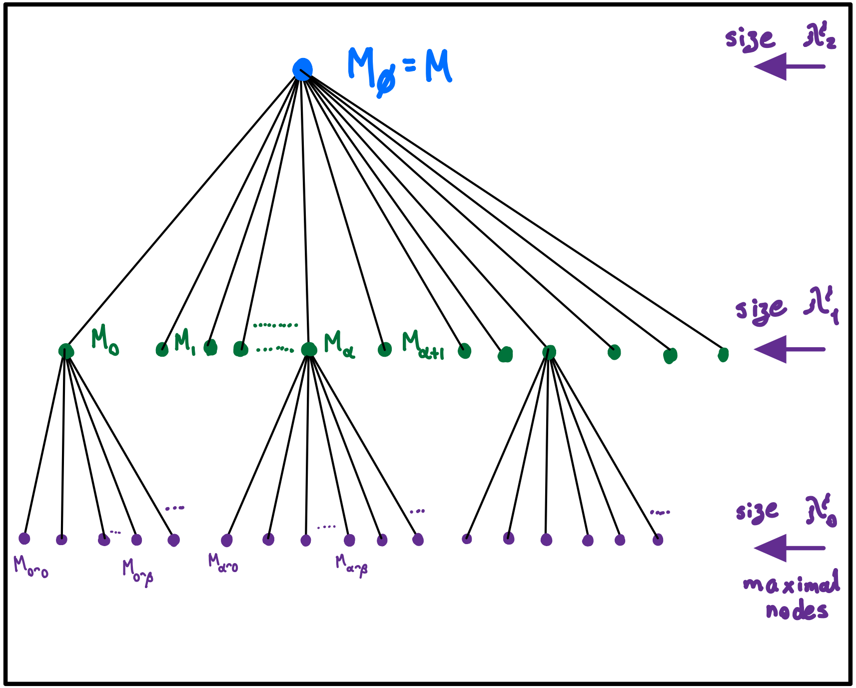
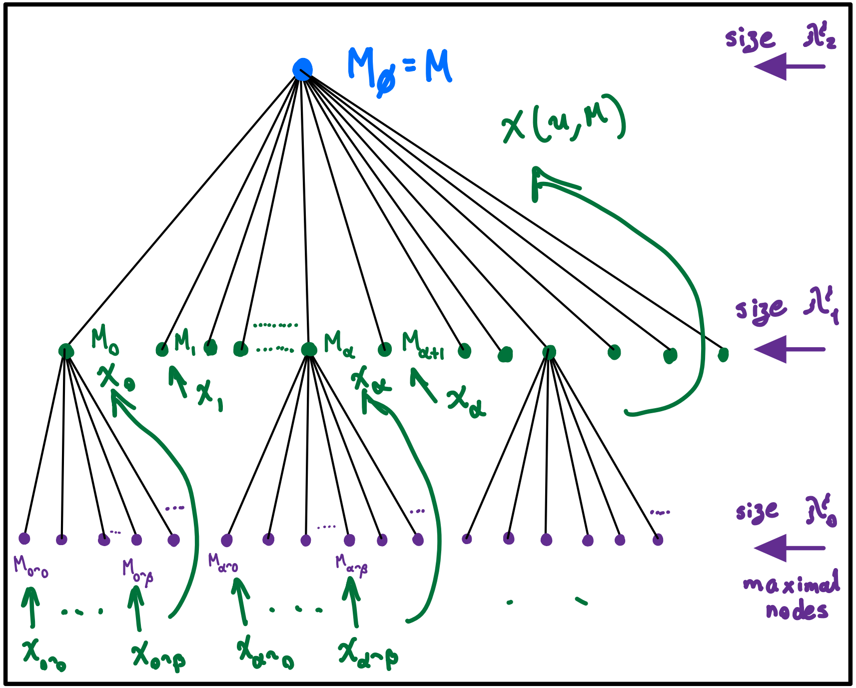


$M_\emptyset = M$
size $\aleph_2$
M_0
M_1
M_α
$M_{\alpha+1}$
size $\aleph_1$
$M_{0^\frown 0}$
$M_{0^\frown \beta}$
$M_{\alpha^\frown 0}$
$M_{\alpha^\frown \beta}$
size $\aleph_0$
maximal nodes



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$X(u, M)$
size $\aleph_2'$
M_0
M_1
M_α
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X_0
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Davies Trees

INFINITE COMBINATORICS PLAIN AND SIMPLE

DÁNIEL T. SOUKUP AND LAJOS SOUKUP

Now, we say that a *high Davies-tree for κ over x* is a sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $(H(\theta), \in, \lhd)$ for some large enough regular θ such that

(I) $[M_\alpha]^\omega \subset M_\alpha$, $|M_\alpha| = \mathfrak{c}$ and $x \in M_\alpha$ for all $\alpha < \kappa$,

(II) $[\kappa]^\omega \subset \bigcup_{\alpha<\kappa} M_\alpha$, and

(III) for each $\beta < \kappa$ there are $N_{\beta,j} \prec H(\theta)$ with $[N_{\beta,j}]^\omega \subset N_{\beta,j}$ and $x \in N_{\beta,j}$ for $j < \omega$ such that

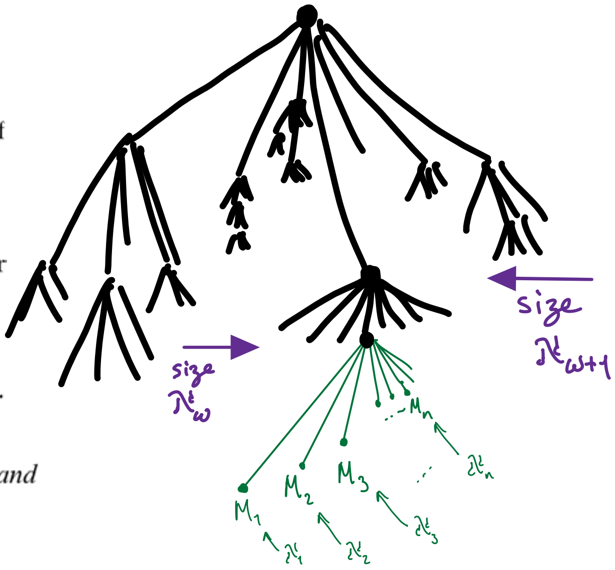
$$\bigcup\{M_\alpha : \alpha < \beta\} = \bigcup\{N_{\beta,j} : j < \omega\}.$$

THEOREM 8.1. *There is a high Davies-tree $\langle M_\alpha : \alpha < \kappa \rangle$ for κ over x whenever*

1. *$\kappa = \kappa^\omega$ and*
2. *μ is ω-inaccessible, $\mu^\omega = \mu^+$ and $\square_\mu$ holds for all μ with $\mathfrak{c} < \mu < \kappa$ and $cf(\mu) = \omega$.*

Moreover, the high Davies-tree $\langle M_\alpha : \alpha < \kappa \rangle$ can be constructed so that

3. *$\langle M_\alpha : \alpha < \beta \rangle \in M_\beta$ for all $\beta < \kappa$ and*
4. *$\bigcup\{M_\alpha : \alpha < \kappa\}$ is also a countably closed elementary submodel of $H(\theta)$.*



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Remark

- $\omega_1 \leq \mathfrak{pse} \leq \mathfrak{s}$ (2^ω is pseudoradial)
- If $\mathfrak{s} > \omega_1$ if and only if $\mathfrak{pse} > \omega_1$ (2^{ω_1} is sequentially compact if and only if it is pseudoradial)

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Question

- Where is $\mathfrak{pse}$ located in van Douwen diagram?
- Is $\mathfrak{pse}$ regular?
- Can we replace $\mathfrak{s}$ by $\mathfrak{pse}$ in...?

Theorem (Dow, Juhász, Soukup, Szentmiklóssy, 1996)

If X compact and not pseudoradial, then there is $Y \in [X]^{<\mathfrak{s}}$ such that $\overline{Y}$ is not pseudoradial.

Let X be compact, then

Remark

$$\begin{gathered}(1)\ X \text{ is pseudoradial}\\ \Downarrow \\ (2)\ \text{'closed separable subspaces are pseudoradial'}\\ \Downarrow \\ (3)\ \text{'radial closures of countable sets are closed'} == \textit{weakly pseudoradial}\\ \Downarrow \\ (4)\ X \text{ is sequentially compact}\end{gathered}$$

(4) $\Rightarrow$ (1): Under CH (Šapirovskiĭ)
(4) $\Rightarrow$ (3): Under $\mathfrak{p} = \mathfrak{c}$
(4) $\not\Rightarrow$ (3): In progress (MATRIX ITERATION) Forcing again >< :(
(3) $\not\Rightarrow$ (2): It is consistent with $\mathfrak{c} = \mathfrak{p} = \mathfrak{s} = \aleph_3$ that 2^{ω_2} is not pseudoradial.
($\mathfrak{s} = \min\{\kappa : 2^\kappa$ is not sequentially compact$\}$)
(2) $\not\Rightarrow$ (1): ??? (For this we need model in which 2^{ω_2} is pseudoradial)

First, the ingredients

Theorem

It is consistent that there is a compactification X of ω and a (descending) mod finite family $\{a_\alpha : \alpha \in \omega_1\} \subset [\omega]^\omega$ s.t.:

1. *the weight of X is less than $\mathfrak{s}$,*
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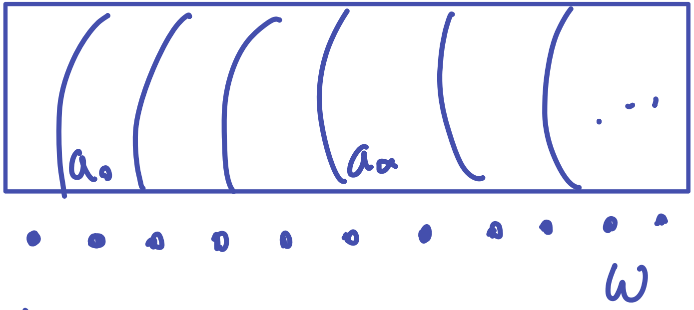
4. *for each cub $C \subset \omega_1$, there is a a clopen $U_C \subset X$ satisfying that each of $\{\delta \in C : a_\delta \setminus a_{\delta_C^+} \subset U_C\}$ and $\{\delta \in C : a_\delta \setminus a_{\delta_C^+} \cap U_C = \emptyset\}$ are uncountable.*

If a space X is as in Theorem. then it follows that X is CSC and the radial closure of ω is not closed.

$\bar{X}$ is CSC



$Y_\alpha = cl_X(\omega \setminus a_\alpha)$, $Y = \bigcup_{\alpha < \omega_1} Y_\alpha = \bar{\omega}^r$

Definition

A matrix iteration is a $\kappa \times \lambda$-system
$\underline{\mathbf{s}} = \{\langle P^{\underline{\mathbf{s}}}_{\alpha,\gamma}, \dot{Q}^{\underline{\mathbf{s}}}_{\alpha,\gamma} : \alpha \leq \kappa, \gamma \leq \lambda\rangle$
satisfying the following conditions:

- For each $\alpha \leq \kappa$ and $\gamma < \lambda$, $\dot{Q}^{\underline{\mathbf{s}}}_{\alpha,\gamma}$ is a $P^{\underline{\mathbf{s}}}_{\alpha,\gamma}$-name of a ccc poset.
- For each $\alpha \leq \kappa$, $\langle P^{\underline{\mathbf{s}}}_{\alpha,\gamma}, \dot{Q}^{\underline{\mathbf{s}}}_{\alpha,\gamma} : \gamma < \lambda\rangle$ is a finite support iteration and $P^{\underline{\mathbf{s}}}_{\alpha,\lambda}$ is the limit.
- For $\gamma < \lambda$ and $\alpha < \beta \leq \kappa$, $P^{\underline{\mathbf{s}}}_{\beta,\gamma}$ forces that $\dot{Q}^{\underline{\mathbf{s}}}_{\alpha,\gamma}$ is a subposet of $\dot{Q}^{\underline{\mathbf{s}}}_{\beta,\gamma}$ in which each $P^{\underline{\mathbf{s}}}_{\alpha,\gamma}$-name of a maximal antichain of $\dot{Q}^{\underline{\mathbf{s}}}_{\alpha,\gamma}$ is a $P^{\underline{\mathbf{s}}}_{\beta,\gamma}$-name of a maximal antichain of $\dot{Q}^{\underline{\mathbf{s}}}_{\beta,\gamma}$.
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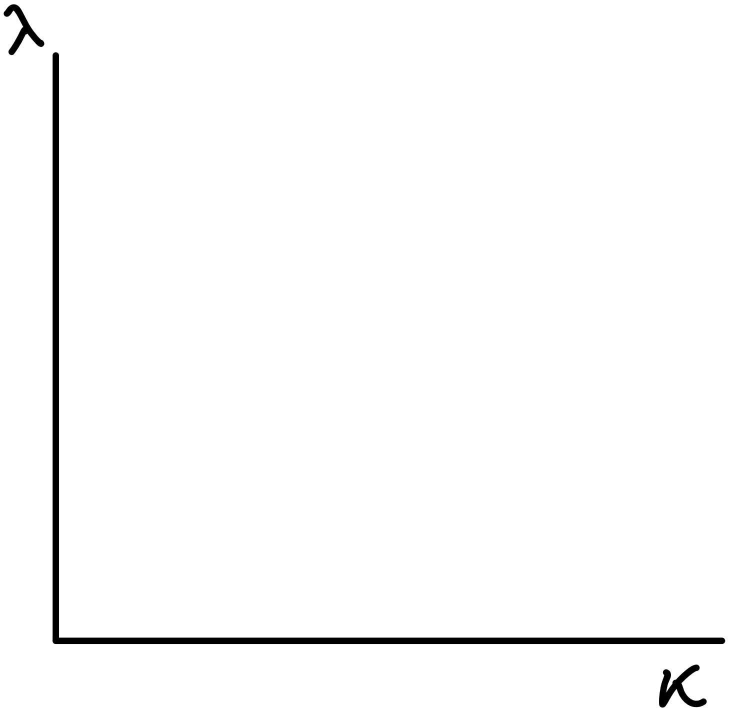

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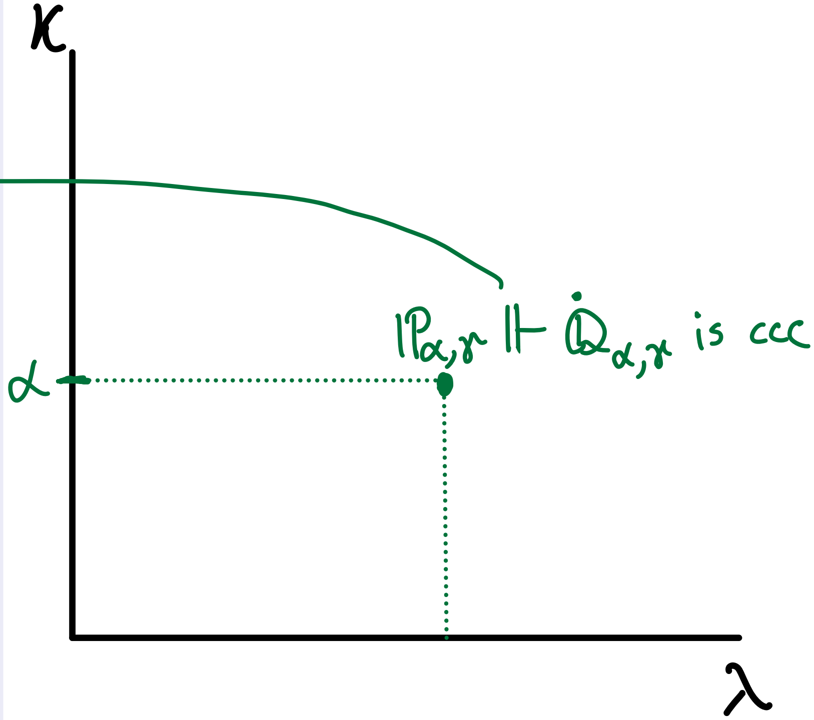

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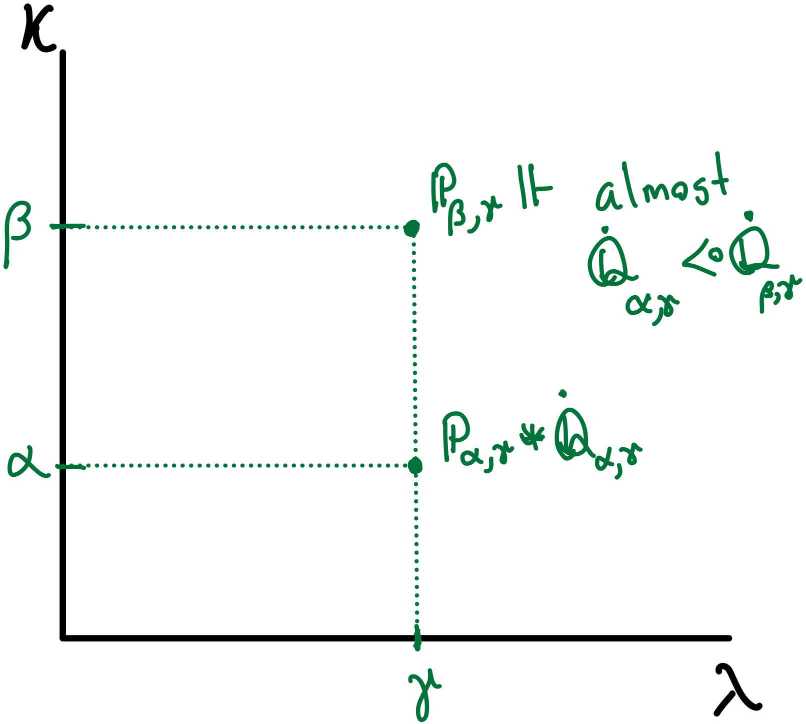

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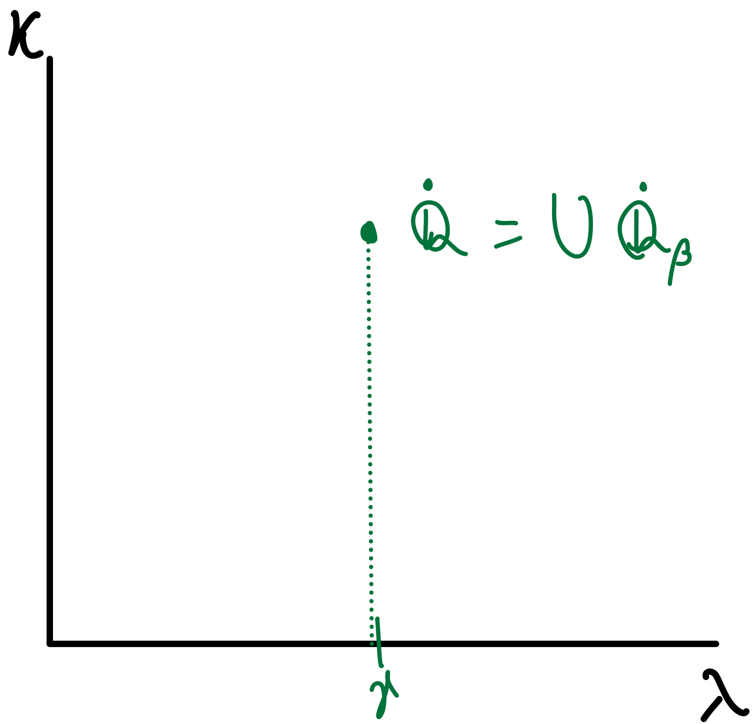
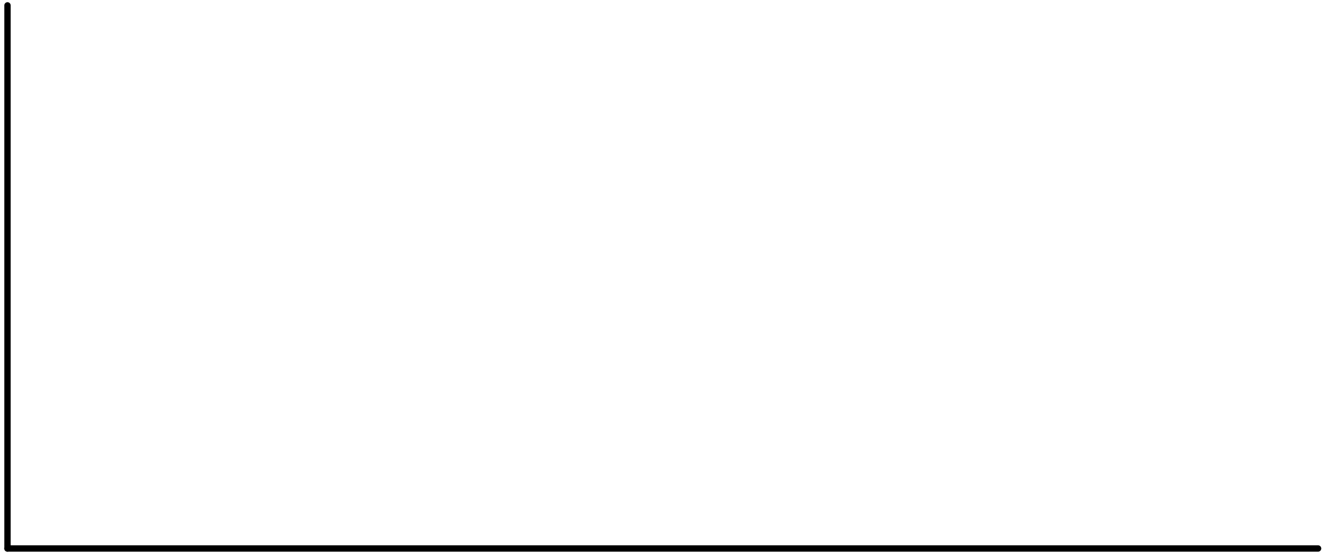


The posets



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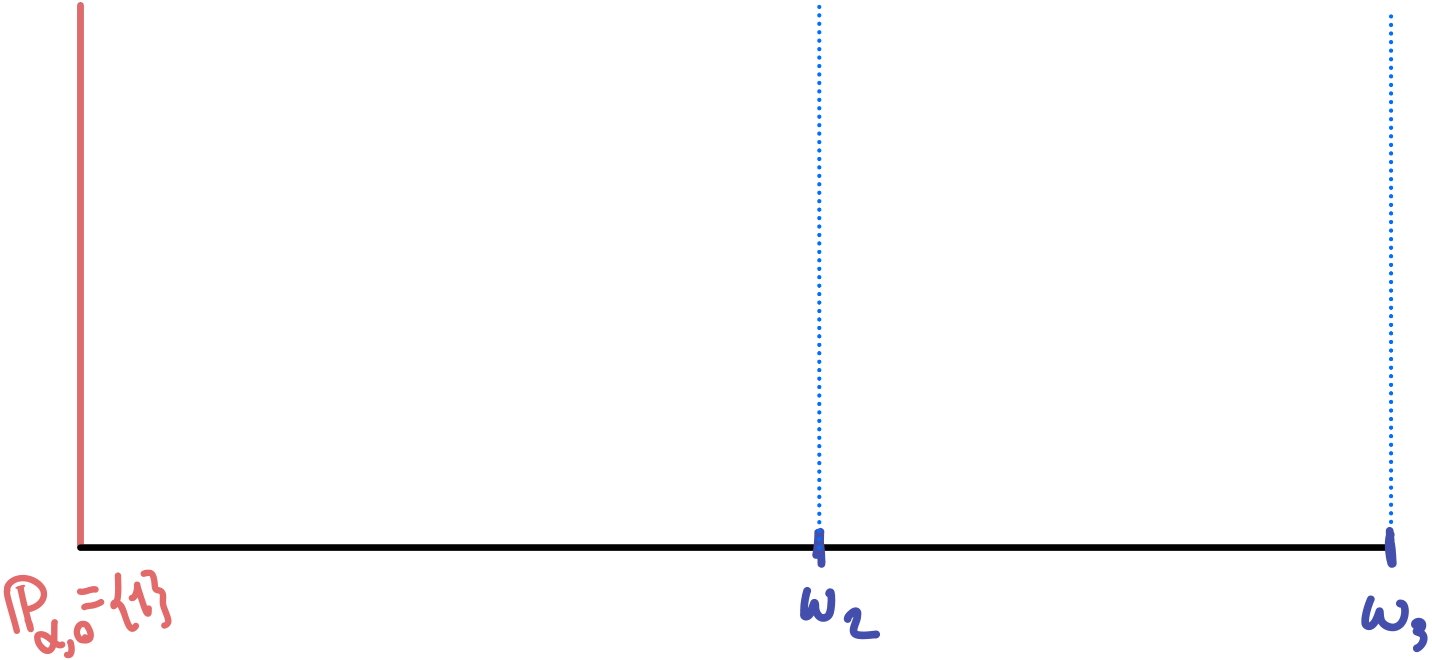

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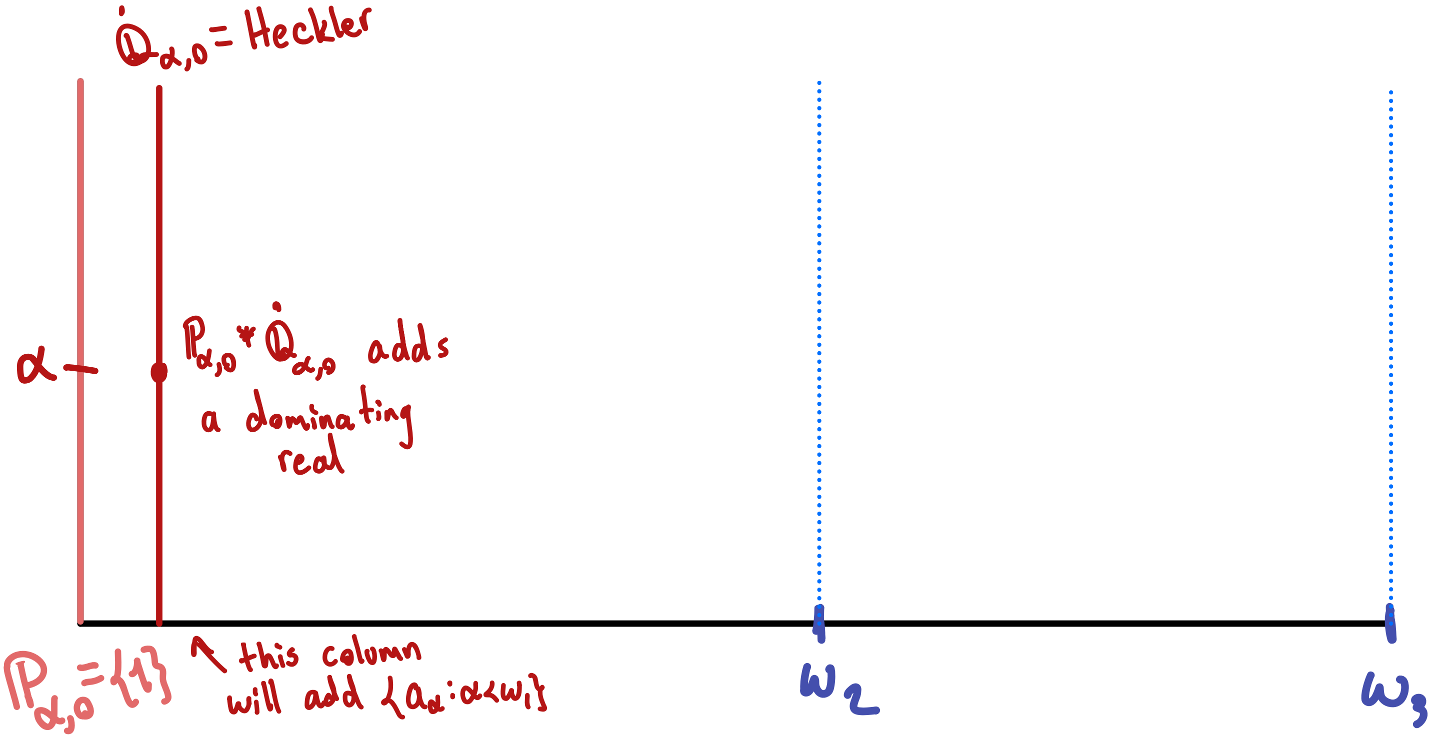
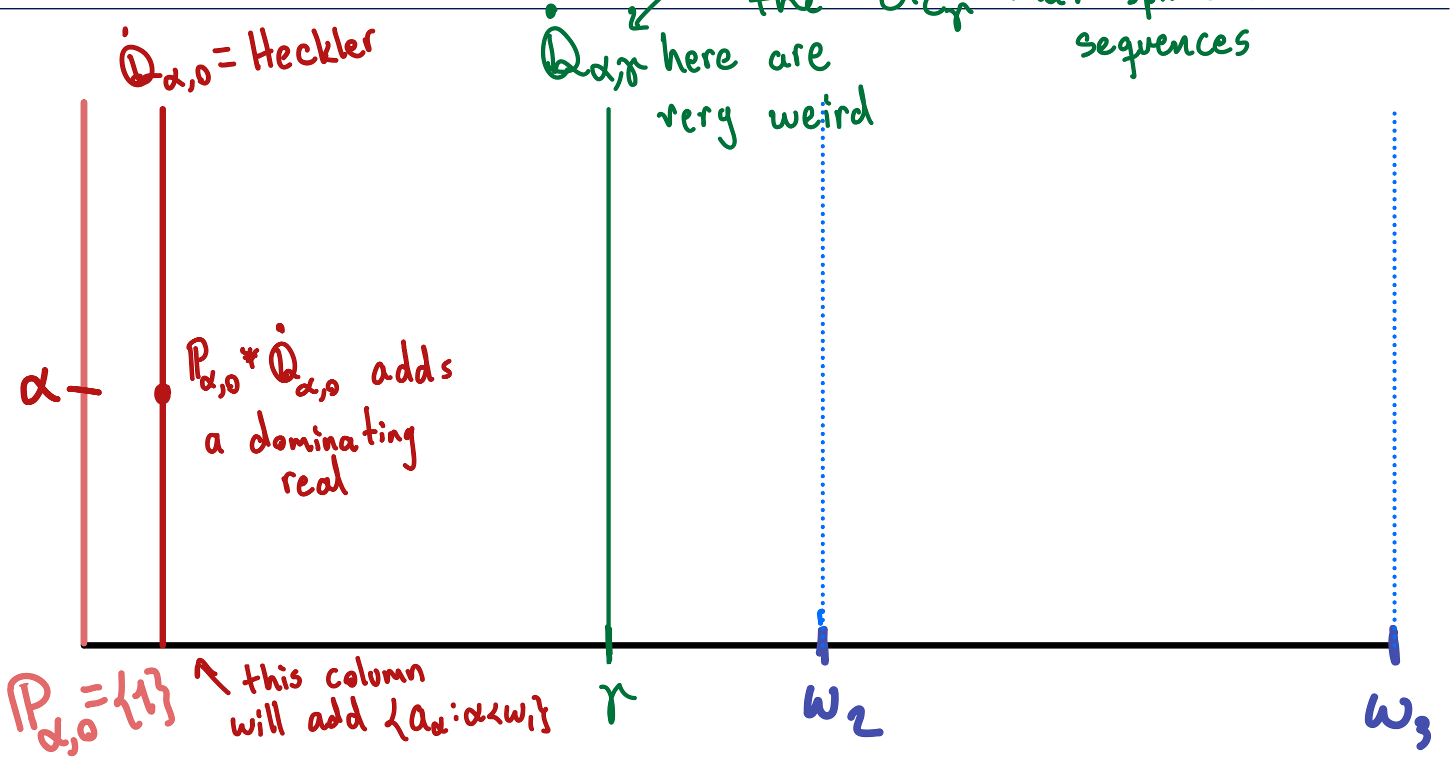

$\dot{\mathbb{Q}}_{\alpha,0}$ = Heckler
$\mathbb{P}_{\alpha,0} * \dot{\mathbb{Q}}_{\alpha,0}$ adds a dominating real
α
$\mathbb{P}_{\alpha,0} = \{1\}$
this column will add $\{a_\alpha : \alpha < \omega_1\}$
ω_2
ω_3

The posets

They add the $\mathcal{U}_{\alpha,\gamma}$ that splits the sequences



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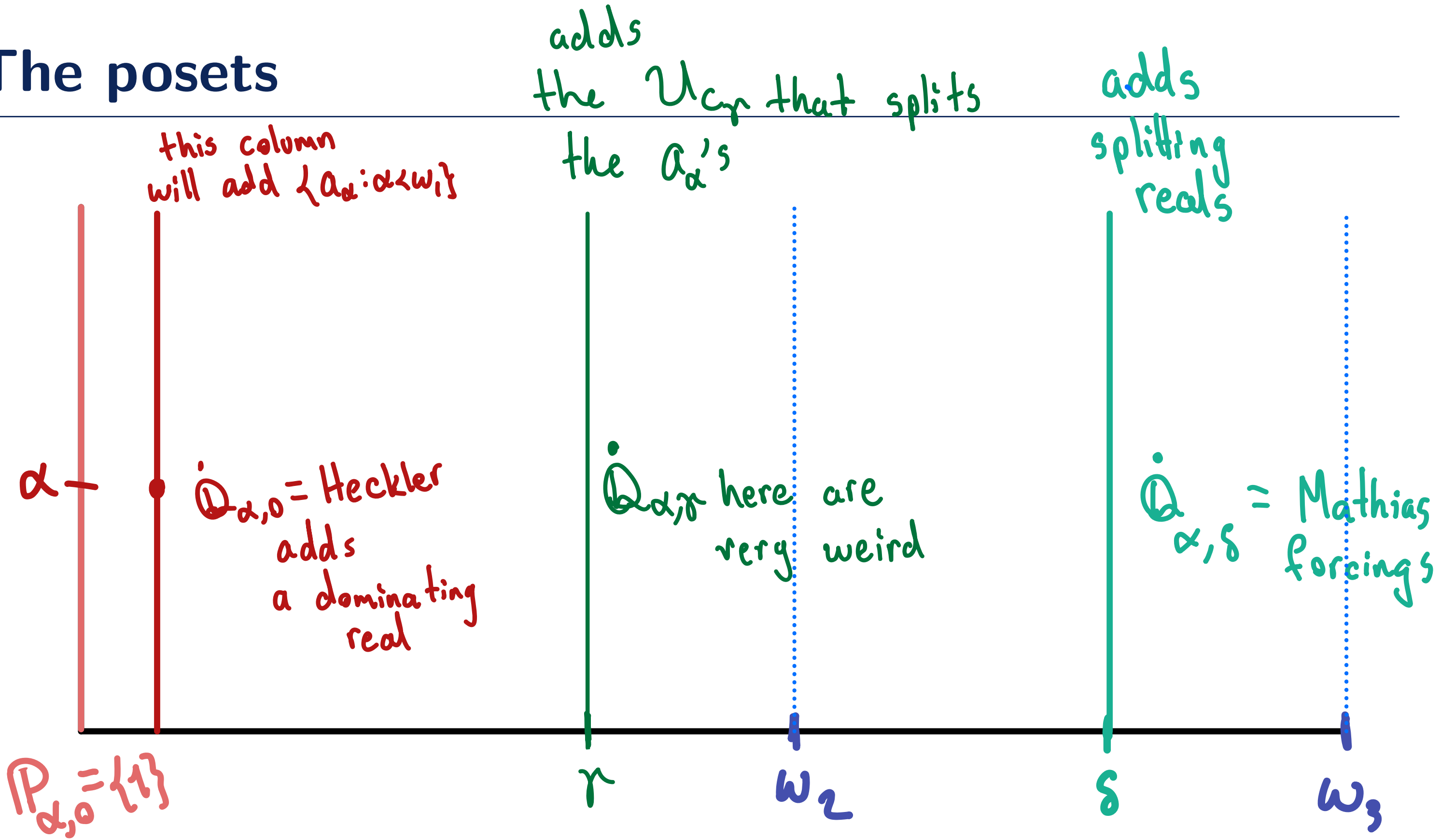
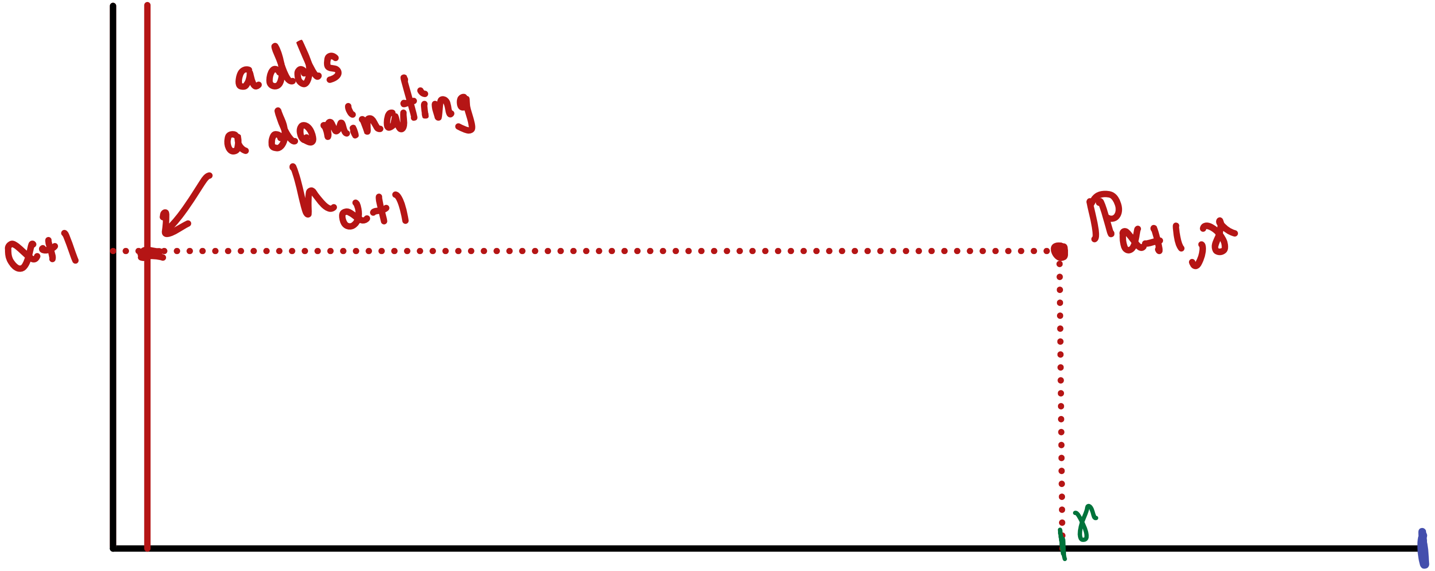


adds the $\mathcal{U}_{c_\gamma}$ that splits the a_α's
adds splitting reals
this column will add $\{a_\alpha : \alpha < \omega_1\}$
α
$\dot{\mathbb{Q}}_{\alpha,0}$ = Heckler adds a dominating real
$\dot{\mathbb{Q}}_{\alpha,\gamma}$ here are very weird
$\dot{\mathbb{Q}}_{\alpha,\delta}$ = Mathias forcings
$\mathbb{P}_{\alpha,0} = \{1\}$
γ
ω_2
δ
ω_3

Lemma

For each $\alpha < \omega_1$ and each P_{α,ω_3}-name $\dot{f}$ of an element of ω^ω, it is forced by $P_{\alpha+1,\omega_3}$ that $\dot{h}_{\alpha+1}$ is not dominated by $\dot{f}$.



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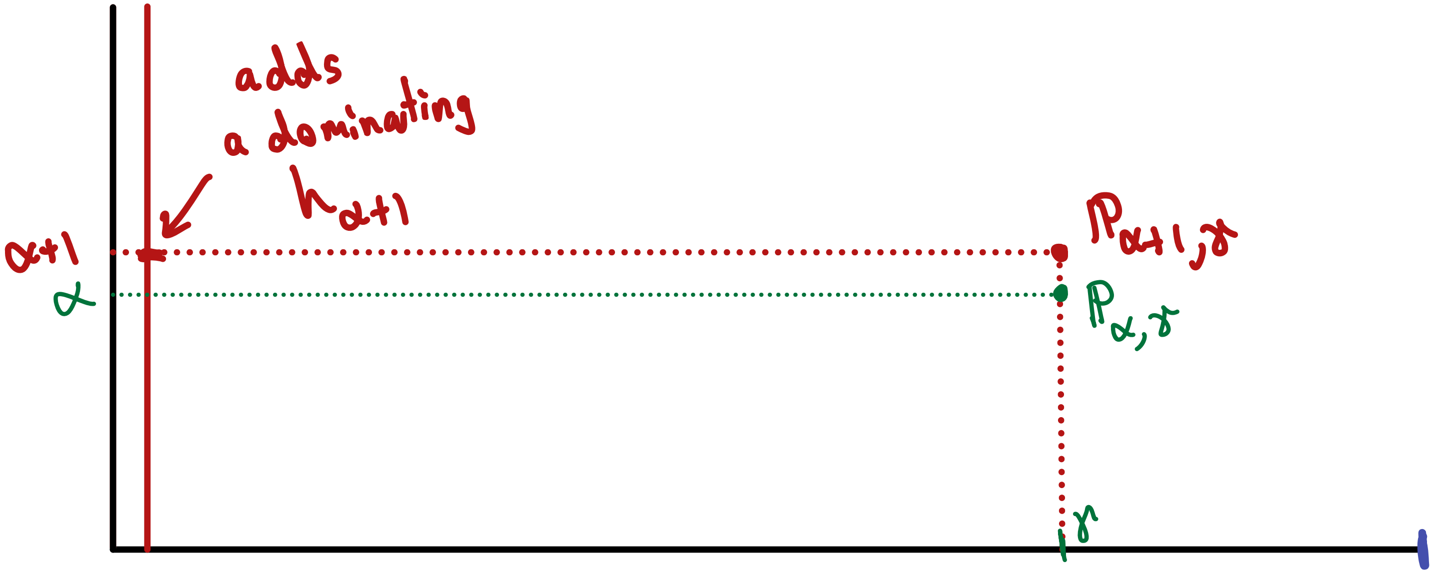

Thank you!