Putting cats together

Completions of mereological models

Imanol Mozo Carollo

imanol.mozo@ehu.eus

University of the Basque Country UPV/EHU

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

What's mereology?

What's mereology?

"Mereology (from the Greek μερος, 'part') is the theory of parthood relations: of the relations of part to whole and the relations of part to part within a whole."

— Achille Varzi , *Mereology*, The Stanford Encyclopedia of Philosophy

What's mereology?

"Mereology (from the Greek μερος, 'part') is the theory of parthood relations: of the relations of part to whole and the relations of part to part within a whole."

— Achille Varzi , *Mereology*, The Stanford Encyclopedia of Philosophy

- From presocratics, through medieval scholastic philosophy, to Leibniz and Kant, but. . .

What's mereology?

"Mereology (from the Greek μερος, 'part') is the theory of parthood relations: of the relations of part to whole and the relations of part to part within a whole."

— Achille Varzi , *Mereology*, The Stanford Encyclopedia of Philosophy

- From presocratics, through medieval scholastic philosophy, to Leibniz and Kant, but. . .
- . . . as a formal theory, it is relatively recent.

What's mereology?

"Mereology (from the Greek μερος, 'part') is the theory of parthood relations: of the relations of part to whole and the relations of part to part within a whole."

— Achille Varzi , *Mereology*, The Stanford Encyclopedia of Philosophy

- From presocratics, through medieval scholastic philosophy, to Leibniz and Kant, but. . .
- . . . as a formal theory, it is relatively recent.

E. Husserl,
Logical Investigation (1901).

What's mereology?

"Mereology (from the Greek μερος, 'part') is the theory of parthood relations: of the relations of part to whole and the relations of part to part within a whole."

— Achille Varzi , *Mereology*, The Stanford Encyclopedia of Philosophy

- From presocratics, through medieval scholastic philosophy, to Leibniz and Kant, but. . .
- . . . as a formal theory, it is relatively recent.

E. Husserl,
Logical Investigation (1901).

S. Leśniewski,
Foundations of the General Theory of Sets (1916).

S. Leśniewski,
Foundations of Mathematics (1927–1931).

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

> *As a formal theory of parts and wholes, mereology should tell us three sorts of thing. It should say:*

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

> *As a formal theory of parts and wholes, mereology should tell us three sorts of thing. It should say:*
>
> 1. *what sort of relation parthood is;*

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

> *As a formal theory of parts and wholes, mereology should tell us three sorts of thing. It should say:*
>
> 1. *what sort of relation parthood is;*
> 2. *what sort of conditions govern mereological decomposition, i.e., intuitively, what happens when we "subtract things" from a given whole;*

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

> *As a formal theory of parts and wholes, mereology should tell us three sorts of thing. It should say:*
>
> 1. *what sort of relation parthood is;*
> 2. *what sort of conditions govern mereological decomposition, i.e., intuitively, what happens when we "subtract things" from a given whole;*
> 3. *what sort of conditions govern mereological composition, i.e., intuitively, what it takes to form a whole by "adding things" together.*

What is mereology as a formal theory?

In words of A. J. Cotnoir and A. Varzi,

> *As a formal theory of parts and wholes, mereology should tell us three sorts of thing. It should say:*
>
> 1. *what sort of relation parthood is;*
> 2. *what sort of conditions govern mereological decomposition, i.e., intuitively, what happens when we "subtract things" from a given whole;*
> 3. *what sort of conditions govern mereological composition, i.e., intuitively, what it takes to form a whole by "adding things" together.*

So. . . where are the cats?

So... where are the cats?

So. . . where are the cats?

"To be sure, if we accept mereology, we are committed to the existence of all manner of mereological fusions. But given a prior commitment to cats, say, a commitment to cat-fusions is not a further commitment. The fusion is nothing over and above the cats that compose it. It just is them. They just are it. Take them together or take them separately, the cats are the same portion of Reality either way. Commit yourself to their existence all together or one at a time, it's the same commitment either way."

— David Lewis , *Parts of classes*, Basil Blackwell (1991), pp. 81–82 .

So. . . where are the cats?

"To be sure, if we accept mereology, we are committed to the existence of all manner of mereological fusions. But given a prior commitment to cats, say, a commitment to cat-fusions is not a further commitment. The fusion is nothing over and above the cats that compose it. It just is them. They just are it. Take them together or take them separately, the cats are the same portion of Reality either way. Commit yourself to their existence all together or one at a time, it's the same commitment either way."

— David Lewis , *Parts of classes*, Basil Blackwell (1991), pp. 81–82 .

Universalism is the thesis that mereological composition is unrestricted.

Core principles

Core principles

Parthood $\leq$ *is a partial order*.

Core principles

Parthood $\leq$ *is a partial order*.

Basic definitions

Core principles

Parthood $\leq$ is a partial order.

Basic definitions

- Proper parthood

$$x < y :\equiv x \leq y \,\&\, \neg x = y.$$

Core principles

Parthood $\leq$ is a partial order.

Basic definitions

- Proper parthood

$$x < y :\equiv x \leq y \,\&\, \neg x = y.$$

- Overlap

$$x \circ y :\equiv \exists z (z \leq x \,\&\, z \leq y).$$

Core principles

Parthood $\leq$ is a partial order.

Basic definitions

- Proper parthood

$$x < y :\equiv x \leq y \,\&\, \neg x = y.$$

- Overlap

$$x \circ y :\equiv \exists z (z \leq x \,\&\, z \leq y).$$

- Disjointness

$$x \wr y :\equiv \neg(x \circ y)$$

Core principles

Parthood $\leq$ is a partial order.

Basic definitions

- Proper parthood
$$x < y :\equiv x \leq y \,\&\, \neg x = y.$$
- Overlap
$$x \circ y :\equiv \exists z(z \leq x \,\&\, z \leq y).$$
- Disjointness
$$x \wr y :\equiv \neg(x \circ y)$$

Remark

Classical mereology rejects the existence of the *empty part*.

Decomposition of parts

Decomposition of parts

Supplementation

Decomposition of parts

Supplementation

(WS) $x < y \rightarrow \exists z(z \leq y \,\&\, x \wr z)$, (*Weak supplementation*)

Decomposition of parts

Supplementation

(WS) $x < y \rightarrow \exists z(z \leq y \,\&\, x \wr z)$, (*Weak supplementation*)

(SS) $y \nleq x \rightarrow \exists z(z \leq y \,\&\, x \wr z)$, (*Strong supplementation*)

Decomposition of parts

Supplementation

(WS) $x < y \to \exists z(z \leq y \,\&\, x \wr z)$, (*Weak supplementation*)

(SS) $y \not\leq x \to \exists z(z \leq y \,\&\, x \wr z)$, (*Strong supplementation*)

Models of Core Principles + StrongSup are precisely *separative posets*.

Composition of parts

Composition of parts

Definition of compositions + Existence of compositions

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x))$, (*Fusion*)

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x)),$ (*Fusion*)
2. $S_A(t) :\equiv \forall x(x \in A \to x \leq t) \,\&\, \forall y(y \leq t \to \exists x(x \in A \,\&\, y \circ x)),$ (*Sum*)

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x)),$ (*Fusion*)
2. $S_A(t) :\equiv \forall x(x \in A \rightarrow x \leq t) \,\&\, \forall y(y \leq t \rightarrow \exists x(x \in A \,\&\, y \circ x)),$ (*Sum*)

Theorem

Core Principles + WeakSup + $\exists$ *Second Order Sums* $\equiv$ *Closed Classical Mereology*

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x))$, (*Fusion*)
2. $S_A(t) :\equiv \forall x(x \in A \rightarrow x \leq t) \,\&\, \forall y(y \leq t \rightarrow \exists x(x \in A \,\&\, y \circ x))$, (*Sum*)

Theorem

Core Principles + WeakSup + $\exists$ *Second Order Sums* $\equiv$ *Closed Classical Mereology*

Theorem

Core Principles + StrongSup + $\exists$ *Second Order Fusions* $\equiv$ *Closed Classical Mereology*

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x))$, (*Fusion*)
2. $S_A(t) :\equiv \forall x(x \in A \to x \leq t) \,\&\, \forall y(y \leq t \to \exists x(x \in A \,\&\, y \circ x))$, (*Sum*)

Theorem

Core Principles + WeakSup + $\exists$ *Second Order Sums* $\equiv$ *Closed Classical Mereology*

Theorem

Core Principles + StrongSup + $\exists$ *Second Order Fusions* $\equiv$ *Closed Classical Mereology*

Theorem (Pontow and Schubert, 2006)

Models of Closed Classical Mereology are precisely complete Boolean algebras with the bottom removed.

Composition of parts

Definition of compositions + Existence of compositions

Second order compositions

1. $F_A(t) :\equiv \forall y(y \circ t \leftrightarrow \exists x(x \in A \,\&\, y \circ x)),$ (*Fusion*)
2. $S_A(t) :\equiv \forall x(x \in A \rightarrow x \leq t) \,\&\, \forall y(y \leq t \rightarrow \exists x(x \in A \,\&\, y \circ x)),$ (*Sum*)

Theorem

Core Principles + WeakSup + ∃ Second Order Sums ≡ Closed Classical Mereology

Theorem

Core Principles + StrongSup + ∃ Second Order Fusions ≡ Closed Classical Mereology

Theorem (Pontow and Schubert, 2006)

Models of Closed Classical Mereology are precisely complete Boolean algebras with the bottom removed.

Mereological completions

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.
- f should preserve and reflect the overlap relations.

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.
- f should preserve and reflect the overlap relations.
- f should preserve existing compositions.

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.
- f should preserve and reflect the overlap relations.
- f should preserve existing compositions.
- f should reflect compositions.

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.
- f should preserve and reflect the overlap relations.
- f should preserve existing compositions.
- f should reflect compositions.
- $f(P)$ should be *composition-dense* in M:

Mereological completions

Objective

To extend models P of mereological theories into models M that satisfy universalism.

$$f : P \to M$$

We want to add compositions to P while modifying its mereological structure as little as possible:

- M should satisfy universalism (M is *composition-complete*)
- f should be a order-embedding.
- f should preserve and reflect the overlap relations.
- f should preserve existing compositions.
- f should reflect compositions.
- $f(P)$ should be *composition-dense* in M: for any $x \in M$ there exists $A \subseteq P$ such that x is a composition of $f(A)$.

Mereological completion of separative posets

Mereological completion of separative posets

Core Principles + Strong Supplementation

Mereological completion of separative posets

Core Principles + Strong Supplementation

Theorem

Let P be a separative poset. Up to isomorphism, there is at most one separative fusion-completion and one separative sum-completion of P and they coincide.

Mereological completion of separative posets

Core Principles + Strong Supplementation

Theorem

Let P be a separative poset. Up to isomorphism, there is at most one separative fusion-completion and one separative sum-completion of P and they coincide.

Let P be a poset. For $x \in P$ and $A \subseteq P$, let

$$\mathrm{O}(x) = \{y \mid x \circ y\} \quad \text{and} \quad \mathrm{O}(A) = \bigcup\{\mathrm{O}(x) \mid x \in A\}.$$

Mereological completion of separative posets

Core Principles + Strong Supplementation

Theorem

Let P be a separative poset. Up to isomorphism, there is at most one separative fusion-completion and one separative sum-completion of P and they coincide.

Let P be a poset. For $x \in P$ and $A \subseteq P$, let

$$\mathrm{O}(x) = \{y \mid x \circ y\} \quad \text{and} \quad \mathrm{O}(A) = \bigcup \{\mathrm{O}(x) \mid x \in A\}.$$

Let

$$\mathrm{O}P = \{\mathrm{O}(A) \mid \varnothing \neq A \subseteq P\}.$$

Mereological completion of separative posets

Core Principles + Strong Supplementation

Theorem

Let P be a separative poset. Up to isomorphism, there is at most one separative fusion-completion and one separative sum-completion of P and they coincide.

Let P be a poset. For $x \in P$ and $A \subseteq P$, let

$$\mathrm{O}(x) = \{y \mid x \circ y\} \quad \text{and} \quad \mathrm{O}(A) = \bigcup\{\mathrm{O}(x) \mid x \in A\}.$$

Let

$$\mathrm{O}P = \{\mathrm{O}(A) \mid \varnothing \neq A \subseteq P\}.$$

Theorem

Let P be a separative poset. Up to isomorphism,

$$\omega \colon (P, \leq) \to (\mathrm{O}P, \subseteq)$$
$$x \mapsto \mathrm{O}(x)$$

is the unique separative fusion- and sum-completion of P.

Fusions completion of arbitrary posets

Fusions completion of arbitrary posets

First attempt

$$F(P) = P \cup \{f_A \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_A \iff x \in A.$$

Fusions completion of arbitrary posets

First attempt

$$F(P) = P \cup \{f_A \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_A \iff x \in A.$$

Theorem

The inclusion map $P \hookrightarrow F(P)$ is a fusion-completion of P.

Fusions completion of arbitrary posets

First attempt

$$F(P) = P \cup \{f_A \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_A \iff x \in A.$$

Theorem

The inclusion map $P \hookrightarrow F(P)$ is a fusion-completion of P.

Example

$P = \{a, b, c\}$ antichain.

Fusions completion of arbitrary posets

First attempt

$$F(P) = P \cup \{f_A \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

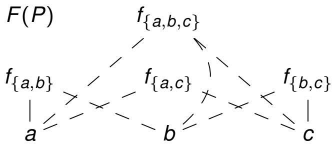
$$x \leq f_A \iff x \in A.$$

Theorem

The inclusion map $P \hookrightarrow F(P)$ is a fusion-completion of P.

Example

$P = \{a, b, c\}$ antichain.



Fusions completion of arbitrary posets

First attempt

$$F(P) = P \cup \{f_A \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

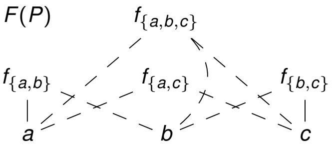
$$x \leq f_A \iff x \in A.$$

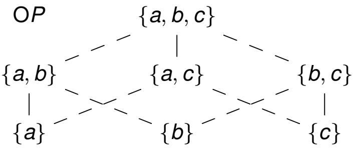
Theorem

The inclusion map $P \hookrightarrow F(P)$ is a fusion-completion of P.

Example

$P = \{a, b, c\}$ antichain.





Fusion completions of arbitrary posets

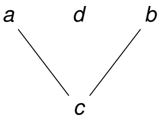
Example

$F(P)$ is not usually a very frugal fusion-completion.

Fusion completions of arbitrary posets

Example

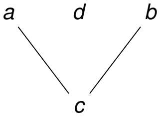
$F(P)$ is not usually a very frugal fusion-completion. Let P be the following poset:

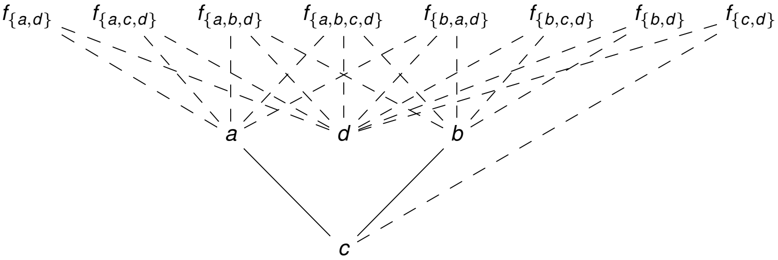


Fusion completions of arbitrary posets

Example

$F(P)$ is not usually a very frugal fusion-completion. Let P be the following poset:





Fusion completions of arbitrary posets

Fusion completions of arbitrary posets

A more economical approach

$$G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_{\mathrm{O}(A)} \iff x \in P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Fusion completions of arbitrary posets

A more economical approach

$$G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_{\mathrm{O}(A)} \iff x \in P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Theorem

The inclusion map $P \hookrightarrow G(P)$ is a fusion-completion of P.

Fusion completions of arbitrary posets

A more economical approach

$$G(P) = P \cup \{f_{O(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

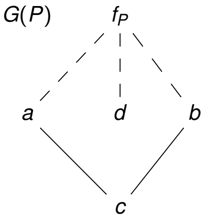
and

$$x \leq f_{O(A)} \iff x \in P \text{ and } O(x) \subseteq O(A).$$

Theorem

The inclusion map $P \hookrightarrow G(P)$ is a fusion-completion of P.

Example



Fusion completions of arbitrary posets

A more economical approach

$$G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

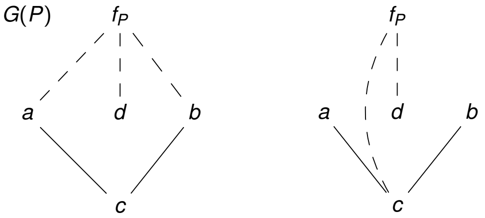
and

$$x \leq f_{\mathrm{O}(A)} \iff x \in P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Theorem

The inclusion map $P \hookrightarrow G(P)$ is a fusion-completion of P.

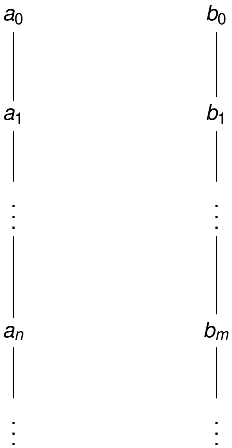
Example



When does a poset have a least fusion-completion?

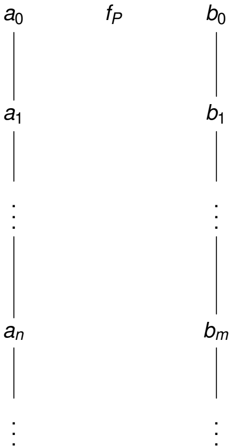
When does a poset have a least fusion-completion?

Example



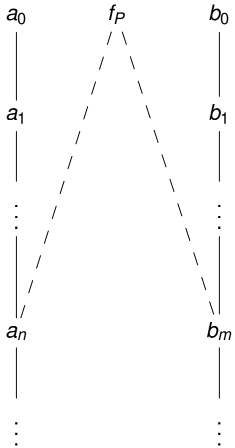
When does a poset have a least fusion-completion?

Example



When does a poset have a least fusion-completion?

Example



Completion of atomic posets

Completion of atomic posets

$$H(P) = G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_{\mathrm{O}(A)} \iff x \text{ is an atom in } P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Completion of atomic posets

$$H(P) = G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

$$x \leq f_{\mathrm{O}(A)} \iff x \text{ is an atom in } P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Theorem

Let P be an atomic poset. Then the inclusion map $\alpha\colon P \hookrightarrow H(P)$ is a fusion-completion of P.

Completion of atomic posets

$$H(P) = G(P) = P \cup \{f_{\mathrm{O}(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

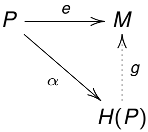
$$x \leq f_{\mathrm{O}(A)} \iff x \text{ is an atom in } P \text{ and } \mathrm{O}(x) \subseteq \mathrm{O}(A).$$

Theorem

Let P be an atomic poset. Then the inclusion map $\alpha\colon P \hookrightarrow H(P)$ is a fusion-completion of P.

Theorem

Let P be an atomic poset and let $e\colon P \to M$ be a dense fusion-completion. Then there is an injective fusion- and order-preserving map $g\colon H(P) \to M$ such that



commutes.

Completion of atomic posets

$$H(P) = G(P) = P \cup \{f_{O(A)} \mid \varnothing \neq A \text{ has no fusion in } P\}$$

and

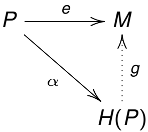
$$x \leq f_{O(A)} \iff x \text{ is an atom in } P \text{ and } O(x) \subseteq O(A).$$

Theorem

Let P be an atomic poset. Then the inclusion map $\alpha\colon P \hookrightarrow H(P)$ is a fusion-completion of P.

Theorem

Let P be an atomic poset and let $e\colon P \to M$ be a dense fusion-completion. Then there is an injective fusion- and order-preserving map $g\colon H(P) \to M$ such that



commutes.

Thank you for your attention

imanol.mozo@ehu.eus