# The definable content of (co)homological invariants. Part III: Theory and Examples.

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These series of lectures are about an ongoing research program that is in joint work with Jeff Bergfalk and Martino Lupini.


[1] The definable content of homological invariants I: $\mathrm{Ext}$ & $\lim^1$, (2020), arXiv:2008.08782.

[2] The definable content of homological invariants II: cohomology and homotopy classication, (2022), in preparation.

# Recall...

**Goal**. Enriching various classical invariants of **homological algebra** and **algebraic topology** with additional structures pertinent to "definability".

This results into much **finer** invariants for:

(1) classifying spaces up to homotopy equivalence

Steenrod homology $\rightsquigarrow$ **definable homology**

Čech cohomology $\rightsquigarrow$ **definable cohomology**

(2) classifying various algebraic structures up to isomorphism

$\mathrm{Ext}(-,-)$-bifunctor $\rightsquigarrow$ **definable** $\mathbf{Ext}(-,-)$

$\lim^1(-)$-functor $\rightsquigarrow$ **definable** $\mathbf{lim^1}(-)$

# Recall...

**Classical** invariants are constructed as (co)homology groups of appropriate (co)chain complexes.

One starts from a **cochain complex** $C^\bullet$

$$0 \to C^0 \longrightarrow \cdots \to C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \to \cdots$$

- Each $C^n$ is an abelian group recording the pertinent $n$-dim data.
- Coboundary maps $\delta^n \colon C^{n-1} \to C^n$ are group homomorphism.

The associated $n$-dimensional cohomology **group** is:

$$\mathrm{H}^n := \ker(\delta^{n+1})/\mathrm{im}(\delta^n)$$

# Recall...

**Definable invariants** are constructed as (co)homology groups with a Polish cover of appropriate Polish (co)chain complexes.

One starts from a **Polish cochain complex** $C^\bullet$

$$0 \to C^0 \longrightarrow \cdots \to C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \to \cdots$$

- Each $C^n$ is an abelian **Polish group**.
- Maps $\delta^n \colon C^{n-1} \to C^n$ are **continuous**. group homomorphism.

The associated $n$-dimensional cohomology **group with a Polish cover** is:

$$\mathbf{H}^n := \ker(\delta^{n+1})/\mathrm{im}(\delta^n)$$

# Recall...

**Standard setup.**

**Objects**. Abstract abelian groups $\mathrm{H}^n := \ker(\delta^{n+1})/\mathrm{im}(\delta^n)$.
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Fact

*There are* **uncountably** *many pairwise homotopy-inequivalent solenoids with the same Steenrod (or singular) homology.*

Theorem (Bergfalk, Lupini, P.)

**Definable homology** *classifies solenoids up to homeomorphism.*

# Today...

Convince that these categories provide a natural/convenient frameworks

- **Polish cochain complexes** with **continuous cochain maps**.
- **Groups with a Polish cover** with **definable homomorphisms**

Provide examples of how one passes from the **classical** to the **definable**.

# **Definable** Homological Algebra

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By definition this means that $\partial^{n+1} \circ \partial^n = 0$ for all $n$.
In other words, we always have that $\mathrm{Ker}(\delta^{n+1})/\mathrm{im}(\delta^n)$ is well defined.

A **continuous cochain map** $f^\bullet \colon A^\bullet \to B^\bullet$ is a sequence of continuous homomorphisms $f^n \colon A^n \to B^n$ so that each square commutes:

$$\begin{tikzcd}
\cdots A^{-2} \arrow[r] \arrow[d, "f^{-2}"] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"] & A^{0} \arrow[r] \arrow[d, "f^{0}"] & A^{1} \arrow[r] \arrow[d, "f^{1}"] & A^{2} \cdots \arrow[d, "f^{2}"] \\
\cdots B^{-2} \arrow[r] & B^{-1} \arrow[r] & B^{0} \arrow[r] & B^{1} \arrow[r] & B^{2} \cdots
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That is, $f^n \circ \partial_A = \partial_B f^{n-1}$ for all $n \in \mathbb{N}$.

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Then one gets a **long exact sequence** of the associated cohomology groups:

$$\cdots \to \mathrm{H}^{n-1}(A^\bullet) \to \mathrm{H}^{n-1}(B^\bullet) \to \mathrm{H}^{n-1}(C^\bullet) \xrightarrow{\boldsymbol{\partial}} \mathrm{H}^{n}(A^\bullet) \to \mathrm{H}^{n}(B^\bullet) \to \mathrm{H}^{n}(C^\bullet) \to \cdots$$

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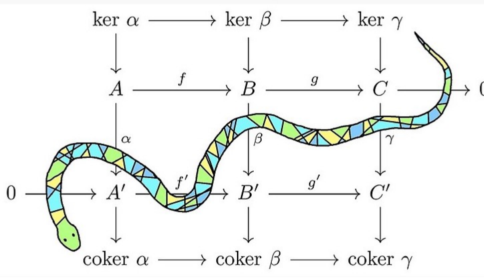

# The snake lemma

Consider the following diagram of abelian groups where squares commute:

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If both horizontal rows are **short exact** then there is a homomorphism $\partial\colon \mathrm{Ker}(\gamma) \to A'/\mathrm{im}(\alpha)$ which fits into a long exact sequence:

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$c \longrightarrow 0$

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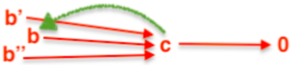


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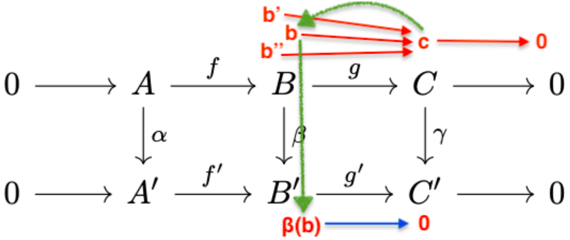


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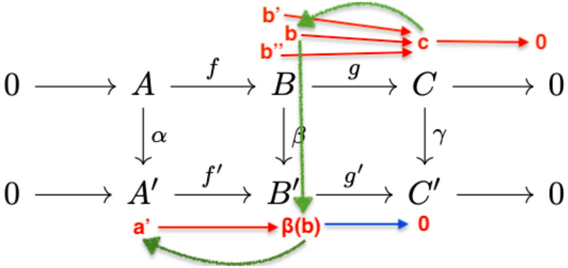


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By $g' \circ \beta(b) = \gamma \circ g(b) = \gamma(c) = 0$ and exactness get $a' \in A'$ with $f'(a') = \beta(b)$.

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If both horizontal rows are **short exact** and $\partial \colon \mathrm{Ker}(\gamma) \to A'/\mathrm{im}(\alpha)$ is the homomorphism that fits in the exact sequence:

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Then $\partial \colon \mathrm{Ker}(\gamma) \to A'/\mathrm{im}(\alpha)$ admit a **Borel lift** $\widehat{\partial} \colon \mathrm{Ker}(\gamma) \to A'$.

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**Lemma**

*If $f\colon G \to H$ is a continuous homomorphism between Polish groups then there exists a Borel function $s\colon H \to G$ so that $f(s(x)) = x$ for all $x \in H$.*

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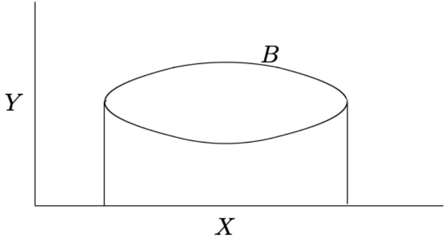
This relies heavily on "Large Section" Uniformization Results from DST.

# Uniformization

Let $B$ be a Borel subset of $X \times Y$ where $X, Y$ are Polish spaces.



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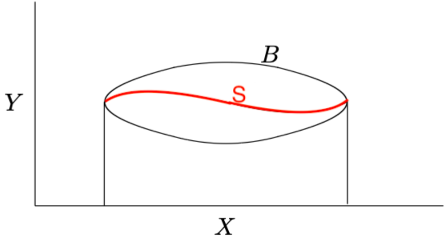


We say that $B$ admits a **Borel uniformization** if there exists a Borel $S \subseteq B$ so that for all $x \in X$ we have: "there exists $y \in Y$ with $(x, y) \in B$" if and only if "there exists **precisely one** $y \in Y$ with $(x, y) \in S$".

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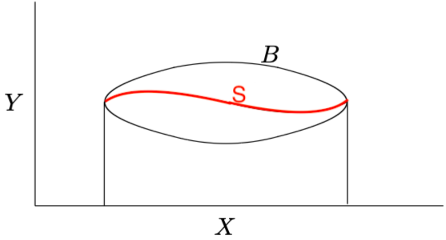


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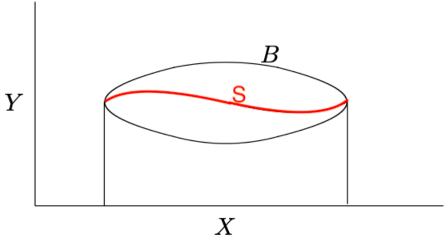


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**Fact 2.** Not every Borel $B$ with $\mathrm{proj}(B)$ Borel admits a Borel uniformization.

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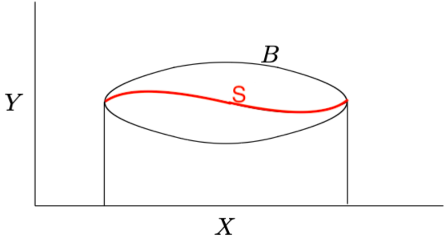


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**Fact 1.** Not every Borel $B$ admits a Borel uniformization.
**Fact 2.** Not every Borel $B$ with $\mathrm{proj}(B)$ Borel admits a Borel uniformization.
**Fact 3.** If $B$ is Borel and the slice $B_x \subseteq Y$ is non-meager for all $x \in X$, $B$ admits a Borel uniformization.

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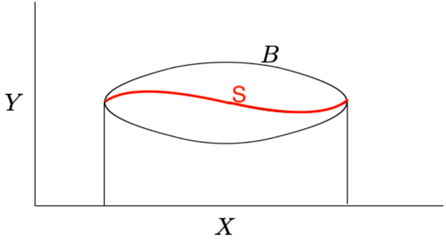


We say that $B$ admits a **Borel uniformization** if there exists a Borel $S \subseteq B$ so that for all $x \in X$ we have: "there exists $y \in Y$ with $(x, y) \in B$" if and only if "there exists **precisely one** $y \in Y$ with $(x, y) \in S$".

**Fact 1.** Not every Borel $B$ admits a Borel uniformization.
**Fact 2.** Not every Borel $B$ with $\mathrm{proj}(B)$ Borel admits a Borel uniformization.
**Fact 3.** If $B$ is Borel and the slice $B_x \subseteq Y$ is non-meager for all $x \in X$, $B$ admits a Borel uniformization.

This is where Polishability of $N$ in $G/N$ is used!

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**Example.** Here are two non-isomorphic extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z}) \to 0 \text{ and } 0 \to \mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z}) \to 0$$

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For **definable** $\mathbf{Ext}(B,F)$ use the product topology on $F^{B\times\cdots\times B}$.

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Theorem (Bergfalk, Lupini, P.)

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**Compare:** there are uncountably many such $\Lambda$ with isomorphic $\mathrm{Ext}(\Lambda, \mathbb{Z})$.

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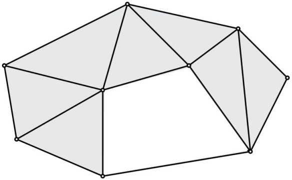

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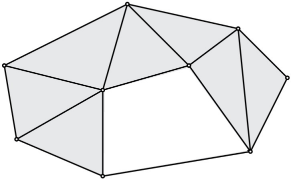

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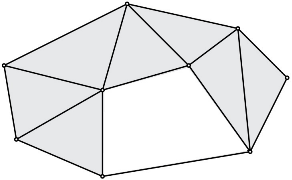

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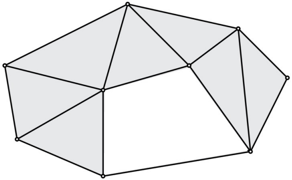

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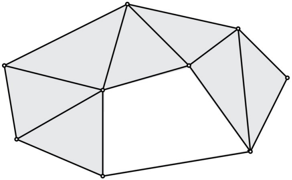

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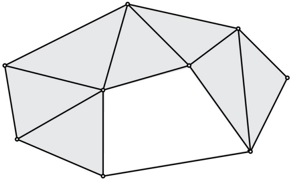

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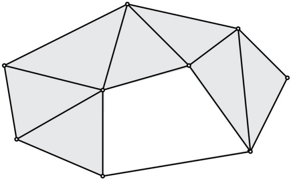

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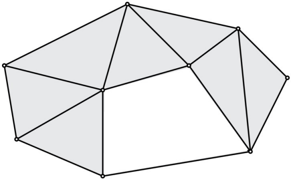

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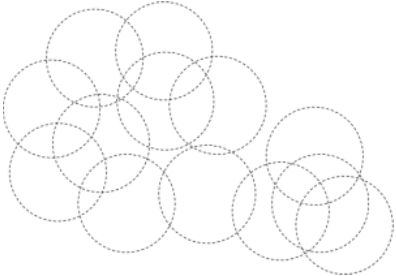

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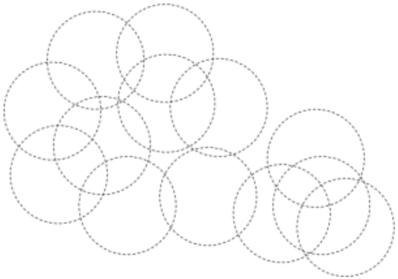

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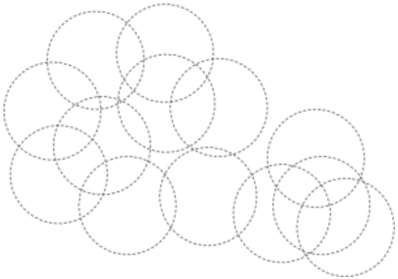

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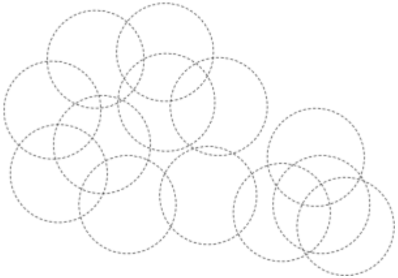

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2. The definition above is **not** in terms of a cochain complex.

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*Proof.* $\boldsymbol{\mathcal{U}} = \mathcal{A}(\boldsymbol{\mathcal{U}}_{\mathrm{fin}})$ for some scheme $\boldsymbol{\mathcal{U}}_{\mathrm{fin}}$ of finite covers.

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**Question.** Does $\mathbf{H}^n(X, G)$ depend on $\boldsymbol{\mathcal{U}}$?

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*Proof.* We use (E1) to prove a "definable simplicial approximation" lemma.

# Application 1

**Theorem (Bergfalk, Lupini, P.)**

*The isomorphism* $j\colon [X, K(G,n)] \to \mathbf{H}^n_{\boldsymbol{\mathcal{U}}}(X,G)$ *is definable.*

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**Theorem (Bergalk, Lupini, P.)**

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This relies on a “definable homotopy extension” theorem

# Application 2

**Theorem (Bergfalk, Lupini, P.)**

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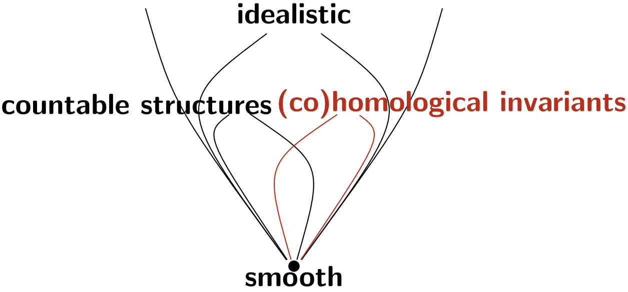


**Corollary**

$(\mathrm{Maps}(X,Y), \simeq)$ *is* **classifiable by cohomological invariants** *whenever* $Y$ *is an Eilenberg-MacLane space, e.g. when* $Y = S^1$ *or* $Y = S^2$*.*