Presenting Quantitative Inequational Theories

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BLAST 2022

Motivation: Processes with Algebraic Branching

(SRSR, 2021) Framework for process types

$$\text{States} \longrightarrow M(\text{Labels} \times \text{States})$$

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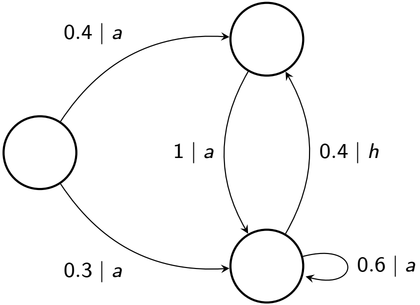
Eg. $M = \mathcal{D}$ the finitary probability distribution functor



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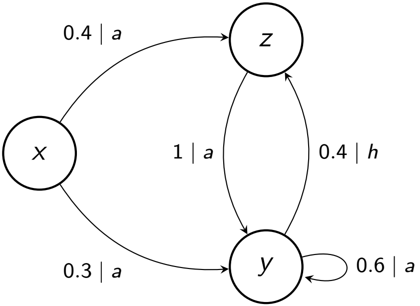

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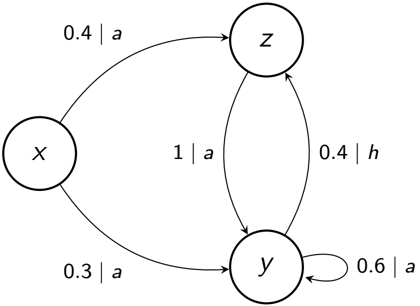
Recursive Specification:

$x \mapsto 0.3(a, y) + 0.4(a, z)$

$y \mapsto 0.6(a, y) + 0.4(h, z)$

$z \mapsto 1(a, y)$



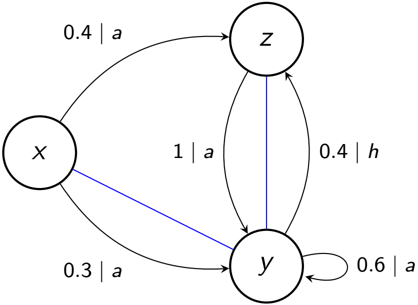
Motivation: Processes with Algebraic Branching

(S, 2022) Extends framework to partially ordered states

$$(\text{States}, \leq) \xrightarrow{\text{(monotone)}} M((\text{Labels}, \leq) \times (\text{States}, \leq))$$

where M = free *ordered* algebra construction.

Questions:



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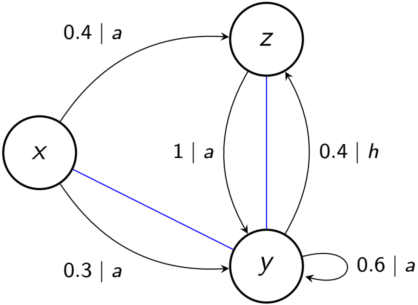
Questions:

1. What is the free *ordered* $\mathbb{R}^+$-module?



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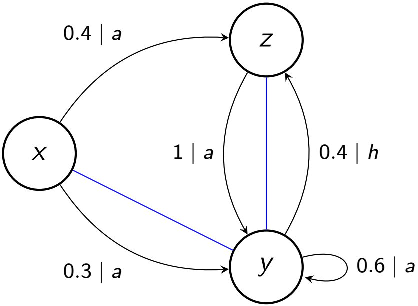
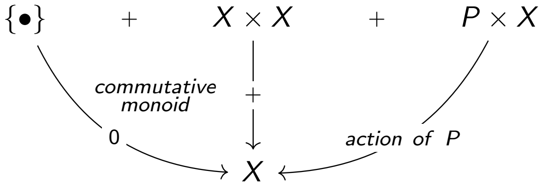
Questions:

1. What is the free *ordered* $\mathbb{R}^+$-module?
2. How do we construct free ordered P-modules?



Modules over a Semiring

Given a semiring $(P, 0, 1, +, \cdot)$, a *P-module* consists of



An *action* is required to satisfy

$$(r+s)x = rx + sx \qquad (rs)x = r(sx) \qquad r(x+y) = rx + ry$$
$$0x = 0 \qquad 1x = x$$

The Free P-module Construction

The free P-module on a set X is

$$\mathcal{O}_P X = \left\{ X \xrightarrow{\theta} P \;\middle|\; \theta^{-1}(P \setminus 0) \text{ is finite} \right\}$$

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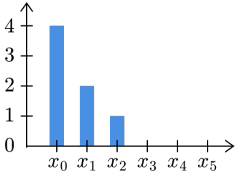

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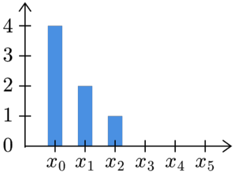

Defining Property: $\forall\, f : X \to Y$ into $(Y, 0, +, act)$,

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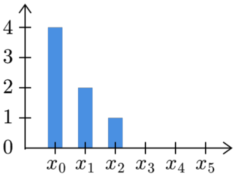

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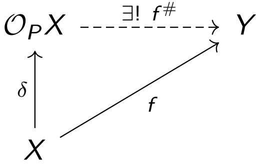

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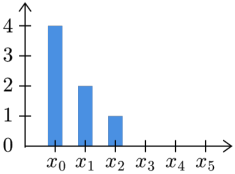

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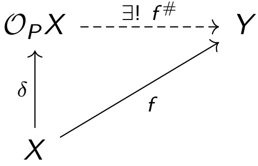

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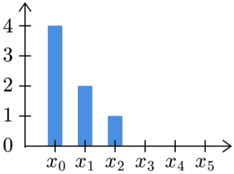

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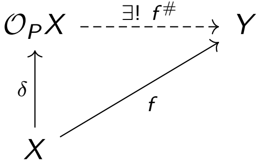

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Theorem

Write $\mu = \mathcal{O}_P(\mathrm{id}_X)^\#$. P-modules present the monad $(\mathcal{O}_P, \delta, \mu)$.

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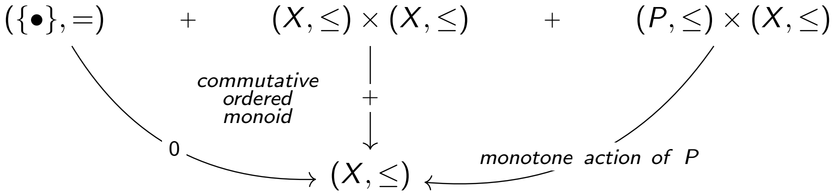
An *ordered P-module* is a monotone structure



The Free Ordered P-module Construction...?

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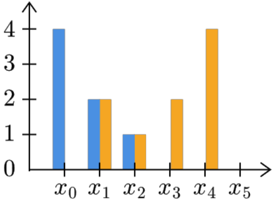

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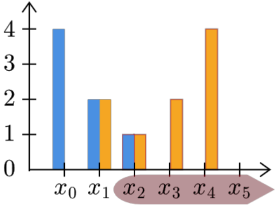

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What we need:

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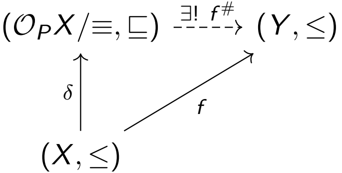

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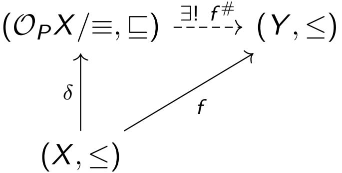

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Theorem

If P is cancellative and difference ordered, or idempotent, then the free ordered P-module is $(\mathcal{O}_P X/\equiv, \sqsubseteq)$.

The cancellative and difference-ordered case

An ordered semiring P is (*sum*) *cancellative* if

$$r + t \leq s + t \implies r \leq s$$

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$\mathcal{O}_{\mathbb{R}^+}(\{0,1,2\}, =) \cong ((\mathbb{R}^+)^3, \leq^+)$ where

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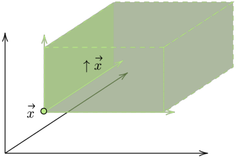

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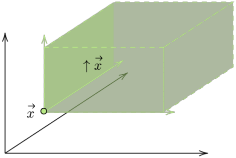

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Note

If P is idempotent, $(\mathcal{O}_P, \delta, \mu)$ does *not* lift to $(\bar{\mathcal{O}}_P, \delta, \mu)$

Some observations

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New Question

Is difference-ordered a necessary condition?

Some references

(Golan, 1999) Semilattices and their Applications

(SRSR, 2022) Processes Parametrised by an Algebraic Theory

(Shaked & Shanthikumar, 1994)
Stochastic Orders and Their Applications

(Bonchi, Sokolova, Vignudelli, 2019)
The Theory of Traces for Systems with Nondeterminism and Probability

(Stone, 1949) Postulates for the barycentric calculus

Recap

- Looking for the free ordered P-module on a poset
- The free (unordered) P-module on a set is

$$\mathcal{O}_P X = \left\{ X \xrightarrow{\theta} P \;\middle|\; X \setminus \theta^{-1}(0) \text{ is finite} \right\}$$

- $\mathcal{O}_P X$ modulo the heavier-higher order

$$\theta_1 \sqsubseteq \theta_2 \iff (\forall \uparrow \text{closed } U)\ \theta_1(U) \leq \theta_2(U)$$

  is the free ordered P-module if
  - P is cancellative and difference-ordered, or
  - P is idempotent
- Interesting example: **N** needs further investigating
- Is the difference-ordered condition necessary?