Finite Base Property for Algebras of Binary Relations

Jaš Šemrl
UCL (University College London)

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Section 1

Introduction

Motivation

- Tarskian Representable Relation Algebras are badly behaved
- Look for better behaviour with reduct languages
- FBP doesn't trivially follow from other properties
- Decidability guarantees in computer science

Proper Relation Algebras

Proper Relation Algebras are $\{0, 1, -, +, 1', \smile, ;\}$-structures with a carrier set $A \subseteq \wp(X \times X)$ where X is called the base set and

1. $0, 1, -, +$ are interpreted as proper Boolean top and bottom, negation, and join
2. $1', \smile, ;$ are interpreted as

$$1' = \{(x, x) \mid x \in X\}$$
$$\breve{S} = \{(y, x) \mid (x, y) \in S\}$$
$$S; T = \{(x, z) \mid \exists y : (x, y) \in S, (y, z) \in T\}$$

for $S, T \subseteq X \times X$

Let τ be a RA-reduct language.

Finite Base Property (FBP)

τ is said to have the Finite Base Property (FBP) if all finite proper τ-structures are isomorphic to a proper τ-structure with a finite base.

Section 2

The Conjecture

Let τ be a RA-reduct signature.

Conjecture

τ has the FBP if and only if

$$\{-, ;\} \not\subseteq \tau \not\supseteq \{\cdot, ;\}$$

Right to Left Implication

- All signatures containing $\{\cdot, ;\}$ have no FBP [Mad16, N^{+}17]
- All signatures containing $\{-, ;\}$ have no FBP

Left to Right Implication

- Composition-free signatures have FBP
- Cayley Representation for Groups works for $\{;\}, \{1', ;\}$
- $\{\leq, ;\}$ have FBP [Zar59]
- $\{D, \smile, ;\} \subseteq \tau \subseteq \{0, 1, \leq, 1', D, R, \smile, ;\}$ have FBP [HE13, Šem21]
- $\{\leq, \backslash, /, ;\}$ have FBP [Rog20]

Section 3

Failure of FBP with Negation and Composition

Theorem

Any signature τ containing $\{-, ;\}$ fails to have the FBP.

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Proof:

The Point Algebra is the proper Relation Algebra over the base $\mathbb{Q}$ with the following 8 elements (interpreted arithmetically)

$$\{0, 1, =, \neq, <, \leq, >, \geq\}$$

Observe how all operations in the language of RA are well defined for this algebra.

Assume there existed a proper τ-structure over a finite base X, isomorphic to the Point Algebra via some isomorphism h.

Claim 1

There must exist $x, y \in X$ with $(x, y) \in h(1)$.

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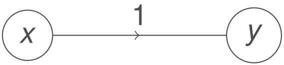

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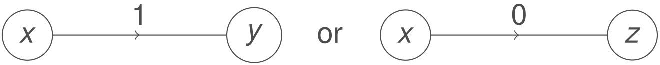

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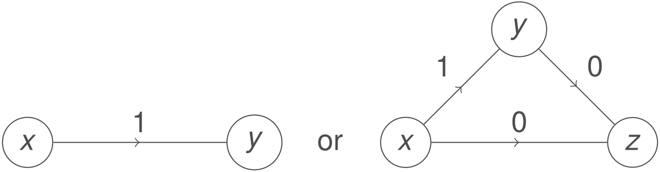

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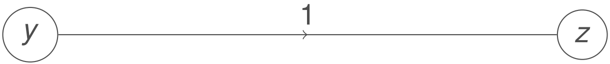

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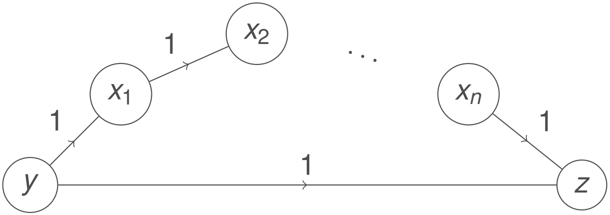
$$1 = 1; 1 = 1; 1; 1 = \ldots = 1^{n+1} = \ldots$$



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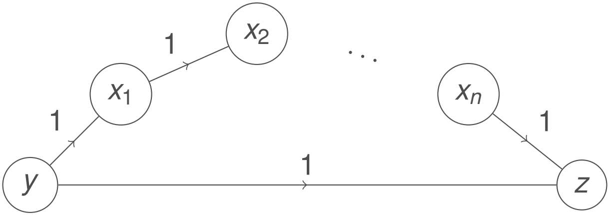

where $|X| = n - 1$.

So, there must exist $i < j$ such that $x_i = x_j = x$ and we get $(x, x) \in h(1)$ as $1 = 1^{j-i}$.

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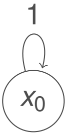

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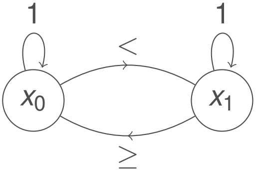

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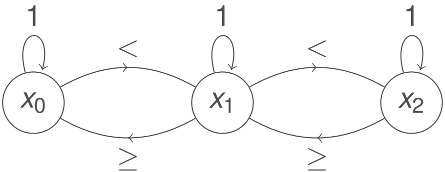

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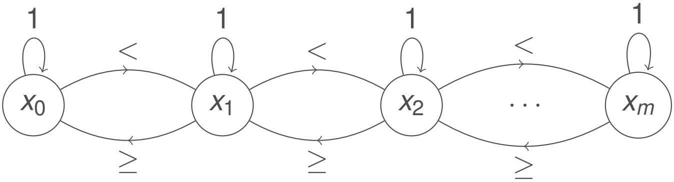

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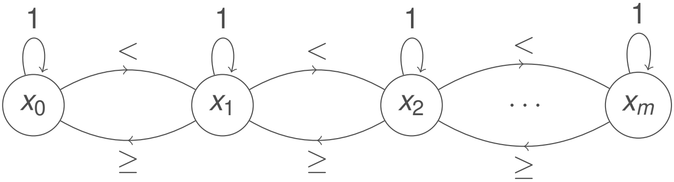


Contradiction! $\square$

Section 4

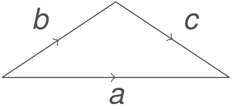
Relation Algebras, 4-Move Games, and FBP

(Abstract) Relation Algebras

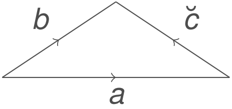
A Relation Algebra is a $\{0, 1, -, +, 1', \smile, ;\}$-structure such that

1. its $\{0, 1, -, +\}$-reduct is a Boolean Algebra
2. 1' is the identity for associative and additive ;
3. $\breve{\breve{a}} = a$ and $(a; b)^\smile = \breve{b}; \breve{a}$
4. the Percian triangle law holds

$$a \cdot (b; c) = 0 \iff b \cdot (a; \breve{c}) = 0$$



$\iff$



Proposition

Every finite τ-structure, where $\{-,;\} \not\subseteq \tau \not\supseteq \{\cdot,;\}$, is finitely representable if and only if it embeds into a finite relation algebra.

Relation Algebras

- Every finite relation algebra can be uniquely defined by its atoms, their converses, the identity, and the set of triangles such that $a \leq b; c$ were a, b, c are atoms
- Example: Point Algebra $\quad Atoms = \{1', l, g\}$

$l = \breve{g}$	$1' = \breve{1}'$	$1' \leq l; g$
$1' \leq g; l$	$1' \leq 1'; 1'$	$l \leq l; 1'$
$l \leq 1'; l$	$g \leq g; 1'$	$g \leq 1'; g$
$g \leq g; g$	$g \leq l; g$	$g \leq g; l$
$l \leq l; l$	$l \leq l; g$	$l \leq g; l$

Representation Games

- A two-player game can be defined for a structure $\mathcal{A}$ where Abelard ($\forall$) challenges Héloïse ($\exists$) to build a representation [HH02]

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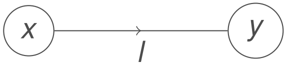

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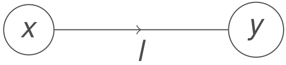
$\forall$: $l \leq l; l$ at (x, y)



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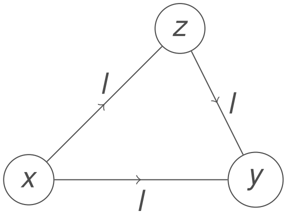

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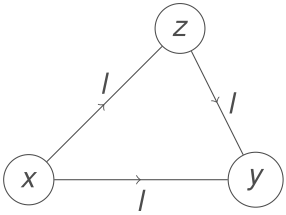

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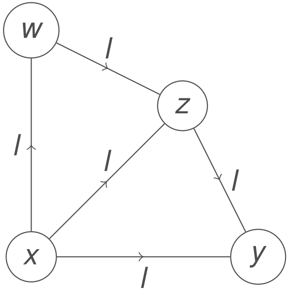

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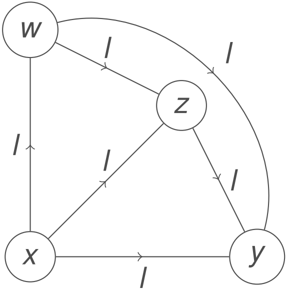

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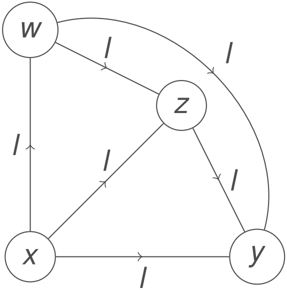

$\forall$ wins iff $\exists$ introduces an inconsistent triangle

Let $\mathcal{A}$ be defined by its atoms with their converses, the identity, and the list of allowed/forbidden triangles.

Proposition

$\mathcal{A}$ is a relation algebra if and only if $\exists$ has a winning strategy for a representation game of length 4.

Proposition

$\mathcal{A}$ is a representable relation algebra if and only if $\exists$ has a winning strategy for the infinite-length representation game.

- A similar representation game can be defined for every $\tau \subseteq \{0, 1, -, +, 1', \smile, ;\}$

Proposition

Every finite τ-structure $\mathcal{A}$, where $\{-, ;\} \not\subseteq \tau \not\supseteq \{\cdot, ;\}$, is finitely representable if and only if it embeds into a finite $\tau \cup \{\cdot, \smile\}$-structure $\mathcal{A}'$ such that $\exists$ has a winning strategy for a 4-move representation game for $\mathcal{A}'$.

Thank You!

ArXiv Identifier

http://arxiv.org/abs/2111.01213

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