Augmented quasigroups and Heyting algebras

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for each object A of $\mathbf{V}$.

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First yanking condition

$$x' \overset{r_A^{-1}}{\longmapsto} x' \otimes 1 \overset{1_A \otimes \mathrm{coev}}{\longmapsto} x' \otimes \sum_{x \in X} \delta_x \otimes x \overset{\mathrm{ev} \otimes 1_A}{\longmapsto} \sum_{x \in X} x'\delta_x \otimes x = 1 \otimes x' \overset{l_A}{\longmapsto} x'$$

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Theorem: Set A with function $A \times A \to 2^A; (x, y) \mapsto x \diamond y$
forms a hypermagma if and only if $(A, \mu, \Delta, \varepsilon)$
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In $(\mathbf{Rel}, \otimes, \top)$, take augmentation $\varepsilon = \{(x, 0) \mid x \in A\}$, comultiplication $\boldsymbol{\Delta} = \{(x, x \otimes x) \mid x \in A\}$, i.e., diagonal relation, and multiplication relation $\{(x \otimes y, z) \mid x, y, z \in A,\ z \in x \diamond y\}$.

Hypermagma: $x \diamond y$ is nonempty for all x, y in A.

Theorem: Set A with function $A \times A \to 2^A; (x, y) \mapsto x \diamond y$ forms a hypermagma if and only if $(A, \mu, \boldsymbol{\Delta}, \varepsilon)$ is an augmented magma in the category $(\mathbf{Rel}, \otimes, \top)$.

Magmas and hypermagmas treated uniformly, regardless of type!

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In the magma case, $(A, \mu, \mathbf{\Delta}, \varepsilon)$ lies in $(\mathbf{Set}, \otimes, \top)$.

Currying and braiding in compact closed categories

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For an object A of $\mathbf{V}$, define

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Augmented quasigroup: Augmented magma $(A, \mu, \Delta, \varepsilon)$
for which $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$ are augmented magmas.

(Quasi-)Group algebras as augmented quasigroups

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Works equally well for a finite quasigroup $(G, \cdot, /, \backslash)$.

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with $\quad x \diamond y = \uparrow(x \wedge y)\,, \quad z \nwarrow y = \downarrow(y \rightarrow z)\,, \quad x \nearrow z = \downarrow(x \rightarrow z)$.

Thank you for your attention!