Fibered Universal Algebra for First-order Logics

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1. A lack of *general* algebraic approaches to providing semantics for first-order logics.
2. A lack of a suitable notion of formula substitution which preserves logical consequence. (A key ingredient of AAL)

Languages, Signatures and Contexts

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a (multi-sorted) **signature** Sg is specified by a collection of:

1. sort symbols $\sigma, \tau, \gamma, \ldots$.
2. typed function symbols $f : \sigma_1, \ldots, \sigma_n \to \tau$,
3. typed relation symbols $R \subseteq \sigma_1, \ldots, \sigma_n$.

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For categorical semantics, well-formed terms and formulas must be *in context*.

That is, they must come with a list $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ of variable/sort pairs that at least include those occurring in the expression.

Point-free Account of Classical Satisfaction via Contexts

Let S is a classical first-order Sg-structure. Then $[\![\sigma]\!]$ is a set, $[\![f]\!] \colon [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\tau]\!]$ is a function and $[\![R]\!] \subseteq [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!]$ is a relation.

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A term $f(M_1, \ldots, M_n)$, in context $f(M_1, \ldots, M_n) : \tau\ [\Gamma]$ is recursively interpreted as the function

$$[\![f]\!] \circ \langle [\![M_1 : \sigma_1\ [\Gamma]]\!], \ldots, [\![M_n : \sigma_n\ [\Gamma]]\!] \rangle\colon [\![\Gamma]\!] \to [\![\tau]\!].$$

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S can then recursively interpret all formulas-in-context as definable relations:

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$$\bigcap_{i=1}^{n} [\![\phi_i\,[\Gamma]]\!] \subseteq [\![\phi\,[\Gamma]]\!].$$

Categorification of Satisfaction I (Lawvere)

Definition

A **prop-category** is a pair $(\mathbf{C}, P)$, where $\mathbf{C}$ is a category with finite products, and $P\colon \mathbf{C} \to \mathbf{Pos}$ is a contravariant functor.

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An Sg**-structure** S **in** $(\mathbf{C}, P)$, assigns:

1. To each sort σ, an object $[\![\sigma]\!] \in \operatorname{Ob}\mathbf{C}$.
2. To each function symbol $f\colon \sigma_1, \ldots, \sigma_n \to \tau$ a morphism $[\![f]\!]\colon [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\tau]\!]$ in $\operatorname{Mor}\mathbf{C}$.
3. To each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$, an element $[\![R]\!] \in P([\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!])$.

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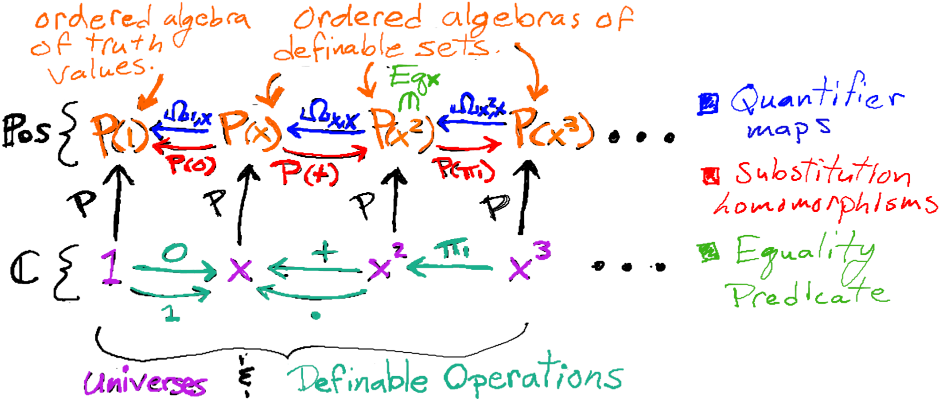
5. For each $\Omega \in \mathscr{L}_q$, and all $c \in \mathrm{Ob}\,\mathbf{C}$, there is a natural transformation $\Omega_{(\cdot),c}\colon UP(\_ \times c) \to UP$, where $U\colon \mathbf{Pos} \to \mathbf{Set}$ is the forgetful functor.

Visualizing Prop-categories



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A collection $\mathcal{K}$ of prop-categories defines a logic $\vDash_{\mathcal{K}}$ via satisfaction. $(\vDash_{\{(\mathbf{Set}, \mathscr{P})\}} = \vdash_{CFL})$

The Literature (To the best of my knowledge!)

1. All other work on prop-categorical semantics only consider $\mathscr{L}_q = \{\forall, \exists\}$ and following Lawvere, fix their interpretations by stronger adjoint conditions.
2. Most other accounts assume the posets $P(c)$ are Heyting algebras, to provide semantics for intuitionistic logic.
3. Shirasu [6] [5] provides complete prop-categorical semantics for substructural predicate logics, and uses this semantics to prove their disjunctive and existence properties.
4. Also used by Maruyama [3], to provide a unified account of logical translations, including Girard's exponential translation in linear logic and Kolmogorov's double negation translation in intuitionistic logic.

Example: t-norm Quantifiers [2, pg. 4]

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min, max and multiplication are t-norms, and $\Omega^{\min}_{A,B} = \forall_{A,B}$ and $\Omega^{\max}_{A,B} = \exists_{A,B}$.

The Logic of all Prop-categories $\mathcal{L}^m$

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$$\frac{}{\phi \vdash \phi \; [\Gamma]}\ \text{Ax} \qquad \frac{\phi \vdash \psi \; [\Gamma] \quad \psi \vdash \theta \; [\Gamma]}{\phi \vdash \theta \; [\Gamma]}\ \text{Cut}$$

$$\frac{\Phi \vdash \psi \; [\Gamma]}{\Phi \vdash \psi \; [\Gamma, x : \sigma]}\ \text{Cwk} \qquad \frac{M = M' : \sigma \; [\Delta] \quad \Phi \vdash \psi \; [\Gamma, x : \sigma, \Gamma']}{\Phi[M/x] \vdash \psi[M'/x] \; [\Gamma, \Gamma']}\ \text{Sub}$$

$$\frac{\phi \dashv\vdash \psi \; [\Gamma, x : \sigma]}{\Omega_{x:\sigma}(\phi) \vdash \Omega_{x:\sigma}(\psi) \; [\Gamma]}\ \Omega\text{-Cong}$$

$$\frac{\phi_1 \dashv\vdash \phi'_1 \; [\Gamma] \quad \ldots \quad \phi_n \dashv\vdash \phi'_n \; [\Gamma]}{\Diamond(\phi_1, \ldots, \phi_n) \vdash \Diamond(\phi'_1, \ldots, \phi'_n) \; [\Gamma]}\ \Diamond\text{-Cong}$$

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Classifying prop-categories. (Indexed LT-algebras)

For each $\mathcal{L}^m$-theory T, one can built a **classifying prop-category** $(\mathbf{C}_T, P_T)$, which has a generic T-model $G \in (\mathbf{C}_T, P_T)$.

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The objects of $\mathbf{C}_T$ are contexts and the morphisms are lists of terms equal up to T.

If Γ is a context, then $P_T(\Gamma)$, is the Lindenbaum-Tarski algebra, of formulas-in-context Γ. (ordered by $\phi\,[\Gamma] \leq \psi\,[\Gamma]$ iff $\phi \vdash \psi\,[\Gamma] \in T$.) For each morphism $\gamma \colon \Gamma' \to \Gamma$, $P_T(\gamma) \colon P_T(\Gamma) \to P_T(\Gamma')$ acts by substitution.

Completeness with Respect to Classifying Prop-categories.

If a logic $\mathcal{L}$ is stronger than $\mathcal{L}^m$, then $\vDash_{\mathcal{K}} \subseteq \vdash_{\mathcal{L}}$, where $\mathcal{K} = \{(\mathbf{C}_T, P_T) : T \text{ is an } \mathcal{L}\text{-theory}\}$.

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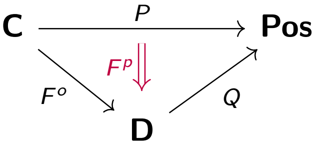
Propositon

Suppose $\mathscr{L}_q = \{\forall, \exists\}$ and $\mathcal{L}$ is an extension of $\mathcal{L}^m$ possibly by $=$-Adj, $\exists$-Adj, $\forall$-Adj and any number of structural and propositional connective rules. Then $\vdash_{\mathcal{L}} = \vDash_{\mathcal{K}}$, where $\mathcal{K} = \{(\mathbf{C}_T, P_T) : T \text{ is an } \mathcal{L}\text{-theory}\}$.

The 2-Categorical of Prop-Categorical Semantics: 1-cells

Morphisms $F\colon (\mathbf{C}, P) \to (\mathbf{D}, Q)$ are determined by the following data: (1) a product preserving functor $F^o\colon \mathbf{C} \to \mathbf{D}$ and (2) a natural transformation $F^p\colon P \Rightarrow Q \circ F^o$,



such that $\forall b, c \in \mathrm{Ob}(\mathbf{C})$, and $\forall \Omega \in \mathscr{L}_q$:

$$F^p_c\colon P(c) \to Q \circ F^o(c) \quad \text{is an } \mathscr{L}_\omega\text{-algebra hom} \tag{1}$$

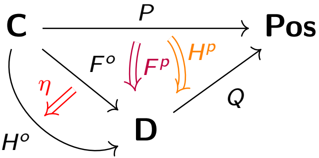
$$F^p_b \circ \Omega_{b,c} = \Omega_{F^o b, F^o c} \circ Q(a^{-1}_{F,b,c}) \circ F^p_{b\times c}. \tag{2}$$

$$F^p_{c\times c}(Eq_c) = Q(a_{F,c,c})(Eq_{F^o c}), \tag{3}$$

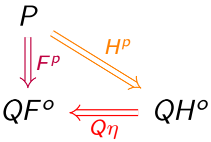
where $a_{F,b,c}\colon F^o(b \times c) \to F^o b \times F^o c$ is the change-in-product isomorphism.

The 2-Category of Prop-Categorical Semantics: 2-cells

For parallel morphisms $F, H\colon (\mathbf{C}, P) \to (\mathbf{D}, Q)$, we define a 2-cell $\eta\colon F \Rightarrow H$ to be a natural transformation $\eta\colon F^o \Rightarrow H^o$



such that $F^p = Q\eta \cdot H^p$.



Big picture: **FA** is a 2-category where objects are theories, morphisms are structures and 2-cells are structure preserving maps.

However, the classifying prop-categories are not generic, they satisfy some extra identities from the definitions:

1. For $\Gamma' = y_1 : \tau_1, \ldots, y_m : \tau_m$ we define $\Omega_{\Gamma,\Gamma'}(\phi\,[\Gamma,\Gamma']) := \Omega_{y_1:\tau_1} \ldots \Omega_{y_m:\tau_m}(\phi)\,[\Gamma]/\sim$.
2. $Eq_\Gamma := \bigotimes_{i=1}^{n} x_i =_{\sigma_i} x_i'\ [\Gamma,\Gamma']/\sim$.

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Thus we impose two analogous conditions on the prop-categories:

1. For all $\Omega \in \mathscr{L}_q$, and all $b, c, d \in \mathrm{Ob}(\mathbf{C})$,

$$\Omega_{b,1} \circ P(\pi_1^{b,1}) = id_{P(b)} \quad \text{and} \quad \Omega_{b,c\times d} \circ P(a) = \Omega_{b,c} \circ \Omega_{b\times c,d}.$$

2. $Eq_1 = e_{1\times 1}$, and for all $c_1, c_2 \in \mathrm{Ob}(\mathbf{C})$, $c = c_1 \times c_2$

$$Eq_c = P(\langle \pi_1\pi_1, \pi_1\pi_2\rangle)Eq_{c_1} \otimes P(\langle \pi_2\pi_1, \pi_2\pi_2\rangle)Eq_{c_2}.$$

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Note: our prior examples satisfy these conditions.

AAL Results

1. Each T-model $S \in (\mathbf{C}, P)$ can be identified with a unique (up to unique 2-isomorphism) morphism $\overline{S}\colon (\mathbf{C}_T, P_T) \to (\mathbf{C}, P)$.

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2. Let the **term prop-category** $(\mathbf{C}_{Sg}, P_{Sg})$, be the classifying prop category determined by the $\mathcal{L}^m$-theorems. Then $(\mathbf{C}_{Sg}, P_{Sg})$ is a suitable analogue of the term algebra determined by a propositional language.

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3. $\mathrm{End}(\mathbf{C}_{Sg}, P_{Sg})$ determines a natural action of formula substitution on Sg-theories, which preserves consequence in $\vDash_{\mathcal{K}}$, for each $\mathcal{K} \subseteq \mathrm{Ob}(\mathbf{FA})$.

How Algebraic? Homomorphism Theorem for $\mathbf{FA}_{\mathcal{L}}$

Fix a logic $\mathcal{L}$ and let $\mathbf{FA}_{\mathcal{L}}$ be the full sub-2-category of **FA** which provides sound semantics for $\mathcal{L}$.

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Also, $\mathscr{E}, \mathscr{M}$ are an orthogonal factorization system [4] for $\mathbf{FA}_{\mathcal{L}}$.

Operators $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$

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$$(\mathbf{C}_{Sg}, P_{Sg}) \xrightarrow{H \circ F \in \mathbb{H}(\mathcal{Y})} (\mathbf{D}, Q)$$

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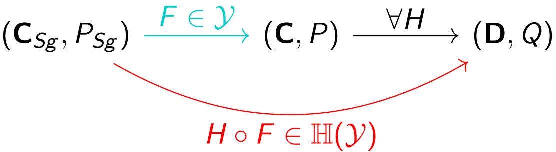

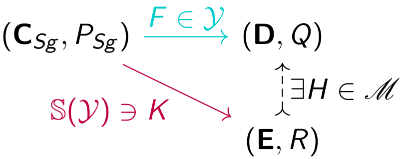

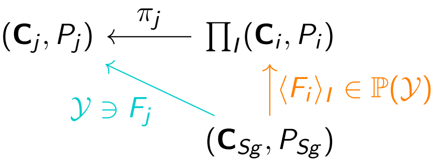

How Algebraic? A First-order HSP Theorem

Theorem (Fibered HSP Theorem)

Let $\mathcal{L}$ be a logic and Sg a signature. For each $\mathcal{Y} \subseteq \mathsf{Mod}^{\mathcal{L}}_{Sg}$, $\mathsf{Mod}_{(\cdot)} \circ \mathsf{Th}(\mathcal{Y}) = \mathbb{HSP}(\mathcal{Y})$.

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Note: Using the internal logic of prop-categories, along with the Fibered Homomorphism Theorems, one can give a proof that closely mirrors a proof of the classical HSP Theorem.

Special thanks to my advisor Constantine Tsinakis and Adam Přenosil for their mentorship and many insightful remarks.

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Thank you!

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