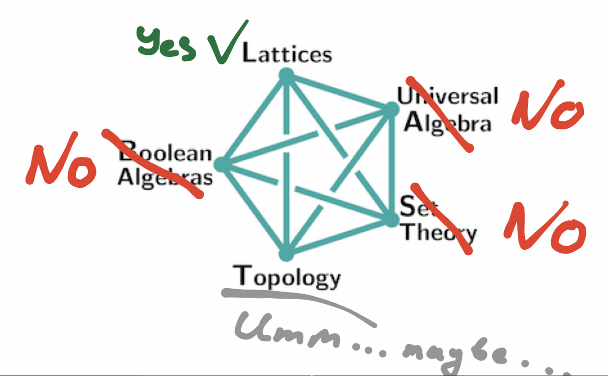
$\text{SSP} \stackrel{?}{=} \text{RC}$

Bogdan Chornomaz



Reminder: Sauer-Shelah-Perles lemma

$\mathcal{F}$

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$Sh(\mathcal{F})$

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Let us fix a base set X and a family $\mathcal{F}$. A set $Y \subseteq X$ is **shattered** by $\mathcal{F}$ iff $\mathcal{F}|_Y = 2^Y$. Stated otherwise:

$$\forall Z \subseteq Y \ \ \exists X \in \mathcal{F} \ \text{ s.t. } \ Z = Y \cap X.$$

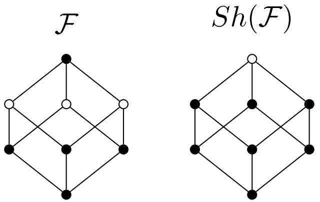
Lemma (Sauer-Shelah-Perles)

Every family $\mathcal{F}$ shatters at least as many elements as it has.

Alternatively, we can say that F is a subset of a boolean lattice B_n, and an element $y \in B_n$ is **shattered** by F if

$$\forall z \leq y \ \ \exists X \in F \ \text{ s.t. } \ z = y \wedge x.$$

Lattices, satisfying SSP.



So, for the original SSP lemma in the background we always have a boolean lattice B_n, which regulates how shattering is defined.

We can change B_n to arbitrary finite lattice L, and say that $F \subseteq L$ **shatters** an element $y \in L$, iff

$$\forall z \leq y \ \ \exists x \in F \ \ \text{s.t.} \ \ z = y \wedge x.$$

We say that L satisfies Sauer-Shelah-Perles property (is SSP), if for any $F \subseteq L$ it holds: $|F| \leq |Sh(F)|$.

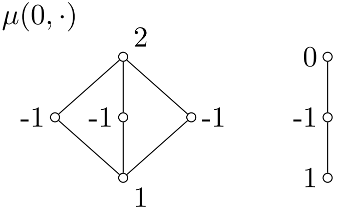
Thus, all B_n are SSP, but, for example, a chain of length at least two is not.

A sufficient condition for SSP.

Which finite lattices are SSP? There is one nice sufficient condition from *László Babai, Péter Frankl. Linear algebra methods in combinatorics.*

Theorem (Babai, Frankl)

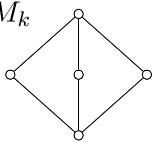
If a lattice L has a non-vanishing Möbius function (is NMF), then it is SSP.

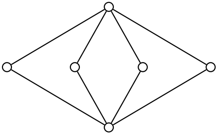


- As we see, M_3 is NMF, so it is SSP. Same argument shows that M_n is SSP for all $n \geq 2$, including $M_2 = B_2$.
- Chains of length at least two are not NMF. Although this condition is not necessary, such chains are not SSP.

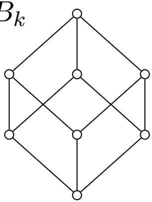
Example: geometric lattices (they are NMF)

M_k

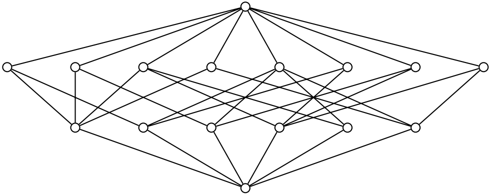




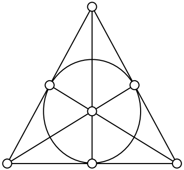
B_k



Also anything modular...



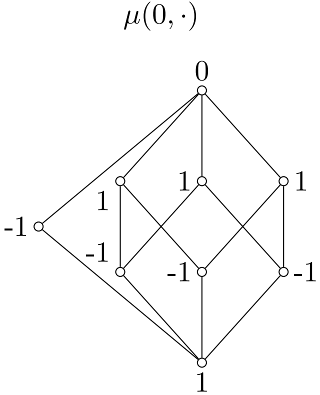
...also some weird things...



...and the Fano plane...

...and so on.

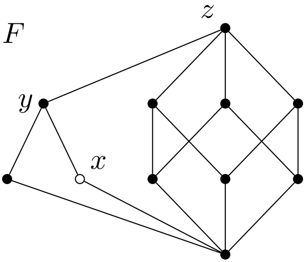
Some examples: NMF is not necessary



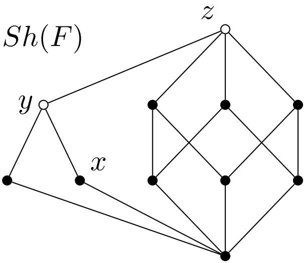
- For a lattice on the picture, the Möbius vanishes on the pair $(0,1)$, however the corresponding lattice is SSP. This example can be covered by a slight strengthening of the NMF condition;
- This example can be generalized by adjoining an element in the similar way to any SSP lattice with $\mu(0,1) = -1$.

Very simple necessary condition

F



$Sh(F)$



Lemma

If L is SSP then it does not have a three-element chain as a subinterval.

A lattice is **relatively complemented** (RC) if for all $x < y < z$ there is w, a *complement of y in $[x, z]$*, s.t. $x = y \wedge w$ and $z = y \vee w$.

Lemma (Björner)

A lattice does not have 3-element intervals iff it is relatively complemented.

Corollary

$SSP \Rightarrow RC$.

Conjecture

$SSP = RC$.

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A simple SSP⇒RC type result.

- Suppose $\mathcal{F} \subseteq L$ is such that an order-filter of elements not shattered by $\mathcal{F}$ contains just one minimal element. That is

$$L - \mathrm{Str}(\mathcal{F}) = [x).$$

- Then x is non-shattered *through* some $y \leq x$. That is

$$\mathcal{F} \subseteq L - \{u \in L \mid u \wedge x = y\}.$$

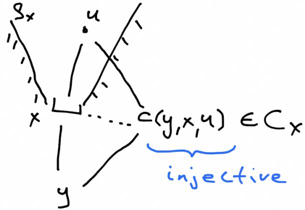
- Call $S = S_x = [x)$ and $C = C_x = \{u \in L \mid u \wedge x = y\}$. It is enough to show that $|C| \geq |S|$, as then

$$|\,\mathrm{Str}(\mathcal{F})| = |L - S| = |L| - |S| \geq |L| - |C| = |L - C| \geq |\mathcal{F}|.$$

- But $\varphi(u) = c(y, x, u)$ is an injective function from S to C, so $|C| \geq |S|$ and we are done.

$S = S_x = [x)$

$C = C_x = \{u \mid u \wedge x = y\}$

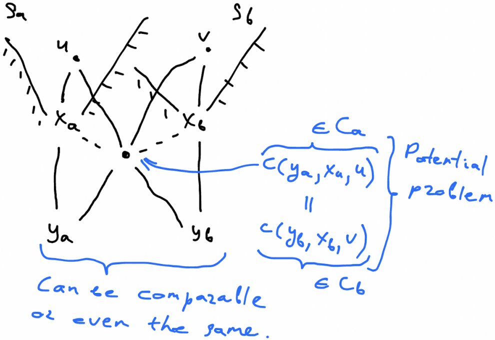


So $|C| \geq |S|$

In general:

$x_1 \dots x_k$ – antichain

$y_1 \dots y_k$ – $y_i \leq x_i$



The first reformulation (antichains)

The SSP=RC conjecture would follow from the following one

Conjecture (a slight strenghtening of SSP=RC)

For an RC-lattice L, let a system of size k be an antichain $x_1, \ldots, x_k$ together with the elements $y_1, \ldots, y_k$ where $y_i \leq x_i$. Let us define $S = \bigcup [x_i)$, and $C = \bigcup \{u \mid u \wedge x_i = y_i\}$.

Then $|S| \leq |C|$.

We can now try to prove it for small k, and:

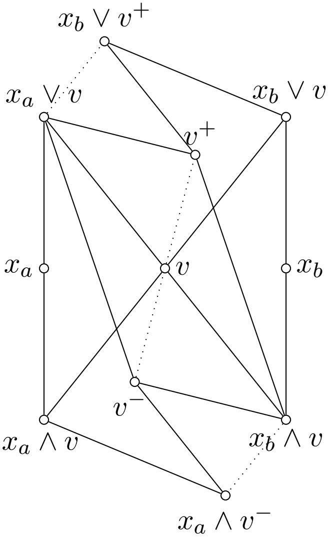
- for $k = 1$ it's trivial, we have just proven it;
- for $k = 2$ it's not hard (but we use a peculiar structural lemma);
- for $k = 3$ it's true, but very hard to prove. The subject of this talk is a subcase of this case, which is generic enough.

The structural lemma

Lemma (Corresponds to E2)

For arbitrary x_a, v, $x_b \in L$ there are elements v^- and v^+, $v^- \leq v \leq v^+$ such that

$$v^- \vee x_a = v \vee x_a = v^+ \vee x_a,$$
$$v^- \wedge x_b = v \wedge x_b = v^+ \wedge x_b,$$
$$v^+ \vee x_b \geq x_a,$$
$$v^- \wedge x_a \leq x_b.$$



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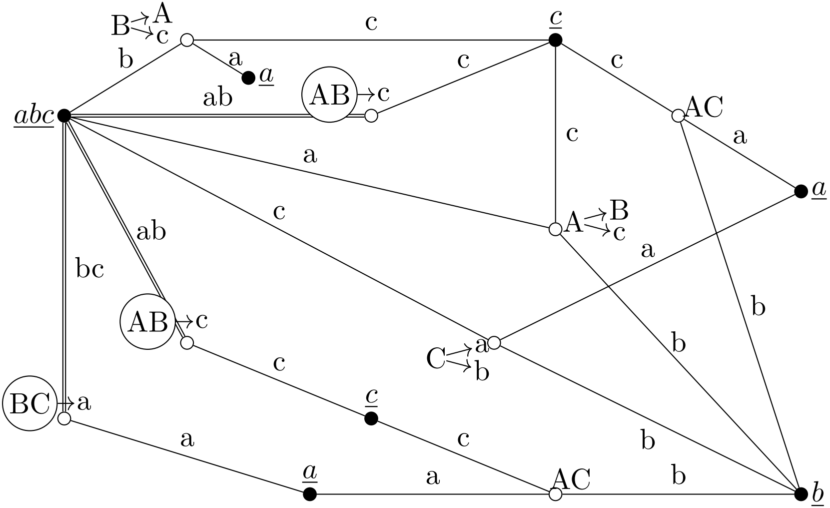
The second reformulation: RC-graphs

An RC-graph (over an index set $\mathcal{I} = \{a, b, c\}$) is "this"

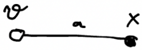


How RC graph tracks a system over RC lattice

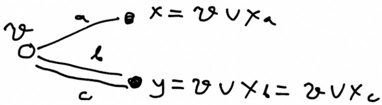
$x \in S \longrightarrow$ black vertex x •

$v \in C \longrightarrow$ white vertex v ○

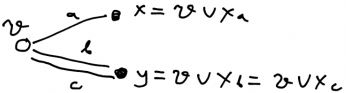
$v \vee X_a = x \longrightarrow$ a-edge between v & x



If $I = \{a, b, c\}$ then each ○ has one a, b, and c edges



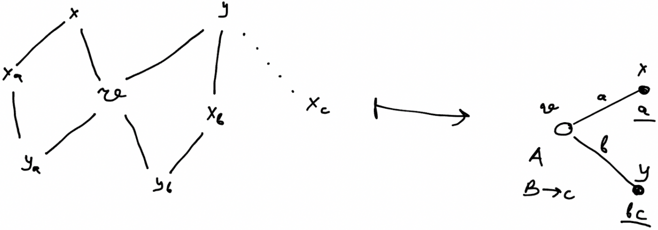
If $I = \{a, b, c\}$ then each $\circ$ has one a, b, and c edges



Black type tracks which x_i's an element in S is above

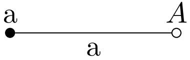
$x \geq x_a, x_b \sim x \in S_a, S_b \longmapsto$

White type tracks if an element in C is in C_a, C_b, C_c & also above which x_i's its joins are



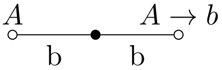
The extension conditions

(E1) For a black vertex u and $a \in \tau(u)$, there is an a-edge from u to (a white vertex) x with $A \in \tau(x)$;



This corresponds to the fact that if $x \geq x_a$ then $u = c(y_a, x_a, u)$ is in C_a and joins x_a to u.

(E2) For a white vertex x and $a, b \in \mathcal{I}$, if $A \in \tau(x)$ then there is a b-b-path from x to (a white vertex) x^+ such that 1) $A \to b \in \tau(x^+)$, and 2) $A_{\tau(x^+)} \supseteq A_{\tau(x)}$, that is, $\tau(x^+)$ has all arrows that $\tau(x)$ has;



This is a somewhat special condition, and it corresponds to a structural lemma about RC-lattices.

The second reformulation: RC-graphs

Conjecture (a big strenghtening of the antichain SSP=RC)

For an RC-graph it holds $|S| \leq |C|$.

Now, if $k = |\mathcal{I}|$, then

- for $k = 1$ it's trivial;
- for $k = 2$ it's easy - by a straightforward use of (E2);
- for $k = 3$ it's true, but complicated (the definition of an RC-graph should be modified);
- for $k = 5$ it's **false**!

So the last conjecture is false, and to make use of this approach, we have to reformulate it as

Lemma (Graph SSP_k=RC_k $\Rightarrow$ SSP_k=RC_k)

For a fixed k, if for any $\mathcal{I}$, $|\mathcal{I}| \leq k$, and for any Γ over $\mathcal{I}$ it holds $|S_\Gamma| \leq |C_\Gamma|$, then for any finite RC lattice L and any $F \subseteq L$ such that $\min(L - Sh(F))$ has at most k elements, then $|Sh(F)| \geq |F|$.

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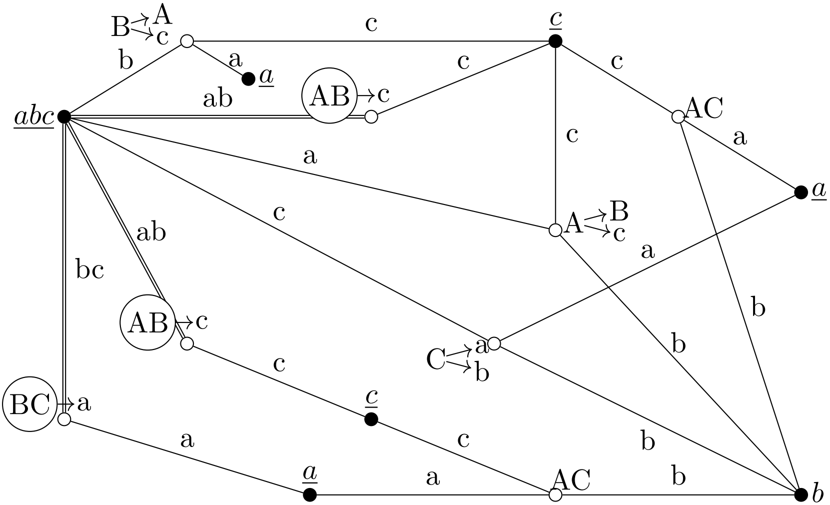
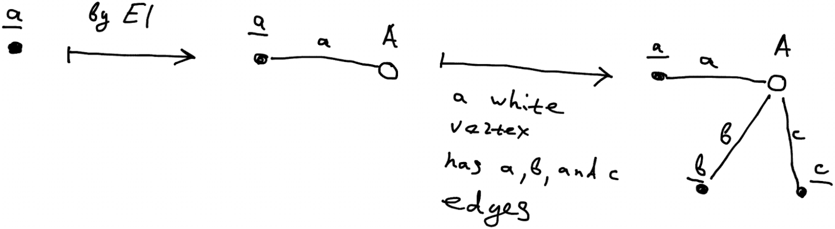
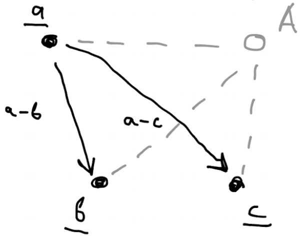
Consider black vertices with types to be fixed, and white ones as varying.



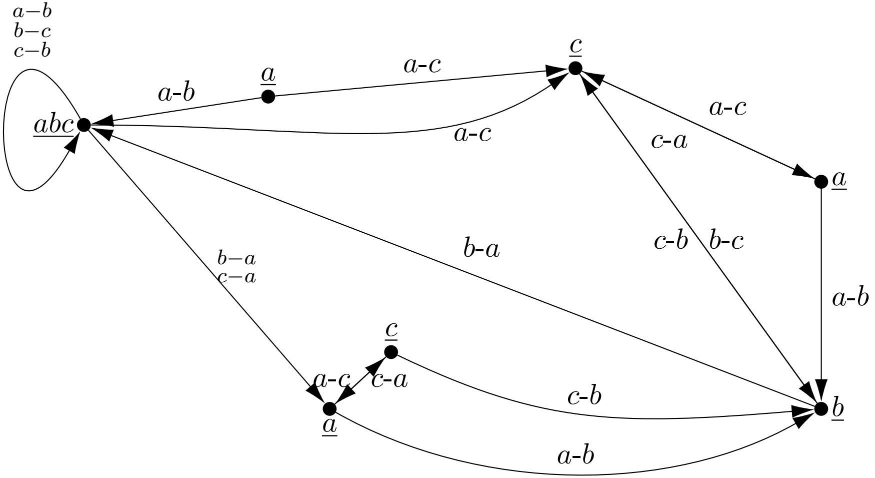


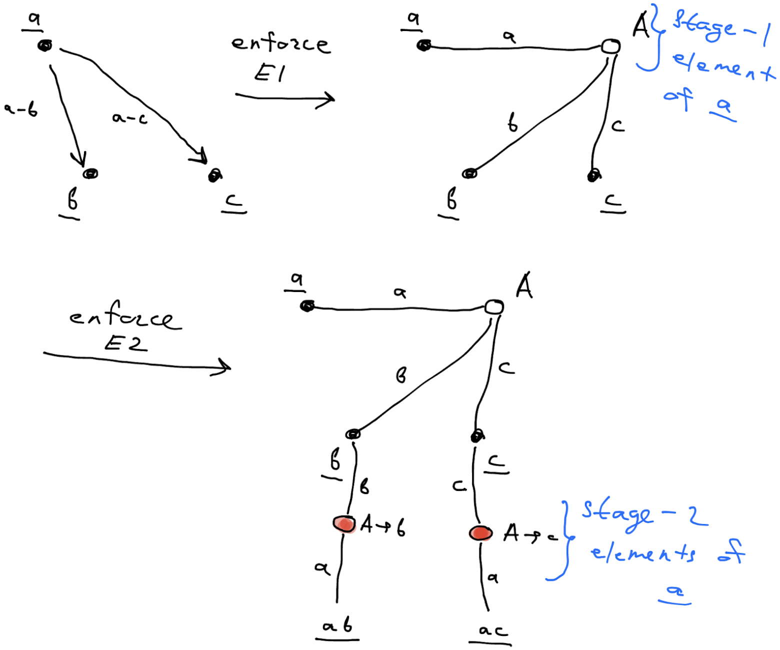
Now, „forget" this white vertex,
but remember its neighbors.
We track it with „arrow structure"



An arrow structure $\mathcal{A}_\Gamma$ of Γ

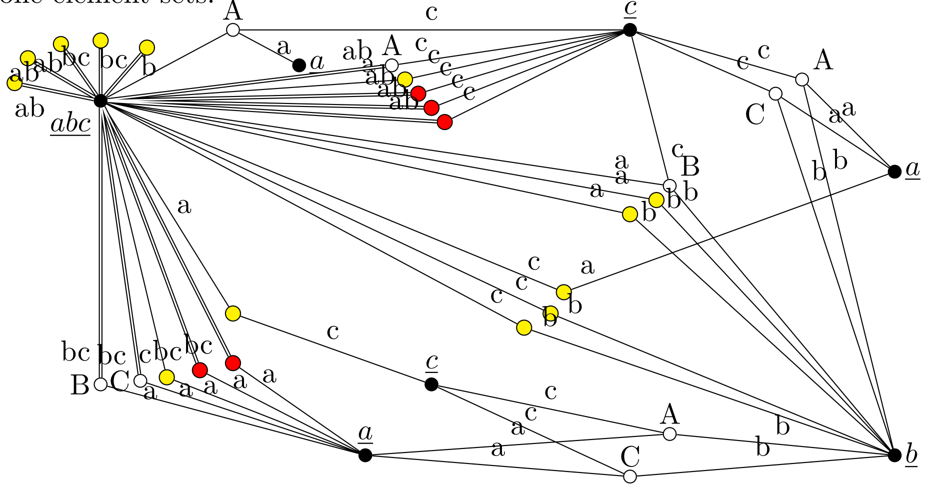
For any $u \in S$, any $a \in \tau(u)$, and any $b \in \mathcal{I} - a$ there is an $u \xrightarrow{a-b} v$ arrow (formally, a tuple (u, a, b, v)) such that $b \in \tau(v)$.



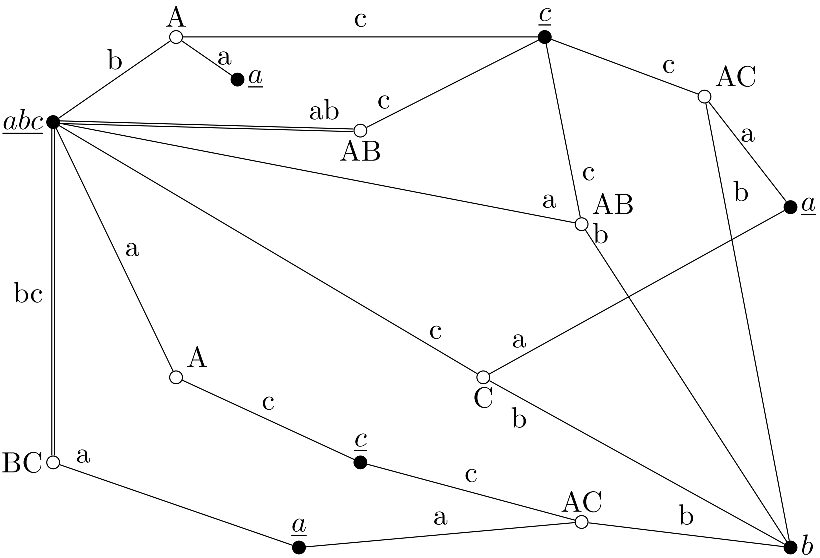

enforce E1
stage-1 element of a
enforce E2
A→b
A→c
stage-2 elements of a
ab
ac

The free RRC-graph $F_{\mathcal{A}}$ of $\mathcal{A} = \mathcal{A}_\Gamma$

Assumed simplification: *trivial closures* - there is exactly one *top* black element t, $\tau(t) = \underline{abc}$, and the types of all other black elements are one-element sets.



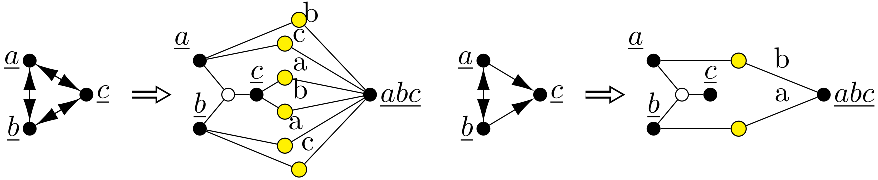
The image of $F_{\mathcal{A}}$ under a congruence Θ



Proof strategy: collapsing white vertices in $F_{\mathcal{A}}$

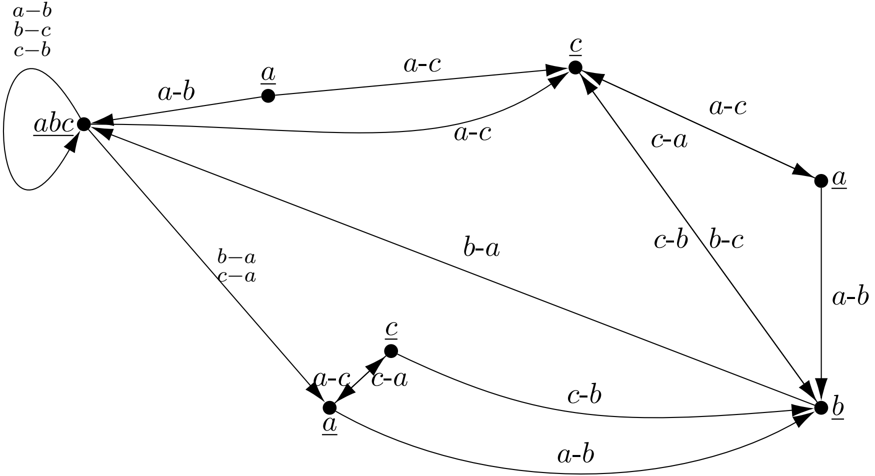
1. We want to prove that $|S| \leq \|\Theta\|$;
2. There is at least one stage-1 vertex for every black vertex, so if none of them are collapsed, we're already done. And it's not easy to collpase them, but sometimes they do;
3. Also, there is a lot of stage-2 vertices, but they are easily collapsible;
4. So, we will track the special cases when stage-1 vertices collapse. These situations will force some stage-2 vertices to be hard to collapse. We will track those, and ignore all others.

The special cases when stage-1 vertices collapse are *triangles* and *pyramids*.

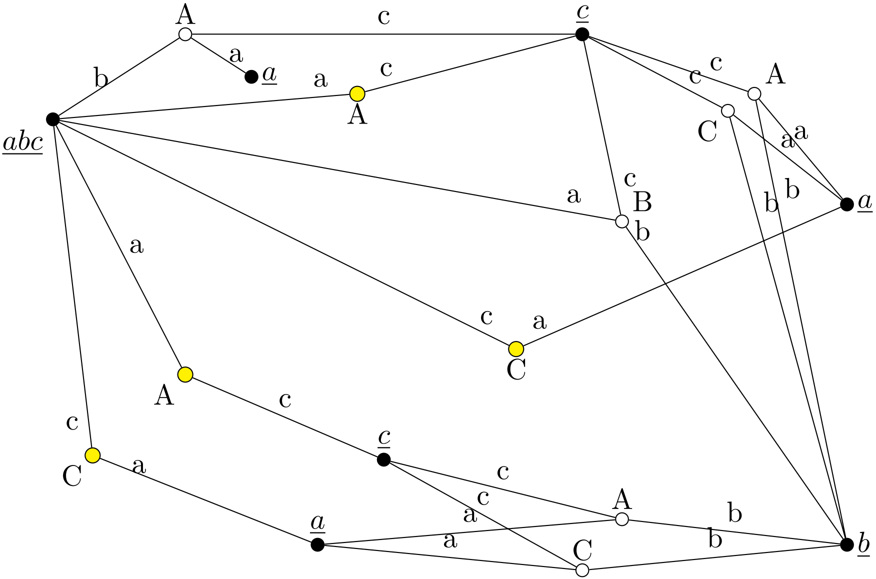


An arrow structure $\mathcal{A}_\Gamma$ of Γ

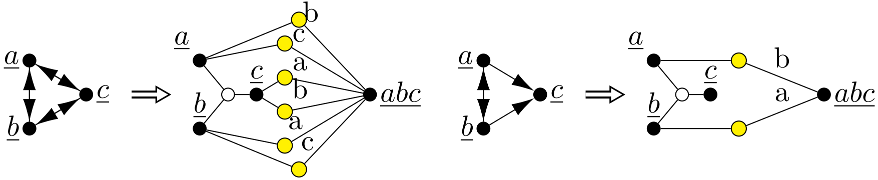
We will look for special patterns: *triangles* and *pyramids*.



Stage-1 and principal stage-2 vertices of $F_{\mathcal{A}}$



For a congruence Θ of $F_{\mathcal{A}}$, $|S| \leq \|\Theta\|$



- We (temporarily) ignore the top vertex t.

- There are r triangles, p pyramids, and s *singletones*, i.e. non-top black vertices not in a triangle or in a base of a pyramid. Then $|S| = 3r + 2p + s + 1$;

- Triangles and pyramids produce one stage-1 vertex each, so there are $r + p + s$ stage-1 verticies after contraction. Also, they produce 6 and 2 *principal* stage-2 vertices respectively;

- All white vertices produced this way are incomparable (almost, some trickery about special stage-2 vertices is done here);

- Principal stage-2 vertices can contract, but at most in pairs. So, we get at least $r + p + s + (6r + 2p)/2 = 4r + 2p + s$ white elements.

For a congruence Θ of $F_{\mathcal{A}}$, $|S| \leq \|\Theta\|$

We are almost done: we want to prove that $|S| \leq \|\Theta\|$, but we know that $|S| = 3r + 2p + s + 1$ and $4r + 2p + s \leq \|\Theta\|$. So the only possible "bad situation" can happen if $r = 0$, and

$$|S| = 2p + s + 1 > 2p + s = \|\Theta\|.$$

But also notice that the lower bound for Θ is acheivable only if we ignore all arrows from t, all non-principal stage-2 vertices, and only if all principal stage-2 vertices collapse precisely in pairs. That is, the fact that it's acheived gives us a lot of structural information. Then, consequtively, we get:

- there are no triangles;
- no arrow in $\mathcal{A}$ goes to the top;
- there are no pyramids;

After that we reach a contradiction because the stage-1 elements of t cannot be contracted with anything.

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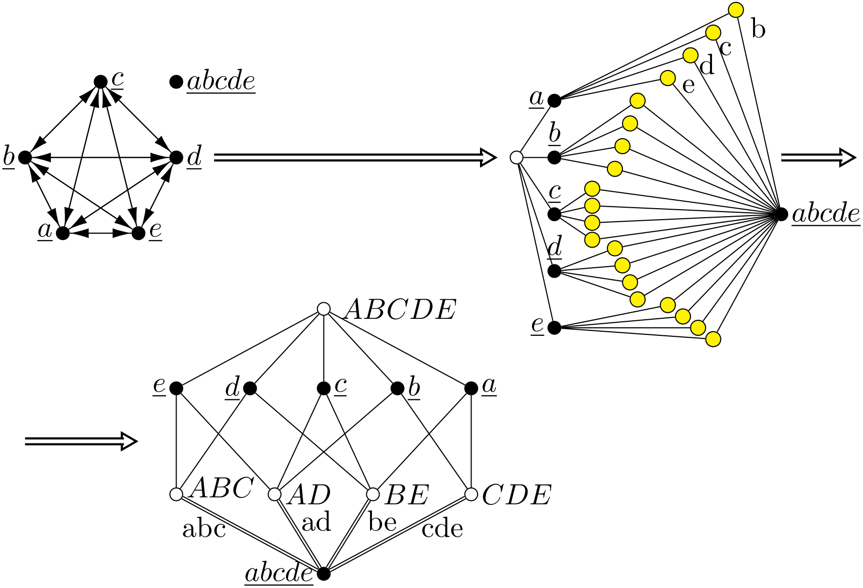
A counterexample for $k = 5$ ($\mathcal{I} = \{a, b, c, d, e\}$)



A related question

Question. Let $k > 0$, and let G_k be a “normal” graph with $V_{G_k} = \{(a, b) \mid a \neq b\}$ and $E_{G_k} = \{(a_1, b_1), (a_2, b_2) \mid b_1 \neq a_2, b_2 \neq a_1\}$, with $a, b \in \overline{k}$. What is the clique covering number of G_k?

In the construction of G_k vertices are all stage-2 vertices of a “generalized triangle”, and edges capture the compatibility relation, that is, which edges can be contracted to which. In particular, the counterexample above comes from the fact that for $k = 5$ there is a clique covering if size 4, namely:

- $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, and $(2, 5)$;
- $(1, 2)$, $(3, 2)$, $(3, 5)$, $(4, 2)$, and $(4, 5)$;
- $(2, 1)$, $(3, 1)$, $(3, 4)$, $(5, 1)$, and $(5, 4)$;
- $(4, 1)$, $(4, 3)$, $(5, 2)$, and $(5, 3)$.

Thank you!