Quasi-Engel varieties of lattice-ordered groups

Michael R. Darnel

Indiana University, Emeritus Professor

August 8, 2022

Contents

Background

In the following, $\mathcal{A}$ denotes the variety of abelian ℓ-groups, which is the smallest nontrivial variety of ℓ-groups. $\mathcal{N}$ denotes the variety of *normal-valued* groups, –those ℓ-groups satisfying the law

$$(x \vee e)(y \vee e) \wedge (y \vee e)^2(x \vee e)^2 = (x \vee e)(y \vee e)$$

– and is the largest proper variety.

Background

In the following, $\mathcal{A}$ denotes the variety of abelian ℓ-groups, which is the smallest nontrivial variety of ℓ-groups. $\mathcal{N}$ denotes the variety of *normal-valued* groups, –those ℓ-groups satisfying the law

$$(x \vee e)(y \vee e) \wedge (y \vee e)^2(x \vee e)^2 = (x \vee e)(y \vee e)$$

– and is the largest proper variety.
$\mathcal{R}$ denotes the variety of *representable* ℓ-groups, those generated by all totally ordered groups and satisfying the law

$$y^{-1}(x \vee e)y \wedge (x^{-1} \vee e) = e.$$

An ℓ-group G is representable if and only if every minimal prime subgroup of G is normal.

Background

In the following, $\mathcal{A}$ denotes the variety of abelian ℓ-groups, which is the smallest nontrivial variety of ℓ-groups. $\mathcal{N}$ denotes the variety of *normal-valued* groups, –those ℓ-groups satisfying the law

$$(x \vee e)(y \vee e) \wedge (y \vee e)^2(x \vee e)^2 = (x \vee e)(y \vee e)$$

– and is the largest proper variety.

$\mathcal{R}$ denotes the variety of *representable* ℓ-groups, those generated by all totally ordered groups and satisfying the law

$$y^{-1}(x \vee e)y \wedge (x^{-1} \vee e) = e.$$

An ℓ-group G is representable if and only if every minimal prime subgroup of G is normal.

Generalizing this, for a positive integer n, $\mathcal{R}_n$ is the variety of ℓ-groups satisfying the law

$$y^{-n}(x \vee e)y^n \wedge (x^{-1} \vee e) = e.$$

An ℓ-group G is in $\mathcal{R}_n$ if and only if for any $y \in G$ and minimal prime subgroup M of G, $y^{-n}My^n = M$.

If $\mathcal{V}$ is a variety of ℓ-groups, then for any ℓ-group G, there exists a unique maximal convex ℓ-subgroup of G, $\mathcal{V}(G)$, which is in $\mathcal{V}$. Given two varieties $\mathcal{U}, \mathcal{V}$ of ℓ-groups, the class of ℓ-groups G such that $G/\mathcal{V}(G) \in \mathcal{U}$ is also a variety, and is denoted $\mathcal{U}\mathcal{V}$. This multiplication of varieties is associative and so powers of varieties are well defined.

If $\mathcal{V}$ is a variety of ℓ-groups, then for any ℓ-group G, there exists a unique maximal convex ℓ-subgroup of G, $\mathcal{V}(G)$, which is in $\mathcal{V}$. Given two varieties $\mathcal{U}, \mathcal{V}$ of ℓ-groups, the class of ℓ-groups G such that $G/\mathcal{V}(G) \in \mathcal{U}$ is also a variety, and is denoted $\mathcal{U}\mathcal{V}$. This multiplication of varieties is associative and so powers of varieties are well defined.

If $\mathcal{V}$ is a variety of ℓ-groups such that $\mathcal{V} \subset \mathcal{N}$, there exists a positive integer n such that $\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{R})^n$.

If $\mathcal{V}$ is a variety of ℓ-groups, then for any ℓ-group G, there exists a unique maximal convex ℓ-subgroup of G, $\mathcal{V}(G)$, which is in $\mathcal{V}$. Given two varieties $\mathcal{U}, \mathcal{V}$ of ℓ-groups, the class of ℓ-groups G such that $G/\mathcal{V}(G) \in \mathcal{U}$ is also a variety, and is denoted $\mathcal{U}\mathcal{V}$. This multiplication of varieties is associative and so powers of varieties are well defined.

If $\mathcal{V}$ is a variety of ℓ-groups such that $\mathcal{V} \subset \mathcal{N}$, there exists a positive integer n such that $\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{R})^n$.

For a variety $\mathcal{V}$ of ℓ-groups, $\mathcal{A}^2 \not\subseteq \mathcal{V}$ if and only if $\mathcal{V} \subseteq \mathcal{R}_n$ for some positive integer n.

If $\mathcal{V}$ is a variety of ℓ-groups, then for any ℓ-group G, there exists a unique maximal convex ℓ-subgroup of G, $\mathcal{V}(G)$, which is in $\mathcal{V}$. Given two varieties $\mathcal{U}, \mathcal{V}$ of ℓ-groups, the class of ℓ-groups G such that $G/\mathcal{V}(G) \in \mathcal{U}$ is also a variety, and is denoted $\mathcal{U}\mathcal{V}$. This multiplication of varieties is associative and so powers of varieties are well defined.

If $\mathcal{V}$ is a variety of ℓ-groups such that $\mathcal{V} \subset \mathcal{N}$, there exists a positive integer n such that $\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{R})^n$.

For a variety $\mathcal{V}$ of ℓ-groups, $\mathcal{A}^2 \not\subseteq \mathcal{V}$ if and only if $\mathcal{V} \subseteq \mathcal{R}_n$ for some positive integer n.

$\mathfrak{Wa}$ denotes the variety of *weakly abelian* ℓ-groups, those satisfying the law $(x \vee e)^y \leq (x \vee e)^2$. For a positive integer n, $\mathfrak{Wa}_n$ is the variety of ℓ-groups satisfying the law $(x \vee e)^{y^n} \leq (x \vee e)^2$. If C is a convex ℓ-subgroup of $G \in \mathfrak{Wa}_n$, then for any $y \in G$, $C^{y^n} = C$. $\mathfrak{Wa}_n$ is the largest variety having this property.

For a positive integer n, $\mathcal{L}_n$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^n, y^n] = e$, and for positive integers m, n, $\mathcal{L}_{m,n}$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^m, y^n] = e$. It is known that if $d = \gcd(m, n)$, $\mathcal{L}_{m,n} = \mathcal{L}_d$. For every $n > 1$, $\mathcal{L}_n \cap \mathcal{R} = \mathcal{A}$ and if $n = p_1 p_2 \ldots p_k$ as a product of prime integers, then $\mathcal{L}_n \subset \mathcal{A}^{k+1} \setminus \mathcal{A}^k$.

For any positive integer n, $\mathcal{L}_n \subset \mathfrak{Wa}_n \subset \mathcal{R}_n$.

For a positive integer n, $\mathcal{L}_n$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^n, y^n] = e$, and for positive integers m, n, $\mathcal{L}_{m,n}$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^m, y^n] = e$. It is known that if $d = \gcd(m, n)$, $\mathcal{L}_{m,n} = \mathcal{L}_d$. For every $n > 1$, $\mathcal{L}_n \cap \mathcal{R} = \mathcal{A}$ and if $n = p_1 p_2 \ldots p_k$ as a product of prime integers, then $\mathcal{L}_n \subset \mathcal{A}^{k+1} \setminus \mathcal{A}^k$.

For any positive integer n, $\mathcal{L}_n \subset \mathfrak{Wa}_n \subset \mathcal{R}_n$.

If $\mathcal{V} \cap \mathcal{R} = \mathcal{A}$, then $\mathcal{V} \subseteq \mathcal{L}_n$ for some positive integer n.

For a positive integer n, $\mathcal{L}_n$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^n, y^n] = e$, and for positive integers m, n, $\mathcal{L}_{m,n}$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^m, y^n] = e$. It is known that if $d = \gcd(m, n)$, $\mathcal{L}_{m,n} = \mathcal{L}_d$. For every $n > 1$, $\mathcal{L}_n \cap \mathcal{R} = \mathcal{A}$ and if $n = p_1 p_2 \ldots p_k$ as a product of prime integers, then $\mathcal{L}_n \subset \mathcal{A}^{k+1} \setminus \mathcal{A}^k$.

For any positive integer n, $\mathcal{L}_n \subset \mathfrak{Wa}_n \subset \mathcal{R}_n$.

If $\mathcal{V} \cap \mathcal{R} = \mathcal{A}$, then $\mathcal{V} \subseteq \mathcal{L}_n$ for some positive integer n.

Suppose that a, b are elements of an ℓ-group G such that for all $1 < i < n$, $a \wedge a^{b^i} = e$ while $a = a^{b^n}$. Then the ℓ-subgroup $\langle a, b \rangle$ is an ℓ-group called the *Scrimger n-group* S_n.

For a positive integer n, $\mathcal{L}_n$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^n, y^n] = e$, and for positive integers m, n, $\mathcal{L}_{m,n}$ denotes the variety of ℓ-groups G in which for every $x, y \in G$, $[x^m, y^n] = e$. It is known that if $d = \gcd(m, n)$, $\mathcal{L}_{m,n} = \mathcal{L}_d$. For every $n > 1$, $\mathcal{L}_n \cap \mathcal{R} = \mathcal{A}$ and if $n = p_1 p_2 \ldots p_k$ as a product of prime integers, then $\mathcal{L}_n \subset \mathcal{A}^{k+1} \setminus \mathcal{A}^k$.

For any positive integer n, $\mathcal{L}_n \subset \mathfrak{Wa}_n \subset \mathcal{R}_n$.

If $\mathcal{V} \cap \mathcal{R} = \mathcal{A}$, then $\mathcal{V} \subseteq \mathcal{L}_n$ for some positive integer n.

Suppose that a, b are elements of an ℓ-group G such that for all $1 < i < n$, $a \wedge a^{b^i} = e$ while $a = a^{b^n}$. Then the ℓ-subgroup $\langle a, b \rangle$ is an ℓ-group called the *Scrimger n-group* S_n.

Theorem

If $\mathcal{V} \cap \mathcal{R} \subseteq \mathfrak{Wa}$, then $\mathcal{V} \subseteq \mathfrak{Wa}_n$ for some positive integer n. The least such n is $\mathrm{lcm}\{k : S_k \in \mathcal{V}\}$.

What if we generalize the law $[x^n, y^n] = e$ to $[x^n, y^n, z^n] = e$ (where $[x^n, y^n, z^n] = [[x^n, y^n], z^n]$)?

What if we generalize the law $[x^n, y^n] = e$ to $[x^n, y^n, z^n] = e$ (where $[x^n, y^n, z^n] = [[x^n, y^n], z^n]$)?

Can we, as in the $\mathcal{L}_n$ varieties, assume that all powers are equal?

$\mathcal{N}_k$ denotes the variety of ℓ-groups satisfying the nilpotent law $[x_1, x_2, \ldots, x_k, x_{k+1}] = e$.

$\mathcal{E}_k$ denotes the variety of ℓ-groups satisfying the Engel law $[x, \underbrace{y, \ldots, y}_{k \text{ times}}] = e$.

We will follow standard practice and denote $[x, \underbrace{y, \ldots, y}_{k \text{ times}}]$ as $[x, {}_k y]$.

Clearly for any integer k, $\mathcal{N}_k \subseteq \mathcal{E}_k$ and if $k > 2$, the containment is proper.

Kim and Rhemtulla proved that for any positive integer k, there exists a positive integer $f(k)$ such that $\mathcal{E}_k \subseteq \mathcal{N}_{f(k)}$. Hollister showed that for any positive integer k, $\mathcal{N}_k \subset \mathcal{R}$, Reilly that $\mathcal{N}_k \subseteq \mathfrak{Wa}$, and Medvedev that $\mathcal{E}_k \subseteq \mathfrak{Wa}$.

It is known that $\mathcal{E}_2 = \mathcal{N}_2$, that $\mathcal{E}_3 \subset \mathcal{N}_4$, and that $\mathcal{E}_4 \subseteq \mathcal{N}_7$.

Varieties satisfying iterated commutator identities

For an ℓ-group word $u(x_1, x_2, \ldots, x_n)$, we will use the notation

$$u(x_1, x_2, \ldots, x_n)\Big|_{\{x_1, x_2, \ldots, x_n\} \to \{g_1, g_2, \ldots, g_n\}}$$

to denote the substitution of group elements g_i for x_i. We will also use the notation $\mathcal{V}_{u(x_1, x_2, \ldots, x_n)}$ to denote the variety of ℓ-groups G satisfying the law $u(x_1, x_2, \ldots, x_n) = e$.

We will define an iterated commutator $[x_1, x_2, \ldots, x_n]$ to be of *Engel-type* if for all $2 < i \leq n$, $x_i = x_1$ or $x_i = x_2$. A commutator $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$ is *quasi-Engel* if the commutator $[x_1, x_2, \ldots, x_n]$ is of Engel-type.

We will define an iterated commutator $[x_1, x_2, \ldots, x_n]$ to be of *Engel-type* if for all $2 < i \leq n$, $x_i = x_1$ or $x_i = x_2$. A commutator $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$ is *quasi-Engel* if the commutator $[x_1, x_2, \ldots, x_n]$ is of Engel-type.

Given an iterated commutator of the form $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$, its *Engelized* form is $[w_1^{k_1}, w_2^{k_2}, \ldots, w_n^{k_n}]$, where

$$w_i = \begin{cases} x_1, & \text{if } x_i = x_1 \\ x_2, & \text{if } x_i \neq x_1 \end{cases}.$$

Theorem

Suppose that $e < a, b$ are elements of an ℓ-group G such that $a \wedge a^{b^i} = e$ for all $1 \leq i < n$ while $a = a^{b^n}$. Then for any commutator $[x_1, \ldots, x_k]$ of Engel-type,

$$[x_1, \ldots, x_k]\Big|_{\{x_1,x_2\} \to \{ba,b\}} \neq e.$$

Proof.

The proof begins by tedious inductive arguments for the Engel case

$$[x_1, {}_{k-1}x_2]\Big|_{\{x_1,x_2\} \to \{ba,b\}}.$$

Another inductive proof shows that for any $2 \leq s \leq n$,

$$[x_1, \ldots, x_s]\Big|_{\{x_1,x_2\} \to \{ba,b\}} = [x_1, {}_{s-1}x_2]\Big|_{\{x_1,x_2\} \to \{ba,b\}}.$$

□

Theorem

For any commutator $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$,

$$\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \cap \mathcal{R} \subseteq \mathfrak{Wa}.$$

Proof.

We can assume that $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$ is quasi-Engel.
If not, there exists an o-group $G \in \mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]}$ and elements $a, b \in G$ such that $e < a \ll a^b$. Inducting on n shows that

$$[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k^n}]\Big|_{\{x_1, x_2\} \to \{ba, b\}} \neq e.$$

□

Corollary

For any commutator $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$, *there exists a positive integer* m *such that* $\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \subseteq \mathfrak{Wa}_m$.

Corollary

For any commutator $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$*, there exists a positive integer* m *such that* $\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \subseteq \mathfrak{W}\mathfrak{a}_m$.

Theorem

$$\mathcal{V}_{[x_1, x_2, \ldots, x_n]} \subseteq \mathfrak{W}\mathfrak{a}.$$

Corollary

For any commutator $[x_1^{k_1}, x_2^{k_2}, \dots, x_n^{k_n}]$*, there exists a positive integer* m *such that* $\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \dots, x_n^{k_n}]} \subseteq \mathfrak{Wa}_m$.

Theorem

$$\mathcal{V}_{[x_1, x_2, \dots, x_n]} \subseteq \mathfrak{Wa}.$$

Proof.

We can assume $[x_1, x_2, \dots, x_n]$ is of Engel-type.
We know there exists a least positive integer m such that $\mathcal{V}_{[x_1, x_2, \dots, x_n]} \subseteq \mathfrak{Wa}_m$. If $m > 1$, there exists $1 < k$ such that $k \mid m$ and $S_k \in \mathcal{V}_{[x_1, x_2, \dots, x_n]}$. Let $e < a, b \in S_k$ such that $a \wedge a^{b^i} = e$ for all $1 < i < k$ and $a = a^{b^k}$. But then

$$[x_1, x_2, \dots, x_n]\Big|_{\{x_1, x_2\} \to \{ba, b\}} \neq e.$$

□

Theorem

$$\mathcal{V}_{[x^n,\,_k y^n]} \subseteq \mathfrak{Wa}_n.$$

Theorem

$$\mathcal{V}_{[x^n, {}_k y^n]} \subseteq \mathfrak{Wa}_n.$$

Theorem

Let $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$ *be an iterated commutator such* $x_i \neq x_1$ *for* $2 \leq i \leq n$*, and let* $m = \mathrm{lcm}\{k_1, k_2, \ldots, k_n\}$*. Then*

$$\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \subseteq \mathfrak{Wa}_m.$$

Theorem

$$\mathcal{V}_{[x^n,\,_k y^n]} \subseteq \mathfrak{wa}_n.$$

Theorem

Let $[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]$ *be an iterated commutator such* $x_i \neq x_1$ *for* $2 \leq i \leq n$*, and let* $m = \operatorname{lcm}\{k_1, k_2, \ldots, k_n\}$*. Then*

$$\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \subseteq \mathfrak{wa}_m.$$

Theorem

If $n > 2$*,*

$$S_{k_n} \in \mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]}.$$

Proof.

Let $\{a_1, a_2, \ldots, a_n\}$ be a substitution for $\{x_1, x_2, \ldots, x_n\}$ into S_{k_n}. Then $[a_1^{k_1}, a_2^{k_2}, \ldots, a_{n-1}^{k_{n-1}}] \in \mathcal{A}(S_{k_n})$, and so commutes with $a_n^{k_n}$. □

Proof.

Let $\{a_1, a_2, \ldots, a_n\}$ be a substitution for $\{x_1, x_2, \ldots, x_n\}$ into S_{k_n}. Then $[a_1^{k_1}, a_2^{k_2}, \ldots, a_{n-1}^{k_{n-1}}] \in \mathcal{A}(S_{k_n})$, and so commutes with $a_n^{k_n}$. □

REMARK

For the case $n = 2$, $\mathcal{A} = \mathcal{L}_1 = \mathcal{L}_{2,3}$, and clearly $S_3 \notin \mathcal{A}$.

Generalized 2-Engel Varieties

In this section and the one after, we narrow our focus from varieties satisfying general iterated commutator identities to look at varieties of ℓ-groups satisfying identities of the type $[a^k, b^m, b^n] = e$ or $[a^k, b^m, c^n] = e$, where now we restrict k, m, n to be positive integers. We will denote the variety of ℓ-groups satisfying $[a^k, b^m, b^n] = e$ by $\mathscr{E}_2(k, m, n)$, and the variety of ℓ-groups satisfying $[a^k, b^m, c^n] = e$ by $\mathscr{N}_2(k, m, n)$. For each of these varieties, we seek to learn:

- its intersection with $\mathscr{R}$.

Generalized 2-Engel Varieties

In this section and the one after, we narrow our focus from varieties satisfying general iterated commutator identities to look at varieties of ℓ-groups satisfying identities of the type $[a^k, b^m, b^n] = e$ or $[a^k, b^m, c^n] = e$, where now we restrict k, m, n to be positive integers. We will denote the variety of ℓ-groups satisfying $[a^k, b^m, b^n] = e$ by $\mathscr{E}_2(k, m, n)$, and the variety of ℓ-groups satisfying $[a^k, b^m, c^n] = e$ by $\mathscr{N}_2(k, m, n)$. For each of these varieties, we seek to learn:

- its intersection with $\mathscr{R}$.
- its *representable rank:* the least positive integer q such that $\mathscr{E}_2(k, m, p) \subseteq \mathscr{R}^q$. q exists since $\mathscr{E}_2(k, m, n) \subset \mathscr{N}$, and is the maximum length of all disjoint conjugate chains in the ℓ-groups in $\mathscr{E}_2(k, m, n)$.

Generalized 2-Engel Varieties

In this section and the one after, we narrow our focus from varieties satisfying general iterated commutator identities to look at varieties of ℓ-groups satisfying identities of the type $[a^k, b^m, b^n] = e$ or $[a^k, b^m, c^n] = e$, where now we restrict k, m, n to be positive integers. We will denote the variety of ℓ-groups satisfying $[a^k, b^m, b^n] = e$ by $\mathscr{E}_2(k, m, n)$, and the variety of ℓ-groups satisfying $[a^k, b^m, c^n] = e$ by $\mathscr{N}_2(k, m, n)$. For each of these varieties, we seek to learn:

- its intersection with $\mathscr{R}$.
- its *representable rank:* the least positive integer q such that $\mathscr{E}_2(k, m, p) \subseteq \mathscr{R}^q$. q exists since $\mathscr{E}_2(k, m, n) \subset \mathscr{N}$, and is the maximum length of all disjoint conjugate chains in the ℓ-groups in $\mathscr{E}_2(k, m, n)$.
- its *disjoint conjugate breadth:* the least positive integer s such that $\mathscr{E}_2(k, m, p) \subseteq \mathscr{R}_s$. s exists since $\mathscr{A}^2 \not\subseteq \mathscr{E}_2(k, m, n)$.

Generalized 2-Engel Varieties

In this section and the one after, we narrow our focus from varieties satisfying general iterated commutator identities to look at varieties of ℓ-groups satisfying identities of the type $[a^k, b^m, b^n] = e$ or $[a^k, b^m, c^n] = e$, where now we restrict k, m, n to be positive integers. We will denote the variety of ℓ-groups satisfying $[a^k, b^m, b^n] = e$ by $\mathscr{E}_2(k, m, n)$, and the variety of ℓ-groups satisfying $[a^k, b^m, c^n] = e$ by $\mathscr{N}_2(k, m, n)$. For each of these varieties, we seek to learn:

- its intersection with $\mathscr{R}$.
- its *representable rank:* the least positive integer q such that $\mathscr{E}_2(k, m, p) \subseteq \mathscr{R}^q$. q exists since $\mathscr{E}_2(k, m, n) \subset \mathscr{N}$, and is the maximum length of all disjoint conjugate chains in the ℓ-groups in $\mathscr{E}_2(k, m, n)$.
- its *disjoint conjugate breadth:* the least positive integer s such that $\mathscr{E}_2(k, m, p) \subseteq \mathscr{R}_s$. s exists since $\mathscr{A}^2 \nsubseteq \mathscr{E}_2(k, m, n)$.
- when one such variety is contained in another.

At this time, the only item in the above list that is completely known is $\mathscr{E}_2(k,m,n) \cap \mathscr{R}$ and $\mathscr{N}_2(k,m,n) \cap \mathscr{R}$, and this author does not have an example of positive integers k, m, and n such that $\mathscr{N}_2(k,m,n) \subset \mathscr{E}_2(k,m,n)$.

Theorem

For any positive integers k, m, n, $\mathscr{N}_2(k,m,n) \subseteq \mathscr{E}_2(k,m,n)$.

At this time, the only item in the above list that is completely known is $\mathscr{E}_2(k,m,n) \cap \mathscr{R}$ and $\mathscr{N}_2(k,m,n) \cap \mathscr{R}$, and this author does not have an example of positive integers k, m, and n such that $\mathscr{N}_2(k,m,n) \subset \mathscr{E}_2(k,m,n)$.

Theorem

For any positive integers k, m, n, $\mathscr{N}_2(k,m,n) \subseteq \mathscr{E}_2(k,m,n)$.

Theorem

For any positive integers k and n,

$$\mathcal{E}_2(k,n,n) \subset \mathfrak{Wa}_n.$$

At this time, the only item in the above list that is completely known is $\mathscr{E}_2(k, m, n) \cap \mathscr{R}$ and $\mathscr{N}_2(k, m, n) \cap \mathscr{R}$, and this author does not have an example of positive integers k, m, and n such that $\mathscr{N}_2(k, m, n) \subset \mathscr{E}_2(k, m, n)$.

Theorem

For any positive integers k, m, n, $\mathscr{N}_2(k, m, n) \subseteq \mathscr{E}_2(k, m, n)$.

Theorem

For any positive integers k and n,

$$\mathcal{E}_2(k, n, n) \subset \mathfrak{Wa}_n.$$

Corollary

For any positive integer k,

$$\mathcal{E}_2(k, 1, 1) \subset \mathfrak{Wa}.$$

Theorem

For any positive integers k, m, and n,

$$\mathcal{E}_2(k, m, n) \cap \mathcal{R} \subseteq \mathcal{E}_2(k, 1, 1).$$

Proof.

Note that $[a^k, b^m, b^n] = [a^k, b^m, b]\,[a^k, b^m, b^{n-1}]^b$, and so by induction on n, $[a^k, b^m, b^n]$ is a product of conjugates of $[a^k, b^m, b]$. So if G is an o-group in $\mathscr{E}_2(k, m, n)$ having elements $a, b \in G$ such that $[a^k, b^m, b] \neq e$, then either $[a^k, b^m, b] > e$ or $[a^k, b^m, b] < e$. In the first case, $[a^k, b^m, b^n]$ is a product of strictly positive elements, and so is strictly positive itself, or $[a^k, b^m, b] < e$, in which case $[a^k, b^m, b^n] < e$.

So $\mathscr{E}_2(k, m, n) \cap \mathscr{R} \subseteq \mathscr{E}_2(k, m, 1) \cap \mathscr{R}$. □

Proof.

So assume that G is an o-group in $\mathscr{E}_2(k, m, 1)$ with elements $a, b \in G$ such that $[a^k, b, b] \neq e$. But

$$\begin{aligned}
[a^k, b^m, b] &= [a^k, b^{m-1}b, b] \\
&= \left[[a^k, b]\,[a^k, b^{m-1}]^b, b\right] \\
&= [a^k, b, b]^{[a^k, b^{m-1}]^b} \left[[a^k, b^{m-1}]^b, b\right] \\
&= [a^k, b, b]^{[a^k, b^{m-1}]^b} [a^k, b^{m-1}, b]^b
\end{aligned}$$

is again by induction on m a product of conjugates of $[a^k, b, b]$, and so $[a^k, b, b] = e$.

□

However, there are o-groups with elements a and b such that $[a^k, b, b] = e]$ for all $k > 1$ while $[a, b, b] \neq e$. So different techniques must be used to prove:

Theorem

For any positive integer k, $\mathscr{E}_2(k, 1, 1) = \mathscr{E}_2$.

Corollary

Let k, m, and n be positive integers. Then

$$\mathscr{N}_2(k, m, n) \cap \mathscr{R} = \mathscr{E}_2(k, m, n) \cap \mathscr{R} = \mathscr{E}_2.$$

Theorem

For any positive integers k, m, and n,

$$\mathcal{E}_2(k, m, n) \subset \mathfrak{Wa}_{mn}.$$

Corollary

For any positive integer n,

$$\mathcal{E}_2(1, 1, n) \subset \mathfrak{Wa}_{n}.$$

Theorem

Let m, n be positive integers such that $\gcd(m,n) = 1$. Then

$$\mathcal{N}_2(1, m, n) \subset (\mathcal{N}_2)^2.$$

For any positive integer n, $S_n \in \mathcal{N}_2(1,1,n)$, but we cannot conclude that $\mathcal{L}_n \subseteq \mathcal{N}_2(1,1,n)$ for all n.

Theorem

For a positive integer n, $\mathscr{L}_n \subseteq \mathscr{N}_2(1,1,n)$ if and only if n is prime.

Theorem

For any positive prime integer p,

$$\mathcal{N}_2(1, p, p) \subset (\mathcal{N}_2)^2.$$

Theorem

For any positive prime integer p,

$$\mathcal{N}_2(p, p, p^2) \subseteq (\mathcal{N}_2)^3.$$

Theorem

For any positive prime integer p,

$$\mathcal{N}_2(p, p, p^2) \subseteq (\mathcal{N}_2)^3.$$

Theorem

For any positive integer s,

$$\mathcal{N}_2(s, s, s) = \mathcal{E}_2(s, s, s).$$

The proof takes the course that

$$\mathcal{V}_{[(x^k)^{y^m}, x^n]} = \mathcal{V}_{[x^k, y^m, x^n]} = \mathcal{E}_2(m, k, n)$$

and then use Reilly's argument to show that in $\mathcal{V}_{[(x^s)^{y^s}, x^s]}$,

$$[x^s, y^s, z^s]^3 = e$$

.

Special Results

1. $\mathcal{N}_2(2,2,2) \not\subseteq \mathcal{N}_2(1,1,2)$
2. $\mathcal{E}_2(1,2,2) \not\subseteq \mathcal{E}_2$
3. $\mathcal{N}_2(2,2,n) \not\subseteq \mathcal{L}_2$
4. $\mathcal{N}_2(2,2,4) \not\subseteq \mathcal{N}_2(2,2,1), \mathcal{N}_2(2,2,2)$, or $\mathcal{N}_2(2,2,3)$

Questions

1. Do there exist positive integers k, m, and n such that $\mathcal{N}_2(k, m, n) \subset \mathcal{E}_2(k, m, n)$?
2. Is $\mathcal{V}_{[x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}]} \cap \mathcal{R} = \mathcal{V}_{[x_1, x_2, \ldots, x_n]}$?
3. Is $\mathcal{V}_{[x_1, x_2, \ldots, x_n]}$ always a nilpotent variety, regardless of how often and where x_1 and x_2 occur in $2 < i \leq n$?

Question 3 is completely solved for $n = 3$. Indeed,

$$\mathcal{N}_2 = \mathcal{E}_2 = \mathcal{V}_{[x,y,x]} = \mathcal{V}_{[x,x^y]}.$$

For $n = 4$, it suffices to consider the four possible Engel-type forms:

- $[x, y, x, x]$
- $[x, y, x, y]$
- $[x, y, y, x]$, and
- $[x, y, y, y]$.

Jabara and Traustason showed that in the torsion-free case,

$$[x, y, y, x] = e \implies [x, y, y, y] = e.$$

Theorem

$$\mathcal{V}_{[x,y,x,y]} \subseteq \mathcal{V}_{[x,y,x,x]}.$$

Proof.

$$\begin{aligned} e = [x, xy, x, xy] &= [x, xy, x, y][x, xy, x, x]^y \\ &= [x, y, x, y][x, y, x, x]^y \\ &= [x, y, x, x]^y, \end{aligned}$$

implying $[x, y, x, x] = e$. □

Theorem

$$\mathcal{V}_{[x,y,x,x]} = \mathcal{E}_3.$$

Proof.

$$\begin{aligned} e = [y^{-1}, x, y^{-1}, y^{-1}] &= \left[[x,y]^{y^{-1}}, y^{-1}, y^{-1}\right] \\ &= [x, y, y^{-1}, y^{-1}]^{y^{-1}}, \end{aligned}$$

implying $[x, y, y^{-1}, y^{-1}] = e$ and $[x, y, y^{-1}, y] = e$.
But then

$$\begin{aligned} e = [x, y, y^{-1}, y] = \left[\left[y, [x,y]\right]^{y^{-1}}, y\right] &= \left[[x,y,y]^{-y^{-1}}, y\right] \\ &= \left[[x,y,y]^{-1}, y\right]^{y^{-1}} = \left[y, [x,y,y]\right]^{[x,y,y]y^{-1}} \\ &= [x,y,y,y]^{-[x,y,y]y^{-1}}. \end{aligned}$$

Reversing the steps to prove the converse is easy. □

Corollary

$$\mathcal{V}_{[x_1,x_2,x_3,x_4]} \subseteq \mathcal{N}_4.$$

The question is open for $n = 5$ and higher.

Bibliography

- Botto Mura, R. and Rhemtulla, A.: Orderable Groups, Marcel Dekker (1977)
- Darnel, M. R.: Varieties Minimal over Representable Varieties of Lattice-ordered Groups. Comm. Alg. **21**(8), 2637-2665 (1993).
- Darnel, M. R.: Theory of lattice-ordered groups, Marcel Dekker (1995).
- Darnel, M. R. and Holland, W. C.: Solvable covers of the Boolean variety of unital ℓ-groups, Algebra Universalis **62**, 185-199, (2009)
- Darnel, M. R. W. Charles Holland, W. C.: Minimal Non-metabelian varieties of ℓ-groups which contain no nonabelian o-groups, Comm. Alg. **42**, 5100-5133, (2014)
- Darnel, M. R., Holland, W. C., Pajoohesh, H.: Generalized commutativity of lattice-ordered groups, II Algebra Universalis, **75**, 51-59, (2016)

- Glass, A. M. W, Holland, W. C., McCleary, S. H.: The structure of ℓ-group varieties, Algebra Universalis, **10**, 1-20, (1980)
- Heineken, H.: Engelsche elemente der länge drei, Illinois J. Math, **5**, 681-707, (1961)
- Hollister, H. A.: Nilpotent ℓ-groups are representable, Algebra Universalis **8**, 65-71, 1978.
- Hollister, H. A.: The varieties of weakly n-abelian ℓ-groups, (1993, draft generously shared with the author).
- Jabara, E., Traustason, G.: On $(n + \frac{1}{2})$-Engel groups J. Group Theory **23**, 503-511, (2020)
- Kappe, L. C., Kappe, W. P.: On three-Engel groups Bull. Aust. Math. Soc. **7**, 391-405., (1972)
- Kim, Y. K., Rhemtulla, A. H.: Orderable groups satisfying an Engel condition Ordered algebraic structures (Gainesville, FL, 1991 Conrad Conference), 73-79. Kluwer Acad. Publ., Dordrecht, (1993)

- Levi, F. W.: Groups in which the commutator operations satisfy certain algebraic conditions, J. Indian Math. Soc. **6**, 87-97, (1942)
- Martinez, J.: Free Products of Varieties of Lattice-ordered Groups, Czech. Math. J. **22**(97), 535-553, 1972.
- McDonald, I. D.: On certain varieties of groups, Math Z., **76**, 270-282, (1961)
- McDonald, I. D.: On certain varieties of groups, II, Math Z. **78**, 175-178. (1962)
- Medvedev, N. Ya.: On o-approximabiliity of bounded Engel ℓ-groups, Algebra i Logika **27**, 418-421, (1988)
- Reilly, N. R.: Nilpotent, weakly abelian and Hamiltonian lattice-ordered groups, Czech. Math. J. **33**(108), 348-353, (1983)
- Reilly, N. R.: Varieties of lattice-ordered groups, In: Lattice-Ordered Groups, Kluwer Academic Publishers 1989, edited by A. M. W. Glass and W. Charles Holland.

Smith, J. E.: The lattice of ℓ-group varieties, Trans. Amer. Math. Soc. **257**, 347-357. (1980)

Weinberg, E.: Free lattice-ordered abelian groups, II, Math. Annalen **159**, 217-222 (1965)

Zel'manov, E. I.: The Solution of the restricted Burnside problem for groups of prime power exponent. Yale University Press, 1990.