Idempotent residuated chains

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Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
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- for all $a, b, c \in A$,

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If $xy = x \wedge y$ then $\mathbf{A}$ is a *Brouwerian algebra*; we write $x \to y$ for $x\backslash y = y/x$. If, further, there is a bottom element $\perp$, then $\mathbf{A}$ is a Heyting algebra. If, further, $\neg\neg x = x$, where $\neg x := x \to \perp$, then $\mathbf{A}$ is a Boolean algebra.

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Furthermore, residuated lattices are the algebraic semantics of *substructural logics*, including relevance, linear and many-valued logic (hence examples also include MV-algebras, BL-algebras, MTL-algebras, etc). Also, they are related to computer science and mathematical linguistics.

Idempotent residuated lattices

A residuated lattice is called

1. *commutative* if it satisfies $xy = yx$
   (In this case we write $x \to y$ for $x\backslash y = y/x$.)
2. *idempotent* if it satisfies $x^2 = x$
3. *linear* or *a chain*, if it is totally ordered.
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- The structure of idempotent residuated chains: nested sums and ordinal sums
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- Failure of amalgamation
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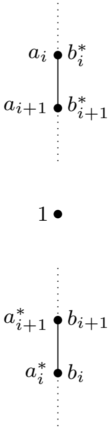
Remark. A residuated lattice is called *conic* if 1 is comparable to every element. Our results extend to the conic idempotent case.

Odd Sugihara monoids

The index set can be any chain, say $\mathbb{Z}$.



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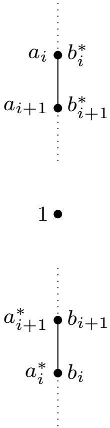

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If there is a tie, then it is the smallest: $b_0 a_0 = a_0 b_0 = b_0$.

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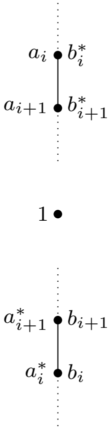

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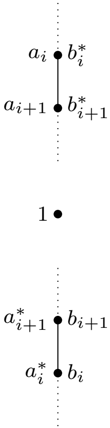

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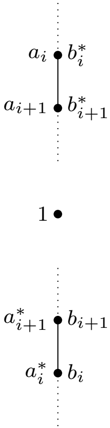

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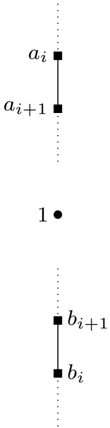
The variety of odd Sugihara monoids is generated by these residuated chains. They are axiomatized as: 1-involutive ($x^{**} = x$), semilinear (generated by chains), idempotent and commutative.

J. Raftery [Ra2007] has studied commutative idempotent residuated structures (partly because of the connections to relevance logic).

Non-commutative chains



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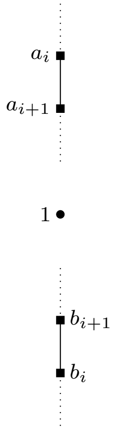

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1. if $i \in J$, then $a_ib_i = a_i$ and $b_ia_i = b_i$ (Left), and
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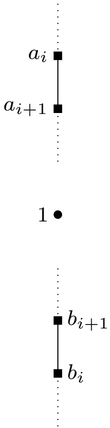

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The operations $x^\ell := 1/x$ and $x^r := x\backslash 1$ (inverses) are defined and their behavior is investigated.

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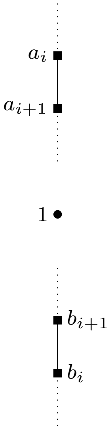

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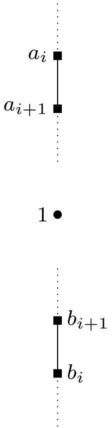

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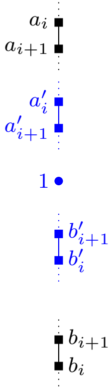
We lose local finiteness and also 1-involutivity ($x^{\ell r} = x = x^{r\ell}$), so $x\backslash y \neq (y^\ell x)^r$, but still divisions are definable

$$x\backslash y = \begin{cases} x^r \vee y & x \leq y \\ x^r \wedge y & y < x \end{cases} \qquad y/x = \begin{cases} x^\ell \vee y & x \leq y \\ x^\ell \wedge y & y < x \end{cases}$$

Nested sums



Nested sums of residuated lattices are defined in [G2002].

Fact. Every odd Sugihara monoid is a nested sum of copies of the three-element Sugihara monoid $\mathbf{S}_3$.

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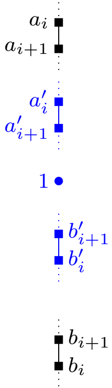

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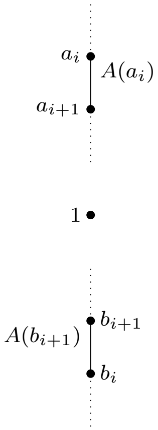
Given two idempotent residuated chains $\mathbf{A}$ and $\mathbf{A}'$, where $a^\ell \neq 1_\mathbf{A}$ and $a^r \neq 1_\mathbf{A}$, for all $a \neq 1$, we define the algebra $\mathbf{A}[\mathbf{A}'] = \mathbf{A} \boxplus \mathbf{A}'$, obtained from $\mathbf{A}$ by replacing $1_\mathbf{A}$ by $\mathbf{A}'$.

Let $\mathbf{A}_i$ for $i \in I$, where I is a chain, be a family of idempotent residuated chains, such that $a^\ell \neq 1_{\mathbf{A}_i}$ and $a^r \neq 1_{\mathbf{A}_i}$, for all $1 \neq a \in \mathbf{A}_i$, for all $i \in I$ except possibly for the top element of I, if it exists. We define $\boxplus_{i \in I} \mathbf{A}_i$ on $\{1\} \cup \bigcup_{i \in I}(A_i \setminus \{1_{\mathbf{A}_i}\})$.

The $\mathbf{A}_i$'s become subalgebras of the nested sum.

Adding dummy elements in Sugihara monoids



We may select an element of an odd Sugihara monoid and add multiple copies of it below it as a chain. So, given an assignment $x \mapsto A(x)$, for $x \in S$, where $A(x)$ is a chain with top element x, we obtain the ordinal sum $\bigoplus_{x \in S} A(x)$.

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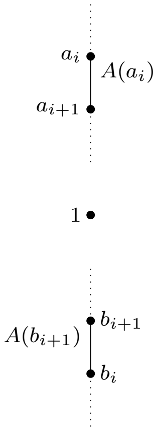

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The associated multiplication yields a residuated lattice $\mathbf{A_S} = \oplus_{x \in S} \mathbf{A}(x)$, that is idempotent, but not 1-involutive. We refer to $\mathbf{S}$ as its *skeleton*.

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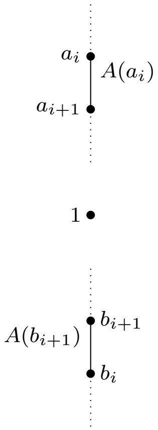

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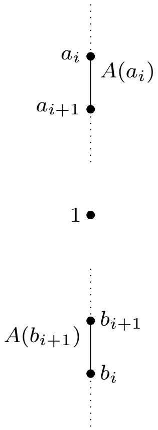

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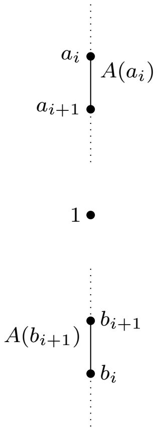

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The image of the nucleus is $A^* = \{a^* : a \in A\} = S$ and it is a subalgebra of $\mathbf{A}$. The blocks are of the form $A(a) = \{x : x^{**} = a\}$, for $a \in A^{**}$.

Nucleus

A *nucleus* on a residuated lattice $\mathbf{A}$ is a closure operator γ on $\mathbf{A}$ satisfying

$$\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y).$$

Then the *nuclear image* of γ is the residuated lattice $\mathbf{A}_\gamma = (\gamma[A], \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, where $x \cdot_\gamma y := \gamma(x \cdot y)$ and $x \vee_\gamma y := \gamma(x \vee y)$.

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Decompisition: non-commutative case

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We define the map γ on A by $\gamma(x) = x^{\ell r} \wedge x^{r\ell}$.

Theorem. If $\mathbf{A}$ is an idempotent residuated chain, then

1. γ is a nucleus.
2. $\gamma[A] = \{x \in A : x = x^{\ell r} \wedge x^{r\ell}\} = A^i$ is the universe of a subalgebra $\mathbf{A}^i$ of $\mathbf{A}$, that further satisfies $x^{\ell r} \wedge x^{r\ell} = x$.
3. The sets $\gamma^{-1}[\{a\}]$, where $a \in A^i$, form subchains of $\mathbf{A}$ with top element a. Also, they are ordered linearly according to the value of a.

A residuated lattice is called *quasi-involutive* if it satisfies $x^{\ell r} \wedge x^{r\ell} = x$.

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Theorem. If $\mathbf{A}$ is an idempotent residuated chain, then

1. γ is a nucleus.
2. $\gamma[A] = \{x \in A : x = x^{\ell r} \wedge x^{r\ell}\} = A^i$ is the universe of a subalgebra $\mathbf{A}^i$ of $\mathbf{A}$, that further satisfies $x^{\ell r} \wedge x^{r\ell} = x$.
3. The sets $\gamma^{-1}[\{a\}]$, where $a \in A^i$, form subchains of $\mathbf{A}$ with top element a. Also, they are ordered linearly according to the value of a.

A residuated lattice is called *quasi-involutive* if it satisfies $x^{\ell r} \wedge x^{r\ell} = x$.

Theorem. We obtain a decomposition (as in the commutative case) as an ordinal sum over a skeleton that is a quasi-involutive ($x^{\ell r} \wedge x^{r\ell} = x$) idempotent residuated chain.

Congruence filters

Given a residuated lattice $\mathbf{A}$, congruences on $\mathbf{A}$ are in bijective correpondence to congruence filters (aka deductive filters):

subsets F such that

1. F is a filter
2. F is submonoid
3. if $x \in F$ and $a \in A$, then $a\backslash xa,\ ax/a \in F$. (closed under conjugation)

The congruence filter associated to a congruence θ is $F_\theta = \uparrow[1]_\theta$.

The congruence associated to a filter F is given by: $x\ \theta_F\ y$ iff $x\backslash y,\ y\backslash x \in F$.

Congruence generation in semilinear

A residuated lattice is called *semilinear* if it is a subdirect product of linear residuated lattices.

Lemma. Every semilinear idempotent residuated lattice satisfies

$$y \wedge y^{\ell\ell} \wedge y^{rr} \leq xy/x \text{ and } y \wedge y^{\ell\ell} \wedge y^{rr} \leq x\backslash yx.$$

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Theorem. The variety of semilinear idempotent residuated lattices has the Congruence Extension Property:
For any algebra $\mathbf{B} \in \mathcal{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if θ is a congruence of $\mathbf{A}$, then there exists a congruence Θ of $\mathbf{B}$ such that $\Theta \cap A^2 = \theta$.

Corollary. The associated substructural logic has a local deduction theorem.

Flow diagrams

Let a be a positive and b a negative element of an idempotent residuated chain $\mathbf{A}$.

$a \; L \; b$ means that $\{a, b\}$ forms a left-zero semigroup.
$a \; R \; b$ means that $\{a, b\}$ forms a right-zero semigroup.

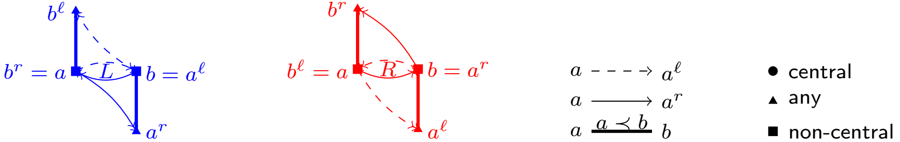
Corollary. Let $\mathbf{A}$ be an idempotent residuated chain.

If a is a positive non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

1. $a^{\ell\ell} \prec a^{\ell r} = a \; L \; a^{\ell} = a^* \succ a^r$.
2. $a^{rr} \prec a^{r\ell} = a \; R \; a^r = a^* \succ a^{\ell}$.

If b is a negative non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

1. $b^{\ell} \prec b^* = b^r \; L \; b = b^{r\ell} \succ b^{rr}$.
2. $b^r \prec b^* = b^{\ell} \; R \; b = b^{\ell r} \succ b^{\ell\ell}$.

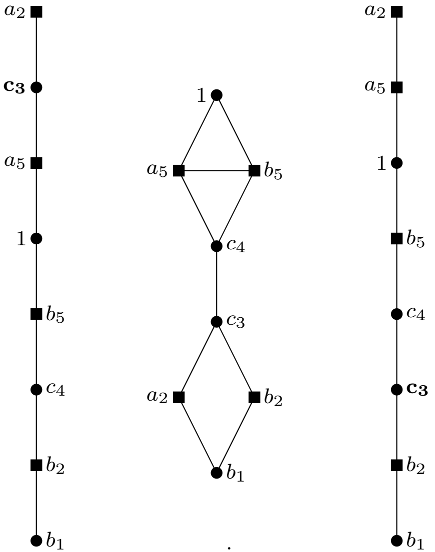


Two algebras with the same monoidal preorder

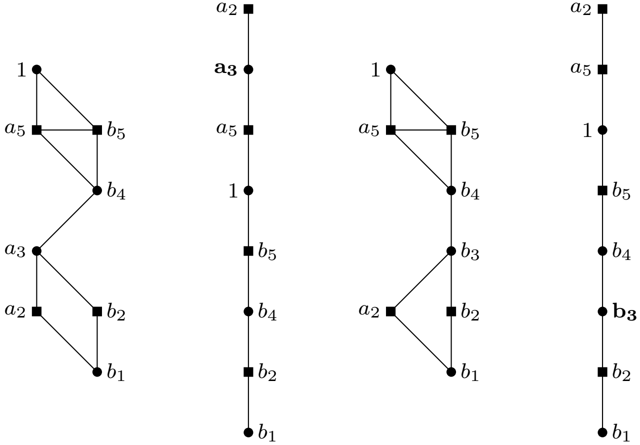
The *monoidal preorder* [GJM2020], is defined by: $x \sqsubseteq y$ iff $xy = x$.

It encodes the multiplication operation, but there are different idempotent residuated structures on the same set that have the same monoidal preorder.

Recall that Hasse diagrams for preordered sets have horizontal line segments connecting mutually comparable elements.



Distinguishing the algebras with the same monoidal order



Enhanced monoidal preorder

Given an idempotent residuated chain $\mathbf{A}$, we define its enhanced monoidal preorder to consist of the following

1. the monoidal preorder $(A, \sqsubseteq)$, as defined earlier,
2. the positive cone A^+ and is the negative cone A^-, and
3. for $a \in A$, $a^\star = a^\ell \wedge a^r$.

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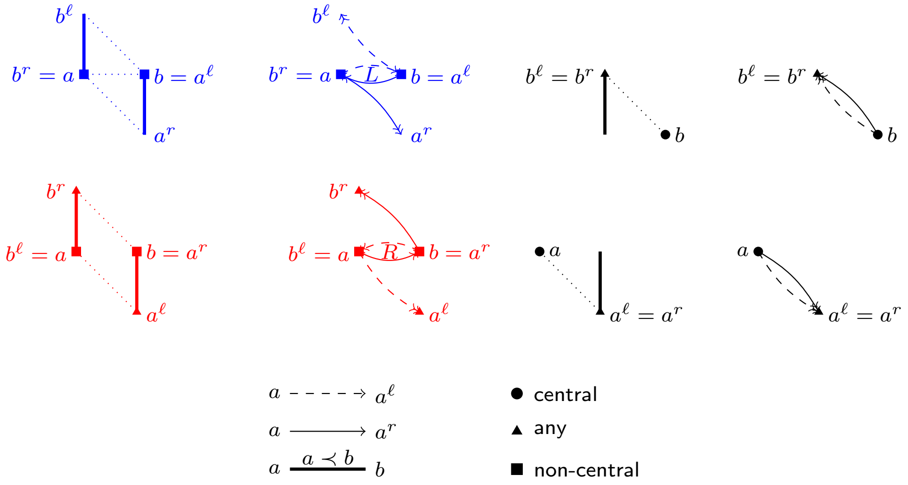
These properties lead to the following definition: $(\mathbf{P}, P^+, P^-, ^\star, 1)$ is an *enhanced monoidal preorder*, if $\mathbf{P} = (P, \sqsubseteq)$ is a pre-ordered set (of width of less than or equal to two), with maximum element 1, P^+ and P^- are totally-ordered subsets such that $P^+ \cup P^- = P$ and $P^+ \cap P^- = \{1\}$, $1^\star = 1$ and

1. for $b \in P^-$, $b^\star$ is the smallest element of P^+ such that $b \sqsubset b^\star$
2. for $a \in P^+$, $a^\star$ is the largest element of P^- such that $a^\star \sqsubset a$.
3. the preordered is *layered*: if two distinct elements are not related by $\sqsubset$ nor $\sqsupset$, then they have different signs and their $\sqsubset$-upsets and downsets coincide.

Since the width is two, in Hasse diagrams of enhanced monoidal preorders we use a left column for positive elements and a right column for negative elements.

Enhanced monoidal-preorders and the flow diagrams



Correspondence

Conversely, given an enhanced monoidal preorder. We define

1. $A = P$,
2. $xy = yx = x$, if $x \sqsubset y$ or $x = y$
3. $xy = x$ and $yx = y$, if $x \sqsubseteq y$ and $y \sqsubseteq x$
4. $xy = y$ and and $yx = x$, if x and y are incomparable
5. $x \leq y$, if $(x, y \in P^-$ and $x \sqsubseteq y)$ or $(x, y \in P^+$ and $y \sqsubseteq x)$
6. x^ℓ and x^r according to the flow diagrams.

Theorem. Idempotent residuated chains are in bijective correspondence to enhanced monoidal preorders.

Proof: We make use of idempotent Galois connections as an intermediate step.

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Theorem. Idempotent residuated chains are in bijective correspondence to enhanced monoidal preorders.

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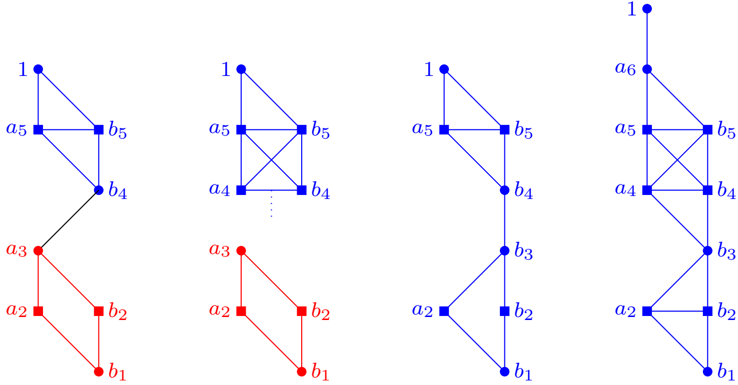
Lemma. Let $\mathbf{A}$ and $\mathbf{B}$ be idempotent residuated chains and let $\mathbf{P_A}$ and $\mathbf{P_B}$ be the corresponding enhanced monoidal preorders. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ iff $\mathbf{P_A}$ is closed under same-level elements, under * and it contains 1.

Subalgebras of enhanced monoidal preorders



Amalgamation

A class $\mathcal{K}$ of similar algebras has the *amalgamation property* if every *V-formation* in $\mathcal{K}$ (an ordered quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f_{\mathbf{B}}, f_{\mathbf{C}})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f_{\mathbf{B}} \colon \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} \colon \mathbf{A} \to \mathbf{C}$ are embeddings) has an *amalgam* $\mathcal{K}$ (an ordered triple $(\mathbf{D}, g_{\mathbf{B}}, g_{\mathbf{C}})$, where $\mathbf{D} \in \mathcal{K}$ and $g_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{D}$ are embeddings such that $g_{\mathbf{B}} \circ f_{\mathbf{B}} = g_{\mathbf{C}} \circ f_{\mathbf{C}}$). For classes closed under isomorphisms, we may assume that $f_{\mathbf{B}}$ and $f_{\mathbf{C}}$ are the inclusion maps (and $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$).

The maps $g_{\mathbf{B}}$ and $g_{\mathbf{C}}$ can also be assumed to inclusions, but only if we identify/rename elements: $b \in \mathbf{B}$ and $c \in \mathbf{C}$ are identified when $g_{\mathbf{B}}(b) = g_{\mathbf{C}}(c)$. This becomes a mute point in the context of the *strong amalgamation property*: $g_{\mathbf{B}}[B] \cap g_{\mathbf{C}}[C] = g_{\mathbf{B}} \circ f_{\mathbf{B}}[A]$.

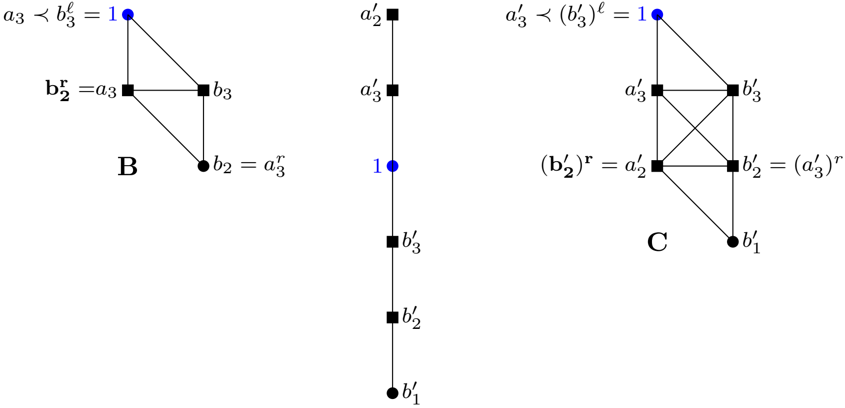
A class of similar algebras $\mathcal{K}$ closed under isomorphisms has the strong amalgamation property iff whenever algebras $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{K}$ intersect at a common subalgebra $\mathbf{A}$, there exists an algebra $\mathbf{D}$ in $\mathcal{K}$ having $\mathbf{B}$ and $\mathbf{C}$ as sublagebras.

Strong amalgamation $\Leftrightarrow$ Amalgamation + Epimorphism Surjectivity (for varieties)

Epimorphism Surjectivity $\Leftrightarrow$ Beth definability. Amalgamation $\Leftrightarrow$ Interpolation.

Failure of amalgamation for idempotent chains

Here $A = \{1\}$, $\mathbf{B}$ is on the left, $\mathbf{C}$ is on the right and they represent quasi-involutive idempotent residuated chains.



Rigidity

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In idempotent residuated chains, the same obstacle in amalgamation appears every time $x^\star$ is in a subalgebra without x also being in the subalgebra.

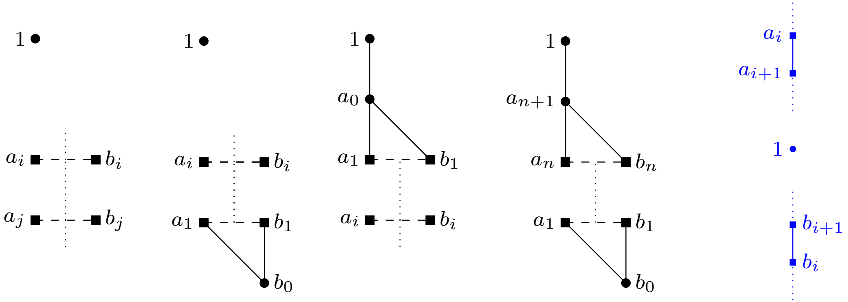
Lemma. In the context of the quasi-involutive skeleton, this is avoided exactly when $x^{\star\star} = x$ for all x, i.e., when it is *-involutive.

Lemma. An idempotent residuated chain has a $\star$-involutive skeleton iff it is *rigid*: it satisfies $x^r = x^{r\star\star}$ and $x^\ell = x^{\ell\star\star}$.

Quasi-involutive + rigid = quasi-involutive + $\star$-involutive = $\star$-involutive

Star-involutive residuated chains

Lemma. The 1-generated *-involutive residuated chains are exactly the *crownian*: their enhaned monoidal preorder is a *vertical crown*. They are exactly the chains in [G2004] (the horizontal lines are optional).



Corollary. The *-involutive residuated chains are nested sums of crownian. Their subalgebras correspond to nested subsums (over a smaller index set).
Proof Nested sums of idempotent residuated chains correspond to nested sums of enhanced monoidal preorders.

Amalgamation for rigid idempotent RL-chains

Theorem. $\star$-involutive idempotent residuated chains have the strong amalgamation property.

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2. the blocks

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Combining results of [Metcalfe-Montagna-Tsinakis] on Amalgamation with results of [Campercholi] on Epimorphism Surjectivity, and using CEP we can prove:

Theorem. Rigid semilinear idempotent residuated lattices have the strong amalgamation property. Hence also epimorphism surjectivity property.

Corollary. The associated substructural logic has deductive interpolation and the Beth definability property.