# On partially ordered algebras and preclones

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BLAST, August 8-12, 2022

Chapman University

# Outline

- Partially ordered algebras
- Clones and preclones
- $S$-preclones
- A Pol-Inv Galois connection
- The lattice of $S$-preclones
- Maximal and minimal Boolean $\pm$preclones

# Partially ordered algebras

**Unisorted universal algebra** has been developed over the last century

**Central concepts:** signature, algebras, homomorphisms, congruences, subalgebras, products, HSP, varieties, free algebras, clones, ...

Every algebra $\mathbf{A} = (A, (f^{\mathbf{A}} \mid f{\in}\mathcal{F}))$ has an underlying **set** $A$ as its universe

A **partially ordered algebra** $\mathbf{A} = (A, \leq^{\mathbf{A}}, (f^{\mathbf{A}} \mid f{\in}\mathcal{F}))$ has an underlying **poset** $A = (A, \leq^{\mathbf{A}})$, as its universe; the **dual poset** $A^\partial$ is $(A, \geq^{\mathbf{A}})$

Each **fundamental operation symbol** $f \in \mathcal{F}$ corresponds to an operation $f^{\mathbf{A}}$ of $\mathbf{A}$ that is order-preserving **or order-reversing** in each argument

This information is part of the **signature** $f^\lambda$ where $\lambda \in \bigcup_{n\geq 1} \{+,-\}^n$

E.g. $f^{(+,-)}$ means $f$ is binary and $f^{\mathbf{A}}{:}A{\times}A^\partial \to A$ is order-preserving

# Partially ordered algebras

In standard universal algebra the base category is the category of **sets**

For **partially ordered algebras (po-algebras)** the base category is **Pos** = the category of posets with order-preserving maps as morphisms

But term-operations on **A** are not necessarily morphisms in **Pos**

For $f^{(+,-)}$, $f^{\mathbf{A}}(x,x)$ may **not** be order-preserving or order-reversing

Varieties of algebras with order-preserving operations have been studied by [Bloom 1976], [Bloom and Wright 1983], [Kurz and Velebil 2017], ...

However for algebraic logic, **negation** and **residuation** are important operations, and they are **not** order-preserving

The study of (nonorder-preserving) po-algebras is due to [Pigozzi 2004]

# Operations with signa from a monoid $S$

The set of term-operations on an algebra form a **clone**

What is a good notion of **clone for partially ordered algebras**?

For a set $A$, let $\mathrm{Op}(A) = \bigcup_{n\geq 1} A^{A^n}$ = the set of all finitary operations on $A$

For a (fixed) **monoid** $(S, \cdot, e)$, let ${}^S\mathrm{Op}(A) = \bigcup_{n\geq 1} A^{A^n} \times S^n$

A pair $(f, \lambda) \in {}^S\mathrm{Op}(A)$ is an $S$-**operation** with **signum** $\lambda$, also written $f^\lambda$

E.g. $S = \{+, -\}$ with $e = +$ and $- \cdot - = +$

$f^{(+,-)}$ is a binary $S$-operation with $+$signum and $-$signum as arguments

# $S$-preclones

A set $F \subseteq {}^S\mathrm{Op}(A)$ is an $S$-**preclone** if

- $F$ contains the **identity** $S$-**operation** $(\mathrm{id}_A, (e))$ and
- $F$ is closed under $\zeta, \tau, \nabla^s, \Delta, \circ$ where $s \in S$ and
  - $\zeta$ **cycles** arguments:
    $(\zeta f^\lambda)(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)$ has signum $(\lambda_n, \lambda_1, \dots, \lambda_{n-1})$
  - $\tau$ **permutes** the first two arguments:
    $(\tau f^\lambda)(x_1, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$ has signum $(\lambda_2, \lambda_1, \lambda_3, \dots, \lambda_n)$
  - $\nabla^s$ **adds a ficticious** argument with signum $s$ at first place:
    $(\nabla^s f^\lambda)(x_1, \dots, x_{n+1}) = f(x_2, \dots, x_{n+1})$ has signum $(s, \lambda_1, \dots, \lambda_n)$
  - $\Delta$ **identifies first two** arguments **if they have equal signa**:
    $(\Delta f^\lambda)(x_1, \dots, x_{n-1}) = f(x_1, x_1, x_2 \dots, x_{n-1})$ has signum $(\lambda_2, \dots, \lambda_n)$ if $\lambda_1 = \lambda_2$, otherwise $\Delta f^\lambda = f^\lambda$
  - $\circ$ **composes** $f^\lambda$ with $g^\mu$ and uses the monoid to get the signum:
    $(f^\lambda \circ g^\mu)(x_1, \dots, x_{m+n-1}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1}$ with **signum** $(\mu_1\lambda_1, \dots, \mu_m\lambda_1, \lambda_2, \dots, \lambda_n)$

# Examples of $S$-preclones $F \subseteq {}^S\mathrm{Op}(A)$

If $S = \{e\}$ then $F$ is an $S$-preclone **iff** $\{f \mid f^\lambda \in F\}$ is a clone

If $S = \{+, -\}$ with $e = +$, $- \cdot - = +$ and if $F$ is all operations that are order-preserving in arguments with $+$signum and order-reversing otherwise, then $F$ is an $S$-preclone, also called a $\pm$**preclone**

${}^S\mathrm{Op}(A)$ is the **largest** $S$-preclone on a set $A$

Let $p_i^n(x_1, \ldots, x_n) = x_i$. Then $\{(p_i^n, \lambda) \mid \lambda_i = e, 1 \leq i \leq n < \omega\}$ is the **smallest** $S$-preclone of **trivial** $S$**-operations** or **projections**

${}^S\langle F\rangle$ is the $S$-preclone **generated by** $F$ (= least one containing $F$)

# Relations, polymorphisms and invariant relations

For $f : A^n \to A$, $\rho \subseteq A^m$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \rho$, consider the column vector

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = (f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{m1}, a_{m2}, \ldots, a_{mn}))^t.$$

$f$ **preserves** $\rho$, written $f \triangleright \rho$, if $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \rho \implies f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \rho$.

In this case $f$ is a **polymorphism** of $\rho$, and $\rho$ is an **invariant relation** of $f$.

For $F \subseteq \mathrm{Op}(A)$ and $R \subseteq \mathrm{Rel}(A) = \bigcup_{m\geq 1} \mathcal{P}(A^m)$, define

$\mathrm{Pol}(R) = \{f \in \mathrm{Op}(A) \mid \forall \rho \in R, f \triangleright \rho\}$ = all **polymorphisms of** $R$ and

$\mathrm{Inv}(F) = \{\rho \in \mathrm{Rel}(A) \mid \forall f \in F, f \triangleright \rho\}$ = all **invariant relations of** $F$

E.g. $\mathrm{Pol}(\{\leq\})$ = all **order-preserving operations** on a poset $(A, \leq)$

# $S$-relations, $S$-polymorphisms and $S$-invariant relations

An $m$-ary $S$-**relation** is of the form $\rho = (\rho_s)_{s\in S}$ where $\rho_s \subseteq A^m$

The set of **all** $S$-**relations** is ${}^S\mathrm{Rel}(A) = \bigcup_{m\geq 1} (P(A^m))^S$

Define $f^\lambda \triangleright \rho$ if $\forall s \in S\ (\mathbf{a}_1 \in \rho_{\lambda_1 s}, \ldots, \mathbf{a}_n \in \rho_{\lambda_n s} \implies f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \rho_s)$

**Abbreviated as:** $\forall s \in S,\ f(\rho_{\lambda_1 s}, \ldots, \rho_{\lambda_n s}) \subseteq \rho_s$

E.g. $S{=}\{+,-\}$: $f^\lambda \triangleright \rho$ if $f(\rho_{\lambda_1}, \ldots, \rho_{\lambda_n}) \subseteq \rho_+$ and $f(\rho_{\lambda_1 -}, \ldots, \rho_{\lambda_n -}) \subseteq \rho_-$

$f^\lambda \triangleright (\leq, \geq)$ **iff** $f$ is order-preserving in $+$args and order-reversing in $-$args

For $F \subseteq {}^S\mathrm{Op}(A)$ and $R \subseteq {}^S\mathrm{Rel}(A)$ define

${}^S\mathrm{Pol}(R) = \{f^\lambda \in {}^S\mathrm{Op}(A) \mid \forall \rho{\in}R, f^\lambda \triangleright \rho\}$ = all $S$-**polymorphisms of** $R$

${}^S\mathrm{Inv}(F) = \{\rho \in {}^S\mathrm{Rel}(A) \mid \forall f^\lambda{\in}F, f^\lambda \triangleright \rho\}$ = all $S$-**invariant relations of** $F$

# Relational clones and the Pol–Inv Galois connection

A set $R \subseteq \mathrm{Rel}(A)$ is a **relational clone** if

- $R$ contains the **identity relation** $\delta_A = \{(x,x) \mid x \in A\}$ and
- $R$ is **closed under** $\zeta, \tau, \mathrm{pr}, \times, \wedge$ where for $\rho, \rho' \in R$

$$\zeta\rho = \{(a_2, a_3, \dots, a_m, a_1) \mid (a_1, a_2, \dots, a_m) \in \rho\}$$ **cyclic shift**

$$\tau\rho = \{(a_2, a_1, a_3, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\}$$ **transposition**

$$\mathrm{pr}\,\rho = \{(a_2, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\}$$ **deletion of 1$^{st}$ row**

$$\rho \times \rho' = \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \mid (a_1, \dots, a_m) \in \rho, (b_1, \dots, b_{m'}) \in \rho'\}$$ **Cartesian product**

$$\rho \wedge \rho' = \rho \cap \rho'$$ **intersection**

$[R]$ is the relational clone **generated by** $R \subseteq \mathrm{Rel}(A)$ (least one $\supseteq R$)

**Theorem (Pol–Inv Galois connection for clones)**

$\mathrm{Pol}(R)$ *is a* **clone** *and* $\mathrm{Inv}(F)$ *is a* **relational clone***.*

*If $A$ is* **finite**, $\langle F \rangle = \mathrm{Pol}(\mathrm{Inv}(F))$ *and* $[R] = \mathrm{Inv}(\mathrm{Pol}(R))$*.*

# $S$-relational clones and the ${}^S\mathrm{Pol}$–${}^S\mathrm{Inv}$ Galois connection

A set $R \subseteq {}^S\mathrm{Rel}(A)$ is a $S$-**relational clone** if

- $R$ contains the $S$-relation $(\delta_A)_{s\in S} = (\delta_A, \dots, \delta_A)$ and
- $R$ is **closed under** $\zeta, \tau, \mathrm{pr}, \times, \wedge, \wedge_{St}, \mu_t$ where for $\rho, \rho' \in R$ and $t \in S$

$\zeta(\rho) = (\zeta\rho_s)_{s\in S}$ **cyclic shift**

$\tau(\rho) = (\tau\rho_s)_{s\in S}$ **transposition**

$\mathrm{pr}(\rho) = (\mathrm{pr}\,\rho_s)_{s\in S}$ **deletion of $1^{\mathrm{st}}$ row**

$\rho \times \rho' = (\rho_s \times \rho'_s)_{s\in S}$ **Cartesian product**

$\rho \wedge_{St} \rho' = (\rho_s \cap \rho'_s \text{ if } s \in St \text{ else } \rho_s)_{s\in S}$ **$St$-intersection,** $St = \{s{\cdot}t \mid s \in S\}$

$\mu_t(\rho) = (\rho_{st})_{s\in S}$ **index translation by $t$**

${}^S[R]$ is the $S$-relational clone **generated by** $R \subseteq {}^S\mathrm{Rel}(A)$ (least one $\supseteq R$)

**Lemma (For $F \subseteq {}^S\mathrm{Op}(A)$ and $R \subseteq {}^S\mathrm{Rel}(A)$)**

${}^S\mathrm{Pol}(R)$ *is an* $S$-**preclone** *and* ${}^S\mathrm{Inv}(F)$ *is an* $S$-**relational clone**.

# Characterizing $S$-preclones and $S$-relational clones

An $m$-**ary diagonal relation** $\delta_\epsilon^m = \{a \in A^m \mid a_i = a_j \text{ for all } (i,j) \in \epsilon\}$ where $\epsilon$ is an equivalence relation on $\{1, \ldots, m\}$

The set of all diagonal relations on $A$ is the smallest relational clone $[\delta_A]$

$(\delta_{\epsilon_s}^m)_{s\in S}$ is an $S$-**diagonal relation** if $\forall s,t \in S,\ Ss \subseteq St \implies \delta_{\epsilon_s}^m \subseteq \delta_{\epsilon_t}^m$

Let ${}^S D_A$ be the set of all $S$-diagonal relations on $A$ (all $m \geq 1$)

**Lemma**

${}^S[(\delta_A)_{s\in S}] = {}^S D_A,\ \ {}^S\mathrm{Pol}(R) = {}^S\mathrm{Pol}({}^S[R])$ *and* ${}^S\mathrm{Inv}(F) = {}^S\mathrm{Inv}({}^S\langle F\rangle)$.

**Theorem**

*Let* $F \subseteq {}^S\mathrm{Op}(A)$. *Then* $\langle\{f \mid f^\lambda \in F\}\rangle = \mathrm{Pol}\{\rho \mid (\rho)_{s\in S} \in {}^S\mathrm{Inv}(F)\}$.

# Characterizing $S$-preclone generation

For $F \subseteq {}^S\mathrm{Op}(A)$ and $\rho \in (\mathcal{P}(A^m))^S$, define

$$\Gamma_F(\rho) = \bigcap \{\sigma \in (\mathcal{P}(A^m))^S \mid \rho \subseteq \sigma \in {}^S\mathrm{Inv}(F)\}$$

where $\bigcup$, $\subseteq$ are applied coordinatewise.

For $|A| = k < \omega$, let $(\varkappa_1, \ldots, \varkappa_n)$ be the $k^n \times n$-matrix with columns $\varkappa_i$ such that the rows are **all** $n$-tuples in $A^n$.

For a signum $\lambda \in S^n$ define $\chi^\lambda = (\{\varkappa_i \mid \lambda_i = s, i \in \{1, \ldots, n\}\})_{s \in S}$

## Theorem

- *For finite $A$, $F \subseteq {}^S\mathrm{Op}(A)$, $\lambda \in S^n$ and $g^\lambda \in {}^S\mathrm{Op}(A)$,*

$$g^\lambda \in {}^S\langle F\rangle \iff g^\lambda \triangleright \Gamma_F(\chi^\lambda).$$

- *For finite $A$ and $F \subseteq {}^S\mathrm{Op}(A)$, ${}^S\langle F\rangle = {}^S\mathrm{Pol}({}^S\mathrm{Inv}(F))$.*

# Characterizing $S$-relational clone generation

So far we have assumed that $(S, \cdot, e)$ is a (finite) **monoid** of signa.

From now on we assume $S$ is a **group** $(S, \cdot, ^{-1}, e)$.

Note that for po-algebras, $S = \{+, -\} \cong \mathbb{Z}_2$ is a group since $- \cdot - = +$

**Theorem**

*For $S$ a group, finite $A$ and $R \subseteq {}^S\mathrm{Rel}(A)$,* $\quad {}^S[R] = {}^S\mathrm{Inv}({}^S\mathrm{Pol}(R))$.

The set ${}^S\mathcal{L}_A$ of all $S$-preclones on $A$ is a **complete lattice** with intersection as meet and $S$-preclone generated by union as join.

**Corollary**

*For a* **group** *$S$ of signa, the lattice ${}^S\mathcal{L}_A$ of $S$-preclones on a* **finite** *set $A$ and the lattice of $S$-relational clones on $A$ are* **dually isomorphic**.

# The lattice of $S$-preclones for a group $S$ and finite $A$

**Theorem**

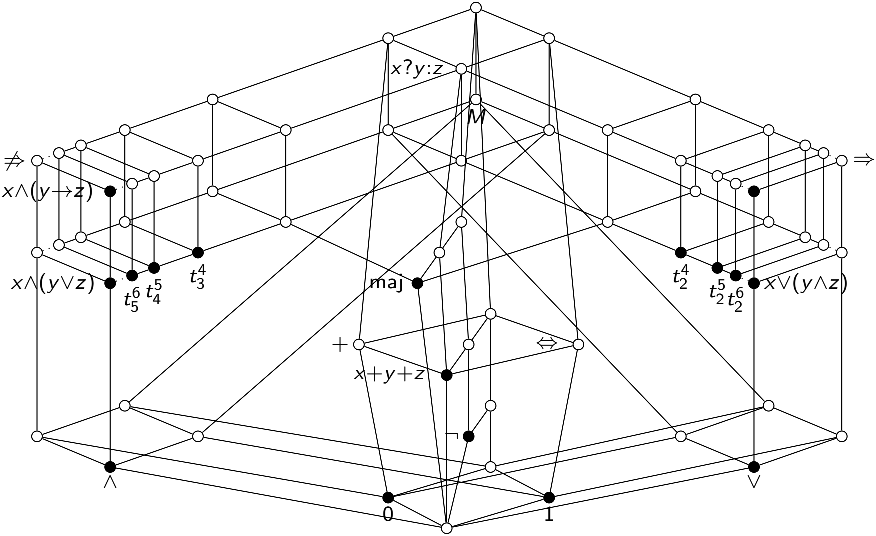
- *The $S$-preclone ${}^S\mathrm{Op}(A)$ is finitely generated.*
- *The $S$-relational clone ${}^S\mathrm{Rel}(A)$ is finitely generated.*
- *The lattice ${}^S\mathcal{L}_A$ is atomic and coatomic.*
- *There are finitely many maximal and finitely many minimal $S$-preclones in ${}^S\mathcal{L}_A$.*

If $A = \{0, 1\}$ then clones on $A$ are called **Boolean clones**

The lattice $\mathcal{L}_{\{0,1\}}$ of Boolean clones is called the **Post lattice** since it was first described by **Emil Post [1941]**.

The Post lattice has 5 maximal clones and 7 minimal clones.

# The Post lattice




$$t_k^n(x_1,\dots,x_n)=\begin{cases}1 & \text{if } |\{i\mid x_i=1\}|\geq k\\ 0 & \text{otherwise.}\end{cases}$$

# Boolean $\{+,-\}$-preclones

For $S = \{+,-\}$ with $- \cdot - = +$, $S$-preclones are also called $\pm$**preclones**

For a poset $(A, \leq)$, ${}^{\pm}\mathrm{Pol}\{(\leq,\geq)\}$ is the $\pm$preclone of all $\pm$operations that are order-preserving in $+$arguments and order-reversing in $-$arguments.

The subpreclones of ${}^{\pm}\mathrm{Pol}\{(\leq,\geq)\}$ abstract from **choosing specific sets of fundamental operations** for varieties of po-algebras.

For $A = \{0, 1\}$ we have found all maximal Boolean $\pm$preclones.

# All nine maximal Boolean ±preclones

## Theorem

*There are* **nine** *maximal Boolean ±preclones listed below. Each such preclone is of the form* $F = {}^{\pm}\mathrm{Pol}\,\rho$ *for some ±-relation* $\rho = (\rho_+, \rho_-)$*:*

(a) ${}^{\pm}\mathrm{Pol}(\sigma, \sigma)$ *with* $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ *where* $\mathrm{Pol}\,\sigma_i$ *is maximal in* $\mathcal{L}_2$ *(0-preserving, 1-preserving, monotone, self-dual, linear operations)*
$\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0,0), (0,1), (1,1)\}$,
$\sigma_3 = \{(0,1), (1,0)\}$, $\sigma_4 = \{(x, y, z, u) \in A^4 \mid x + y + z + u = 0\}$.

(b) ${}^{\pm}\mathrm{Pol}(\leq, \geq) =$ *all functions where each +argument is order-preserving and each −argument is order-reversing.*

(c) ${}^{\pm}\mathrm{Pol}(A, \emptyset) =$ *all functions with positive or mixed signum.*

(d) ${}^{\pm}\mathrm{Pol}(A^2, \delta_A) =$ *all Boolean ±functions, where each negative argument is fictitious (including all negative constants).*

(e) ${}^{\pm}\mathrm{Pol}(\{0\}, \{1\})$.

# There are 23 minimal Boolean ±preclones

## Theorem

*There are twenty three minimal Boolean ±preclones.*

(A) $^{\pm}\langle h_0^\lambda\rangle, {}^{\pm}\langle h_1^\lambda\rangle, {}^{\pm}\langle h_y^\lambda\rangle$ *where* $h_i(x,y,z,u) = \begin{cases} x & \textit{if } x = y \textit{ or } z = u, \\ i & \textit{otherwise,} \end{cases}$
*and* $\lambda = (+,+,-,-)$.

(B) $^{\pm}\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)\rangle, {}^{\pm}\langle x + y + z\rangle$
*where the generators have signum* $\lambda = (+,+,+,-)$.

(C) $^{\pm}\langle x \wedge y\rangle, {}^{\pm}\langle x \vee y\rangle, {}^{\pm}\langle x \vee (y \wedge z)\rangle, {}^{\pm}\langle x \wedge (y \vee z)\rangle,$
$^{\pm}\langle x \vee (y \wedge \neg z)\rangle, {}^{\pm}\langle x \wedge (y \vee \neg z)\rangle, {}^{\pm}\langle (x \wedge \neg z) \vee (y \wedge z)\rangle$
$^{\pm}\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)\rangle, {}^{\pm}\langle (x \wedge y) \vee (y \wedge \neg z) \vee (\neg z \wedge x)\rangle,$
*where the generators have signum* $\lambda = (+,+,-)$

(D) $^{\pm}\langle 0\rangle, {}^{\pm}\langle 1\rangle, {}^{\pm}\langle y\rangle, {}^{\pm}\langle \neg y\rangle, {}^{\pm}\langle \neg x\rangle, {}^{\pm}\langle x \wedge y\rangle, {}^{\pm}\langle x \vee y\rangle, {}^{\pm}\langle x \wedge \neg y\rangle, {}^{\pm}\langle x \vee \neg y\rangle$
*where the generators have signum* $\lambda = (+,-)$

# Some open problems that we hope to solve in the future

Is the lattice of Boolean $\pm$preclones countable?

Classify the maximal $S$-preclones for $|S| > 2$ and $|A| > 2$.

For finite $A$ and a monoid $S$ that is not a group, is ${}^{S}[R] = {}^{S}\mathrm{Inv}({}^{S}\mathrm{Pol}(R))$?

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**THANKS!**