Recent Work on Conjunctive Join-Semilattices

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Outline

1. Background
2. Conjunctivity
3. Jacobson Rings and Subfit Frames
4. Representation Theorem
5. Idempotence
6. Primes & Topology
7. Functoriality
8. Distributivity
9. A Challenge

C Delzell, O Ighedo, & J Madden (2021). *Algebra universalis*, 82(4), 1-30.

Background

A join-semilattice is a set L equipped with an associative, commutative, idempotent operation $\vee$, having an identity element 0. We order L so that $a \leqslant b \iff a \vee b = b$. In this ordering $a \vee b$ is the least upper bound of a and b. (L need not have a top element, but if it does, it is usually denoted by 1.)

Any poset in which any finite subset has a least upper bound is a join-semilattice, with 0 the l.u.b. of the empty set, and $a \vee b$ the l.u.b. of a and b. The order dual of a join-semilattice is a meet-semilattice. (All results about join-semilattices have natural duals for meet-semilattices.)

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An ideal of L is a non-empty downset that is closed under $\vee$.

An element of $\mathrm{JSL}(L, 2)$ is called a character of L. If ϕ is a character of L, then $\phi^{-1}(0)$ is an ideal, and if I is an ideal, then $\phi_I(a) := \begin{cases} 0, & \text{if } a \in I, \\ 1, & \text{if } a \notin I \end{cases}$ is a character. D Papert (1964). *J. London Math. Soc.* 39(1).

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Pontryagin Duality. Note that $2 := \{0, 1\}$ is a discrete topological join-semilattice. It plays the role of the *schizophrenic object* in a duality. $\mathrm{JSL}(L, 2)$ is a closed subset of the Stone space 2^L. If $a \in L$, let $\hat{a} : \mathrm{JSL}(L, 2) \to 2$ be defined by $\hat{a}(\phi) = \phi(a)$. Then $a \mapsto \hat{a} : L \to \mathrm{TJSL}(\mathrm{JSL}(L, 2), 2_d)$ is isomorphism. K Hofmann, M Mislove & A Stralka (1974). Springer LNM 396.

Conjunctivity

Definition. A join-semilattice is said to be conjunctive if it has a top element 1 and for any elements a, b, if $b \nleqslant a$ then there is an element c such that $a \vee c \neq 1$ and $b \vee c = 1$. (The dual condition—disjunctivity—appeared in Wallman, *Ann. of Math.* 39, 1938.)

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If $a \in L$, an ideal of L is a-maximal if it does not contain a and any properly larger ideal does contain a. If I is an ideal of L that does not contain a, then by Zorn's Lemma there is an a-maximal ideal containing I. (If L has 1, "maximal ideal" means 1-maximal ideal.)

Fact. L is conjunctive if and only if: $1 \in L$ and for any $a, b \in L$, if $b \nleq a$ then there is a 1-maximal ideal of L that contains a and does not contain b.

Proof. ($\Rightarrow$) Suppose $b \nleq a$. Pick c so that $a \vee c < 1$ and $b \vee c = 1$. The principal ideal generated by $a \vee c$ contains c and is proper, hence it does not contain b. Therefore, by Zorn's Lemma, there is an ideal $\mathfrak{m}$ that contains $a \vee c$ and is maximal missing b. Any ideal larger than $\mathfrak{m}$ contains b and hence contains 1. Thus, $\mathfrak{m}$ is 1-maximal. ($\Leftarrow$) Suppose $b \nleq a$. By assumption, there is a 1-maximal ideal $\mathfrak{m}$ containing a but not b. Hence $b \vee c = 1$ for some $c \in \mathfrak{m}$, but $a \vee c \in \mathfrak{m}$, so $a \vee c < 1$.

Relationship to Jacobson Rings and Subfit Frames

A commutative ring R with 1 is Jacobson if every prime ideal—hence, every radical ideal—is an intersection of maximal ideals.

A topological space is Jacobson if every closed set is the closure of the set of its closed points. (See the Stacks Project.)

Fact. The following are equivalent:

- R is a Jacobson ring
- $\mathrm{Spec}\, R$ is Jacobson space
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Conjunctive frames are called subfit. Subfitness has several interesting characterizations; see Picado & Pultr, *Separation in Point-Free Topology*. For example, frame F is subfit iff in every proper congruence, the class of 1 has more than one element. The same property characterizes conjunctive join-semilattices.

Problem. Call a ring finitely Jacobson if the radical of every finitely generated ideal is an intersection of maximal ideals. What can be said about such rings?

The Representation Theorem

Let L be a join-semilattice with 1. (We do not assume conjunctivity.)

$\mathrm{Max}\, L :=$ the set of 1-maximal ideals of L in the topology generated by the sets

$\operatorname{coz} a := \{\, \mathfrak{m} \in \mathrm{Max}\, L \mid a \notin \mathfrak{m} \,\}$, $a \in L$.

Theorem. $\mathrm{Max}\, L$ is a compact T_1 space.

Proof. T_1 is clear because maximal ideals are incomparable. For compactness, suppose $B \subseteq L$ and $\{\, \operatorname{coz} b \mid b \in B \,\}$ covers $\mathrm{Max}\, L$. This implies that no maximal ideal contains B, which in turn implies that the ideal generated by B contains 1. Hence, there is a finite $B' \subseteq B$ whose supremum is 1, so $\{\, \operatorname{coz} b \mid b \in B \,\}$ has a finite subcover. Now apply the Alexander Subbase Theorem.

Observation. The map $a \mapsto \operatorname{coz} a : L \to 2^{\mathrm{Max}\, L}$ is injective iff L is conjunctive.

$\operatorname{coz} a = \operatorname{coz} b$ if and only if: $\forall x \in L \quad x \vee a = 1 \iff x \vee b = 1$. (This is the strongest congruence on L in which $\{1\}$ is a class.)

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Aside. For any upset $U \subseteq L$, let $R^U := \{\, (a, b) \mid \forall x \in L \quad x \vee a \in U \iff x \vee b \in U \,\}$. Then R^U is a congruence and L/Θ is conjunctive iff $\Theta = R^U$ for some upset U. Little is known about the relationship between R^U and R^V for different upsets U, V.

Theorem. Suppose X is a compact T_1 space and L is a subbase for the topology of L that is closed under unions. Then (under unions) L is a conjunctive join-semilattice and the map $x \mapsto \mathfrak{m}_x := \{\, a \in L \mid x \notin a \,\}$ is a homeomorphism of X with $\operatorname{Max} L$.

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Example. (The diamond.) Let $X := \{1, 2, 3\}$, and let $L = \{\varnothing, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. The maximal ideals of L are $\{\varnothing, \{1, 2\}\}$, $\{\varnothing, \{1, 3\}\}$, $\{\varnothing, \{2, 3\}\}$. (They are not prime; see next slide.)

Definition. An ideal of L is <u>prime</u> if its complement is a filter.

Proposition. If all elements of $\mathrm{Max}\,L$ are prime, then every finite intersection of cozero sets is a union of cozero sets, i.e., $\{\,\mathrm{coz}\,a \mid a \in L\,\}$ is a base (not merely a subbase) for the topology of $\mathrm{Max}\,L$. Conversely, if $\{\,\mathrm{coz}\,a \mid a \in L\,\}$ is a base for the topology of $\mathrm{Max}\,L$, then all elements of $\mathrm{Max}\,L$ are prime.

Proof. $(\Rightarrow)$ It suffices to show this for binary intersections. Let $\mathfrak{p}$ be a prime ideal of L. If $\mathfrak{p} \in \mathrm{coz}\,a \cap \mathrm{coz}\,b$, then $a, b \in L \backslash \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we may select $c \in L\backslash\mathfrak{p}$ such that $c \leqslant a$ and $c \leqslant b$. Then, $\mathfrak{p} \in \mathrm{coz}\,c \subseteq \mathrm{coz}\,a \cap \mathrm{coz}\,b$. Thus, if all the ideals in $\mathrm{Max}\,L$ are prime, $\mathrm{coz}\,a \cap \mathrm{coz}\,b$ is covered by its cozero subsets.

$(\Leftarrow)$ Suppose $\mathfrak{m} \in \mathrm{Max}\,L$ and $a, b \in L\backslash\mathfrak{m}$. Then $\mathfrak{m} \in \mathrm{coz}\,a \cap \mathrm{coz}\,b$. Because $\{\,\mathrm{coz}\,a \mid a \in L\,\}$ is a base, $\mathfrak{m}$ has a neighborhood $\mathrm{coz}\,c \subseteq \mathrm{coz}\,a \cap \mathrm{coz}\,b$. Then c is a lower bound of a and b that is not in $\mathfrak{m}$.

Functoriality

Recall that if $a \in L$, $\widehat{a} : \mathrm{Max}\, L \to \{0, 1\}$ is defined by: $\widehat{a}(\mathfrak{l}) = 0$ iff $a \in \mathfrak{l}$.
Note that $\mathrm{coz}\, a = \widehat{a}^{\,-1}(1)$.

Given $\psi : L \to M$, define the multifunction $\widehat{\psi} : \mathrm{Max}\, M \to \mathrm{Max}\, L$ by

$$\widehat{\psi}(\mathfrak{m}) := \{\, \mathfrak{l} \in \mathrm{Max}\, L \mid \psi^{-1}(\mathfrak{m}) \subseteq \mathfrak{l} \,\}.$$

Fact. Suppose both L and M are conjunctive, $\psi(1_L) = 1_M$, and $\psi^{-1}(\mathfrak{m})$ is an intersection of maximal ideals of L whenever $\mathfrak{m}$ is a maximal ideal of M. Then for all $a \in L$,

$$\widehat{a} \circ \widehat{\psi} = \widehat{\psi(a)},$$

where $\left(\widehat{a} \circ \widehat{\psi}\right)(\mathfrak{m}) := \bigvee \{\, \widehat{a}(\mathfrak{l}) \mid \psi^{-1}(\mathfrak{m}) \subseteq \mathfrak{l} \,\}$. Moreover, if all the elements of $\mathrm{Max} L$ are prime, then $\widehat{\psi}$ is lower semicontinuous, i.e., $\widehat{\psi}^{-1}(U)$ is open in $\mathrm{Max}\, M$ whenever U is open in $\mathrm{Max}\, L$.

Distributivity

Suppose $u \in L$. We shall examine three "local distributivity" properties at u:

- $\mathcal{D}(u) :\iff \forall a, b \in L : \; u \leqslant a \vee b \implies u = a' \vee b'$ for some $a' \leqslant a$, $b' \leqslant b$.
- $\mathcal{V}(u) :\iff \forall a \in L : \{\, x \in L \mid u \leqslant x \vee a \,\}$ is a filter.
- $\mathcal{P}(u) :\iff$ every u-maximal ideal of L is prime.

Recall that L is said to be distributive if $\mathcal{D}(L)$, i.e., $\mathcal{D}(u)$ holds for all $u \in L$.

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For a conjunctive join-semilattice L, the following are equivalent:

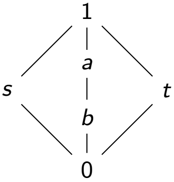
- $\mathcal{V}(1)$;
- the maximal ideals of L are prime;
- $\{\operatorname{coz} a \mid a \in L\,\}$ is a base—not just a subbase—for the topology of $\mathrm{Max}L$.

L may satisfy these conditions and fail to be distributive.

A Challenge

Suppose $s, t \in L$ and $\downarrow s$ and $\downarrow t$ are conjunctive. Is $\downarrow(s \vee t)$ conjunctive?

Answer: Not necessarily!



However, this example is not distributive.

Proposition. Suppose that L is a finite distributive join-semilattice, $s, t \in L$ and $\downarrow s$ and $\downarrow t$ are conjunctive. Then $\downarrow(s \vee t)$ is conjunctive.

Our proof uses the following facts: (1) A finite conjunctive join semilattice is a boolean algebra. (2) If Θ is a non-trivial join-semilattice congruence on $L = \{0, 1\}^X$ in which the the class of every atom is a singleton, then L/Θ is not distributive.

Open Problem. Can we weaken (or even omit) the finiteness hypothesis?