Choice-Free de Vries Duality

Guillaume Massas

UC Berkeley

August 9, 2022

- Stone's representation theorem for Boolean algebras (and therefore also Stone duality) relies on the Boolean Prime Ideal Theorem (**BPI**), a fragment of the Axiom of Choice.

- There exists a constructive, **pointfree** analogue of Stone duality: the Ω-pt adjunction between topological spaces and frames restricts to a duality between Stone spaces and **compact zero-dimensional frames**...

- Stone's representation theorem for Boolean algebras (and therefore also Stone duality) relies on the Boolean Prime Ideal Theorem (**BPI**), a fragment of the Axiom of Choice.

- There exists a constructive, **pointfree** analogue of Stone duality: the Ω-*pt* adjunction between topological spaces and frames restricts to a duality between Stone spaces and **compact zero-dimensional frames**... under BPI!

- Stone's representation theorem for Boolean algebras (and therefore also Stone duality) relies on the Boolean Prime Ideal Theorem (**BPI**), a fragment of the Axiom of Choice.
- There exists a constructive, **pointfree** analogue of Stone duality: the Ω-pt adjunction between topological spaces and frames restricts to a duality between Stone spaces and **compact zero-dimensional frames**... under BPI!
- More recently, Bezhanishvili and Holliday (2020) have developed a fully constructive duality between Boolean algebras and UV**-spaces**.
- Their approach has strong ties with **possibility semantics** for classical and modal logic, and the **Vietoris** hyperspace construction.
- Intuitively, it is a **semi-pointfree** approach, in which classical points are replaced by a partially ordered set of approximations.

- De Vries (1962) generalized Stone duality to a duality between compact Haudorff spaces and **de Vries algebras**, which are complete Boolean algebras endowed with a subordination relation. He also assumed BPI.

- Isbell (1972) showed that the Ω-pt adjunction restricts to a duality between compact Hausdorff spaces and **compact regular frames**...

- De Vries (1962) generalized Stone duality to a duality between compact Haudorff spaces and **de Vries algebras**, which are complete Boolean algebras endowed with a subordination relation. He also assumed BPI.

- Isbell (1972) showed that the Ω-pt adjunction restricts to a duality between compact Hausdorff spaces and **compact regular frames**... under BPI again.

- De Vries (1962) generalized Stone duality to a duality between compact Haudorff spaces and **de Vries algebras**, which are complete Boolean algebras endowed with a subordination relation. He also assumed BPI.
- Isbell (1972) showed that the Ω-pt adjunction restricts to a duality between compact Hausdorff spaces and **compact regular frames**... under BPI again.
- But the choice-free duality of Bezhanishvili and Holliday can be generalized to a duality between de Vries algebras and a category of **de Vries** spaces.
- This work extends the semi-pointfree approach, and is part of a larger research program in **semi-constructive** mathematics ($ZF + DC$).

Algebra	Point-set Topology	Pointfree Topology	Semi-Pointfree Topology
BA	**Stone**	**KZFrm**	**UV**
deV	**KHaus**	**KRFrm**	**dVS**

1. Background
2. Choice-Free Representation of de Vries Algebras
3. *dV* Spaces
4. Compact Regular Frames and the Vietoris Functor
5. Some Applications

Definition

A **compingent algebra** is a pair $(B, \prec)$ such that B is a Boolean algebra with induced order $\leq$, and $\prec$ is a relation on $B \times B$ satisfying the following set of axioms:

(A1) $1 \prec 1$;

(A2) $a \prec b$ implies $a \leq b$;

(A3) $a \leq b \prec c \leq d$ implies $a \prec d$;

(A4) $a \prec b$ and $a \prec c$ together imply $a \prec b \wedge c$;

(A5) $a \prec b$ implies $\neg b \prec \neg a$;

(A6) $a \prec c$ implies that there is $b \in B$ such that $a \prec b \prec c$;

(A7) $a \neq 0$ implies that there is $b \neq 0 \in B$ such that $b \prec a$.

Definition

A **compingent algebra** is a pair $(B, \prec)$ such that B is a Boolean algebra with induced order $\leq$, and $\prec$ is a relation on $B \times B$ satisfying the following set of axioms:

(A1) $1 \prec 1$;

(A2) $a \prec b$ implies $a \leq b$;

(A3) $a \leq b \prec c \leq d$ implies $a \prec d$;

(A4) $a \prec b$ and $a \prec c$ together imply $a \prec b \wedge c$;

(A5) $a \prec b$ implies $\neg b \prec \neg a$;

(A6) $a \prec c$ implies that there is $b \in B$ such that $a \prec b \prec c$;

(A7) $a \neq 0$ implies that there is $b \neq 0 \in B$ such that $b \prec a$.

- A **de Vries algebra** is a compingent algebra $V = (B, \prec)$ such that B is a complete Boolean algebra.
- It is **zero-dimensional** if for any $a \prec b \in V$ there is $c \in V$ such that $a \prec c \prec c \prec b$.

- The **regular open sets** of any compact Hausdorff space form a de Vries algebra, with the subordination relation $\prec$ given by

$$U \prec V \Leftrightarrow \overline{U} \subseteq V.$$

The regular open sets of any **Stone space** form a zero-dimensional de Vries algebra.

- The **regular open sets** of any compact Hausdorff space form a de Vries algebra, with the subordination relation $\prec$ given by

$$U \prec V \Leftrightarrow \overline{U} \subseteq V.$$

  The regular open sets of any **Stone space** form a zero-dimensional de Vries algebra.

Theorem (de Vries 1962)

Every de Vries algebra is isomorphic to the regular open sets of a compact Hausdorff space.

Definition

Let $V = (B, \prec)$ be a de Vries algebra. For any filter F on B, let $\Uparrow F = \{a \in F \mid \exists b \in F : b \prec a\}$. A **concordant filter** on V is a filter F such that $\Uparrow F = F$. An **end** is a maximal concordant filter.

- The dual Stone space of a Boolean algebra B is constructed by endowing the set X_B of all **ultrafilters** on B with the topology generated by the sets $\widehat{a} = \{p \in X_B \mid a \in p\}$ for any $a \in B$.

- De Vries constructed the dual compact Hausdorff space of a de Vries algebra V by taking the set of **ends** of V, which are its **maximal** concordant filters.

- The dual Stone space of a Boolean algebra B is constructed by endowing the set X_B of all **ultrafilters** on B with the topology generated by the sets $\widehat{a} = \{p \in X_B \mid a \in p\}$ for any $a \in B$.

- De Vries constructed the dual compact Hausdorff space of a de Vries algebra V by taking the set of **ends** of V, which are its **maximal** concordant filters.

- By contrast, the dual UV space of B is obtained by taking the set of all **filters** on B endowed with the same Stone-like topology.

- So it is natural to expect our choice-free duals of a de Vries algebra V to be given by the set of all **concordant filters** on V.

- The dual Stone space of a Boolean algebra B is constructed by endowing the set X_B of all **ultrafilters** on B with the topology generated by the sets $\widehat{a} = \{p \in X_B \mid a \in p\}$ for any $a \in B$.

- De Vries constructed the dual compact Hausdorff space of a de Vries algebra V by taking the set of **ends** of V, which are its **maximal** concordant filters.

- By contrast, the dual UV space of B is obtained by taking the set of all **filters** on B endowed with the same Stone-like topology.

- So it is natural to expect our choice-free duals of a de Vries algebra V to be given by the set of all **concordant filters** on V.

- Intuition: "Points" in a semi-pointfree context are coarse **approximations** of classical points.

Definition

Let $V = (B, \prec)$ be a de Vries algebra. The **dual filter space** of V is the topological space (S_V, σ), where:

- S_V is the set of all concordant filters on V;
- σ is the Stone-like topology generated by $\{\widehat{a} = \{F \in S_V \mid a \in F\} \mid a \in V\}$.

- The specialization order $\leq$ induced by σ coincides with the inclusion ordering on S_V. For any $U \subseteq S_V$, let ${\downarrow}U = \{F \in S_V \mid \exists G \supseteq F : G \in U\}$ and define:

$$U \ll V \Leftrightarrow \overline{U} \subseteq {\downarrow}V$$

Definition

Let $V = (B, \prec)$ be a de Vries algebra. The **dual filter space** of V is the topological space (S_V, σ), where:

- S_V is the set of all concordant filters on V;
- σ is the Stone-like topology generated by $\{\widehat{a} = \{F \in S_V \mid a \in F\} \mid a \in V\}$.

- The specialization order $\leq$ induced by σ coincides with the inclusion ordering on S_V. For any $U \subseteq S_V$, let $\downarrow U = \{F \in S_V \mid \exists G \supseteq F : G \in U\}$ and define:

$$U \ll V \Leftrightarrow \overline{U} \subseteq \downarrow V$$

Theorem (M.)

For any de Vries algebra $V = (B, \prec)$, the Stone map $a \mapsto \widehat{a}$ is an isomorphism between V and $\mathrm{RO}(S_V, \ll)$.

1. Background
2. Choice-Free Representation of de Vries Algebras
3. *dV* Spaces
4. Compact Regular Frames and the Vietoris Functor
5. Some Applications

- Given a topological space (X, τ) with specialization order $\leq$, let $\downarrow$ and $\Uparrow$ be the closure and interior operator in the upset topology induced by $\leq$, and let $\mathcal{RO}(X) = \{U \subseteq X \mid U = \Uparrow\downarrow U\}$.

Definition

A topological space (X, τ) is a ***UV*-space** if it satisfies the following conditions:

1. (X, τ) is compact and T_0;
2. $\mathsf{CO}\mathcal{RO}(X)$ is closed under $\cap$ and $-\downarrow$ and forms a basis for τ;
3. Any filter on $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ for some $x \in X$.

- Given a topological space (X, τ) with specialization order $\leq$, let $\downarrow$ and $\Uparrow$ be the closure and interior operator in the upset topology induced by $\leq$, and let $\mathcal{RO}(X) = \{U \subseteq X \mid U = \Uparrow\downarrow U\}$.

Definition

A topological space (X, τ) is a *UV*-**space** if it satisfies the following conditions:

1. (X, τ) is compact and T_0;
2. $\mathsf{CO}\mathcal{RO}(X)$ is closed under $\cap$ and $-\downarrow$ and forms a basis for τ;
3. Any filter on $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ for some $x \in X$.

Theorem (Bezhanishvili and Holliday 2020)

For any UV space (X, τ) the compact open order-regular opens $\mathsf{CO}\mathcal{RO}(X)$ form a Boolean algebra. Conversely, the filter space of any Boolean algebra is a UV space.

Definition

A topological space (X, τ) is **order-normal** if for any closed set A and any regular closed set B such that A is disjoint from $\updownarrow B$, there are disjoint open sets U and V such that $A \subseteq \downarrow U$ and $\updownarrow B \subseteq V$.

- Order-normality is a straightforward generalization of the usual **normality** axiom stating that any two disjoint closed sets can be separated by disjoint open neighborhoods.

Definition

A topological space (X, τ) is **order-normal** if for any closed set A and any regular closed set B such that A is disjoint from $\uparrow B$, there are disjoint open sets U and V such that $A \subseteq {\downarrow} U$ and $\uparrow B \subseteq V$.

- Order-normality is a straightforward generalization of the usual **normality** axiom stating that any two disjoint closed sets can be separated by disjoint open neighborhoods.

Lemma

Let (X, τ) be an order-normal space such that $\mathrm{RO}(X) \subseteq \mathcal{RO}(X)$. For any $U, V \in \mathrm{RO}(X)$, let $U \ll V$ iff $\overline{U} \subseteq {\downarrow} V$. Then $(\mathrm{RO}(X), \ll)$ is a de Vries algebra.

Definition

A **de Vries space** (dV space for short) is a topological space (X, τ) satisfying the following conditions:

1. (X, τ) is T_0, compact and order-normal;
2. $\mathrm{RO}(X)$ is a basis for τ and $\mathrm{RO}(X) \subseteq \mathcal{RO}(X)$;
3. For every $x \in X$, $\mathrm{RO}(x) = \{U \in \mathrm{RO}(X) \mid x \in U\}$ is a concordant filter on $\mathrm{RO}(X)$, and for every filter F on $\mathrm{RO}(X)$, there is $x \in X$ such that $\Uparrow F = \mathrm{RO}(x)$.

Definition

A **de Vries space** (*dV* space for short) is a topological space (X, τ) satisfying the following conditions:

1. (X, τ) is T_0, compact and order-normal;
2. $\mathrm{RO}(X)$ is a basis for τ and $\mathrm{RO}(X) \subseteq \mathcal{RO}(X)$;
3. For every $x \in X$, $\mathrm{RO}(x) = \{U \in \mathrm{RO}(X) \mid x \in U\}$ is a concordant filter on $\mathrm{RO}(X)$, and for every filter F on $\mathrm{RO}(X)$, there is $x \in X$ such that $\upuparrows F = \mathrm{RO}(x)$.

Lemma

For any de Vries algebra V, its dual space (S_V, τ) is a dV space. Moreover, any dV space is homeomorphic to the dual space of its de Vries algebra of regular open sets.

Definition

Let $V_1 = (B_1, \prec_1)$ and $V_2 = (B_2, \prec_2)$ be de Vries algebras. A de Vries morphism from V_1 to V_2 is a function $h : B_1 \to B_2$ satisfying the following set of conditions:

(V1) $h(0) = 0$;

(V2) $h(a \wedge b) = h(a) \wedge h(b)$;

(V3) $a \prec_1 b$ implies $\neg h(\neg a) \prec_2 h(b)$;

(V4) $h(a) = \bigvee\{h(b) \mid b \prec_1 a\}$.

Given two de Vries morphisms $h : V_1 \to V_2$ and $k : V_2 \to V_3$, their composition $k \star h : V_1 \to V_3$ is defined as the map $a \mapsto \bigvee\{kh(b) : b \prec_1 a\}$.

Definition

Let $V_1 = (B_1, \prec_1)$ and $V_2 = (B_2, \prec_2)$ be de Vries algebras. A de Vries morphism from V_1 to V_2 is a function $h : B_1 \to B_2$ satisfying the following set of conditions:

(V1) $h(0) = 0$;

(V2) $h(a \wedge b) = h(a) \wedge h(b)$;

(V3) $a \prec_1 b$ implies $\neg h(\neg a) \prec_2 h(b)$;

(V4) $h(a) = \bigvee \{h(b) \mid b \prec_1 a\}$.

Given two de Vries morphisms $h : V_1 \to V_2$ and $k : V_2 \to V_3$, their composition $k \star h : V_1 \to V_3$ is defined as the map $a \mapsto \bigvee \{kh(b) : b \prec_1 a\}$.

Definition

Let (X, τ_1) and (Y, τ_2) be dV spaces, and let $\leq_1$ and $\leq_2$ be the specialization orders induced by τ_1 and τ_2 respectively. A **de Vries map** (dV map for short) $f : X \to Y$ is a **continuous** function that is also **weakly dense**, i.e., is such that for any $x \in X$, if $f(x) \leq_2 y$ for some $y \in Y$, then there is $x' \geq_1 x$ such that $y \leq_2 f(x')$.

Lemma

- *For any de Vries morphism* $h : V_1 \to V_2$, *the map* $\Lambda(h) : S_{V_2} \to S_{V_1}$ *given by* $F \mapsto {\uparrow} h^{-1}[F]$ *is a* dV *map.*
- *Conversely, given a* dV *map* $f : (X, \tau_1) \to (Y, \tau_2)$, *the map* $U \mapsto (f^{-1}[U])^{\perp\perp}$ *induces a de Vries morphism* $\Phi(f) : \mathrm{RO}(Y) \to \mathrm{RO}(X)$.

- We can therefore define contravariant functors $\Lambda : \mathbf{deV} \to \mathbf{dVS}$ and $\Phi : \mathbf{dVS} \to \mathbf{deV}$.

Lemma

- *For any de Vries morphism* $h : V_1 \to V_2$, *the map* $\Lambda(h) : S_{V_2} \to S_{V_1}$ *given by* $F \mapsto \uparrow h^{-1}[F]$ *is a dV map.*
- *Conversely, given a dV map* $f : (X, \tau_1) \to (Y, \tau_2)$, *the map* $U \mapsto (f^{-1}[U])^{\perp\perp}$ *induces a de Vries morphism* $\Phi(f) : \mathrm{RO}(Y) \to \mathrm{RO}(X)$.

- We can therefore define contravariant functors Λ : **deV** $\to$ **dVS** and Φ : **dVS** $\to$ **deV**.

Theorem (M.)

The functors Φ *and* Λ *establish a dual equivalence between the categories* **deV** *and* **dVS**.

1. Background
2. Choice-Free Representation of de Vries Algebras
3. *dV* Spaces
4. Compact Regular Frames and the Vietoris Functor
5. Some Applications

Definition

- A frame L is **compact** if $1_L = \bigvee B$ for some $B \subseteq L$ implies $1 = \bigvee B'$ for some finite $B' \subseteq B$.
- For any $a, b \in L$, a is **way below** b, $(a \prec b)$, if $b \vee \neg a = 1_L$. A **compact regular frame** is a compact frame L such that for any $a \in L$, $a = \bigvee \{b \in L \mid b \prec a\}$.

Theorem (Isbell 1972)

The Ω-pt adjunction between frames and topological spaces restricts to a duality between compact Hausdorff spaces and compact regular frames.

Definition

- A frame L is **compact** if $1_L = \bigvee B$ for some $B \subseteq L$ implies $1 = \bigvee B'$ for some finite $B' \subseteq B$.
- For any $a, b \in L$, a is **way below** b, $(a \prec b)$, if $b \vee \neg a = 1_L$. A **compact regular frame** is a compact frame L such that for any $a \in L$, $a = \bigvee \{b \in L \mid b \prec a\}$.

Theorem (Isbell 1972)

The Ω-pt adjunction between frames and topological spaces restricts to a duality between compact Hausdorff spaces and compact regular frames.

- Isbell's duality relies on BPI.
- One can show however purely constructively that the category **KRFrm** of compact regular frames is equivalent to **deV**.

Definition

- Let $V = (B, \prec)$ be a de Vries algebra. A set $I \subseteq B$ is a **round ideal** iff $I^\delta = \{\neg a \mid a \in I\}$ is a concordant filter. The frame of round ideals of V is denoted $\mathfrak{R}(V)$.
- Let L be a compact regular frame. An element a of L is **regular** if $a \neg\neg a$. The **Booleanization** of L, denoted $B(L)$, is the subframe $\{a \in L \mid \neg\neg a = a\}$.

Definition

- Let $V = (B, \prec)$ be a de Vries algebra. A set $I \subseteq B$ is a **round ideal** iff $I^{\delta} = \{\neg a \mid a \in I\}$ is a concordant filter. The frame of round ideals of V is denoted $\mathfrak{R}(V)$.
- Let L be a compact regular frame. An element a of L is **regular** if $a \; \neg\neg a$. The **Booleanization** of L, denoted $B(L)$, is the subframe $\{a \in L \mid \neg\neg a = a\}$.

Lemma (Banaschewski, Bezhanishvili)

- *For any de Vries algebra V, $\mathfrak{R}(V)$ is a compact regular frame, and $V \simeq B(\mathfrak{R}(V))$.*
- *For any compact regular frame L, $(B(L), \prec)$ is a de Vries algebra, and $L \simeq \mathfrak{R}(B(L))$.*

Definition

An open set U in a dV space (X, τ) is **well-rounded** if order-normality holds for $-U$, i.e., for any closed set $A \subseteq \downarrow U$, there are disjoint open sets V_1 and V_2 such that $A \subseteq \downarrow V_1$ and $X = V_2 \cup \downarrow U$.

Lemma

The lattice $w\mathcal{ORO}(X)$ of well-rounded order-regular open subsets of a dV space X is isomorphic to $\mathfrak{R}(\mathrm{RO}(X), \ll)$.

- This induces a functor $w\mathcal{ORO} : \mathbf{dVS} \to \mathbf{KRFrm}$.
- What about in the other direction?

Definition

Let L be a compact regular frame. The **upper Vietoris space** of L is the topological space $\Xi(L) = (L^-, \tau_\Box)$, where $\tau_\Box$ is the topology generated by the sets $\Box a = \{b \in L^- \mid a \vee b = 1_L\}$ for any $a \in L$.

Lemma

For any compact regular frame L, $\Xi(L)$ is a dV space, and $(\mathrm{RO}(\Xi(L)), \ll) \simeq (B(L), \prec)$.

Definition

Let L be a compact regular frame. The **upper Vietoris space** of L is the topological space $\Xi(L) = (L^-, \tau_\Box)$, where $\tau_\Box$ is the topology generated by the sets $\Box a = \{b \in L^- \mid a \vee b = 1_L\}$ for any $a \in L$.

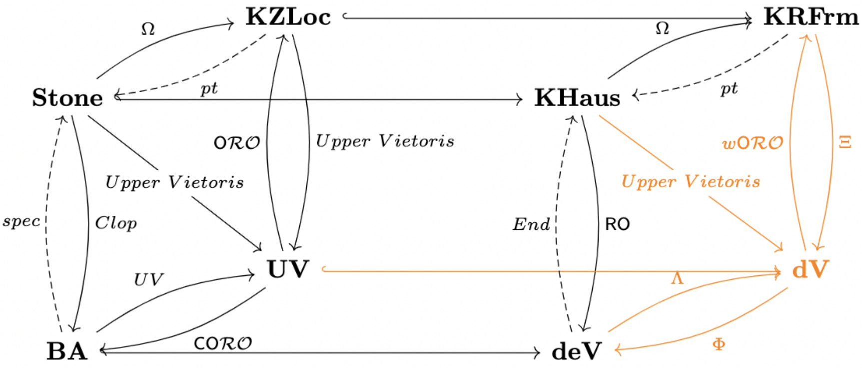
Lemma

For any compact regular frame L, $\Xi(L)$ is a dV space, and $(\mathrm{RO}(\Xi(L)), \ll) \simeq (B(L), \prec)$.

Corollary

The functors $w\mathcal{ORO}$ and Ξ establish an equivalence between **dVS** *and* **KRFrm**.

- Moreover, assuming BPI, the dual *dV* space of a de Vries algebra is homeomorphic to the Upper Vietoris hyperspace of its dual compact Hausdorff space.


KZLoc
KRFrm
Ω
Ω
Stone
pt
KHaus
pt
ORO
Upper Vietoris
wORO
Ξ
Upper Vietoris
Upper Vietoris
spec
Clop
End
RO
UV
UV
dV
Λ
CORO
BA
deV
Φ

1. Background
2. Choice-Free Representation of de Vries Algebras
3. dV Spaces
4. Compact Regular Frames and the Vietoris Functor
5. Some Applications

The Semi-Pointfree Product of Compact Hausdorff Spaces

- It is well known that **Tychonoff's Theorem** relies on the BPI. In fact, Tychonoff's Theorem for compact Hausdorff spaces is equivalent to BPI.

- By contrast, the coproduct of the frames of opens of a family of compact Hausdorff spaces is always compact...

- It is well known that **Tychonoff's Theorem** relies on the BPI. In fact, Tychonoff's Theorem for compact Hausdorff spaces is equivalent to BPI.
- By contrast, the coproduct of the frames of opens of a family of compact Hausdorff spaces is always compact...
- ...but it may fail to be spatial in the absence of BPI.

- It is well known that **Tychonoff's Theorem** relies on the BPI. In fact, Tychonoff's Theorem for compact Hausdorff spaces is equivalent to BPI.
- By contrast, the coproduct of the frames of opens of a family of compact Hausdorff spaces is always compact...
- ...but it may fail to be spatial in the absence of BPI.

Definition

Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of compact Hausdorff spaces. The **semi-pointfree product** of this family is the dV-space $\Xi(\bigoplus_{i\in I} \Omega(X_i))$.

Theorem

The semi-pointfree product of a family of compact Hausdorff spaces $\{(X_i, \tau_i)\}_{i\in I}$ is compact. Moreover, under (BPI), it is homeomorphic to the upper-Vietoris space of $\prod_{i\in I}(X_i, \tau_i)$.

- Bezhanishvili et al. (2019) have used de Vries duality to show that the **Symmetric Strong Implication Calculus** $\mathsf{S}^2\mathsf{IC}$ is sound and complete with respect to the class of compact Hausdorff spaces.

- This calculus is obtained by adding a **strong implication** connective $\rightsquigarrow$, which can be interpreted on any contact algebra $(B, \prec)$ by $a \rightsquigarrow b = 1$ if $a \prec b$ and $a \rightsquigarrow b = 0$ otherwise.

- Bezhanishvili et al. (2019) have used de Vries duality to show that the **Symmetric Strong Implication Calculus** S^2IC is sound and complete with respect to the class of compact Hausdorff spaces.

- This calculus is obtained by adding a **strong implication** connective $\rightsquigarrow$, which can be interpreted on any contact algebra $(B, \prec)$ by $a \rightsquigarrow b = 1$ if $a \prec b$ and $a \rightsquigarrow b = 0$ otherwise.

- We can use dV spaces to provide an alternative, choice-free semantics for S^2IC.

Definition

A **de Vries topological model** is a triple (X, τ, V) such that (X, τ) is a dV-space, and V is a valuation such that for any formulas φ, ψ of $\mathsf{S^2IC}$:

- If φ is propositional letter p, then $V(\varphi) \in \mathrm{RO}(X)$;
- $V(\neg\varphi) = V(\varphi)^{\perp}$ and $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$;
- $V(\varphi \rightsquigarrow \psi) = X$ if $\overline{V(\varphi)} \subseteq {\downarrow}V(\psi)$ and $V(\varphi \rightsquigarrow \psi) = \emptyset$ otherwise.

A formula φ is **valid** on a dV-space (X, τ) iff $V(\varphi) = X$ for any de Vries topological model (X, τ, V).

Theorem (ZF)

The system $\mathsf{S^2IC}$ *is sound and complete with respect to the class of all* dV*-spaces.*

Definition

A **de Vries topological model** is a triple (X, τ, V) such that (X, τ) is a dV-space, and V is a valuation such that for any formulas φ, ψ of $\mathsf{S}^2\mathsf{IC}$:

- If φ is propositional letter p, then $V(\varphi) \in \mathsf{RO}(X)$;
- $V(\neg\varphi) = V(\varphi)^{\perp}$ and $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$;
- $V(\varphi \rightsquigarrow \psi) = X$ if $\overline{V(\varphi)} \subseteq {\downarrow}V(\psi)$ and $V(\varphi \rightsquigarrow \psi) = \emptyset$ otherwise.

A formula φ is **valid** on a dV-space (X, τ) iff $V(\varphi) = X$ for any de Vries topological model (X, τ, V).

Theorem (ZF)

The system $\mathsf{S}^2\mathsf{IC}$ *is sound and complete with respect to the class of all* dV*-spaces.*

- We can think of our choice-free de Vries duality as providing a **possibility semantics** for logics of region-based theories of space, just like choice free Stone duality serves as a categorical foundation for possibility semantics for classical and modal logic.

Thank you!