DUALITIES FROM DOUBLE CATEGORIES OF RELATIONS

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STONE DUALITY

Boolean Algebras — Stone spaces
Distributive lattices — Priestley spaces
Etc
Replacing functions by relations?

STONE DUALITY FOR RELATIONS

Relations as *arrows* (KMJ 2021):

- Rel(DL) — Rel(Priestley)
- Rel(BA) — Rel(Stone)

In this talk: *objects* with relational structure.
Method: A double category

$$dom, cod : \mathbb{C}_1 \to \mathbb{C}_0$$

where *relations are the objects* of the category $\mathbb{C}_1$
(Grandis and Pare, 1999, 2004), (Shulman, 2008)

WEAKENING RELATIONS

Definition: A *relation* $R : A \nrightarrow B$ is a sub-object

$$R \subseteq A \times B$$

and a monotone map

$$A^{op} \times B \to 2$$

where $2 = \{0 < 1\}$.
Remark: Also called a *monotone* or a *weakening* (closed) relation.

A PARADIGMATIC EXAMPLE

A relation in DL (bounded distributive lattices) is similar to a sequent caclulus turnstile:
Taking subobjects in DL amounts to

$$\frac{}{0R0} \qquad \frac{}{1R1} \qquad \frac{aRb \quad a'Rb'}{(a \wedge a')R(b \wedge b')} \qquad \frac{aRb \quad a'Rb'}{(a \vee a')R(b \vee b')}$$

while the monotonicity condition amounts to weakening

$$\frac{a' \leq a \quad aRb \quad b \leq b'}{a'Rb'}$$

DOUBLE CATEGORIES OF RELATIONS

Given a category $\mathcal{C}$ wsp, we can form the double category

$$\S\mathcal{C} = (\S\mathcal{C})_1 \to \mathcal{C}$$

where $(\S\mathcal{C})_1$ has relations as objects and "squares" as arrows $R \to R'$

$$\begin{array}{ccc} A & \overset{R}{\to} & B \\ \downarrow & \Downarrow & \downarrow \\ C & \overset{R'}{\to} & D \end{array} \qquad\qquad a\,R\,b \Rightarrow f(a)\,R'\,g(b)$$

WHY DOUBLE CATEGORIES (1)

(co)limits in Rel($\mathcal{C}$) are not well-behaved …

… but (co)limits in $\S\mathcal{C}$ are

the dual of $\S\mathcal{C} = \mathbb{C}_1 \to \mathbb{C}_0$ we will need is $\mathbb{C}_1^{op} \to \mathbb{C}_0^{op}$ …

… direction of relations not reversed under duality

$Pos^{op} \dashv DL$ lifts to $\S Pos^{op} \dashv \S DL$ …

… but not to Rel(Pos) and Rel(DL)

WHY DOUBLE CATEGORIES (2)

In (KMJ 2021): “vertical truncation” of $\S\mathcal{C} = \mathbb{C}_1 \to \mathbb{C}_0$ ...
... which is a category with **relations as arrows**.

Here: the sub-double-category of $\S\mathcal{C} = \mathbb{C}_1 \to \mathbb{C}_0$ which restricts to the endo-relations in $\mathbb{C}_1$ ...
... which has **relational structures as objects**.

STONE DUALITY FOR RELATIONS (1)

In (KMJ 2021) we give sufficient conditions for an adjunction

$$F \dashv G : \mathcal{A} \to \mathcal{B}$$

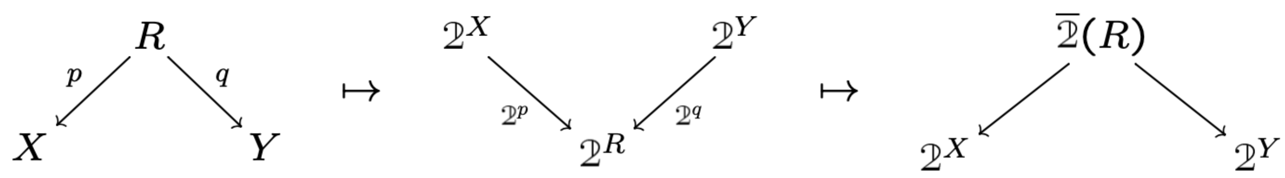
to extend to an adjunction

$$\overline{F} \dashv \overline{G} : \S\mathcal{A} \to \S B$$

Example: Homming into $\mathbb{2}$

Let $\overline{\mathbb{2}}$ be the extension of the contravariant ‘upper set functor’ $\mathbb{2}^{-} : \mathbf{Pos} \to \mathbf{Pos}$

$\overline{\mathbb{2}}$ acts on relations as follows



For all upper sets $a \subseteq X$, $b \subseteq Y$

$$
\begin{aligned}
(a, b) \in \overline{\mathbb{2}}(R) &\iff \mathbb{2}^{p}(a) \subseteq \mathbb{2}^{q}(b) \\
&\iff \forall x \in a \,.\forall y \in Y.\, xRy \Rightarrow y \in b \\
&\iff R[a] \subseteq b
\end{aligned}
$$

STONE DUALITY FOR RELATIONS (2)

Theorem (KMJ 2021): The double category of DL-relations is dually equivalent to the double category of Priestley-relations.
Corollary: The category determined by objects (X, R) where X is a Priestley space and R is a Priestley-relation on X is dually equivalent to the category determined by objects $(A, \prec)$ where A is a bounded distributive lattice and $\prec$ is a DL-relation (aka subordination) on A.

COROLLARIES

Rephrasing the corollary from the previous slide:
Corollary: The category of Priestley spaces with closed weakening relations is dually equivalent to (DL-based) subordination algebras.
We now look at further corollaries that can be obtained by adding conditions on either side of the duality, see **the BLAST 2022 abstract**.

CONCLUSION

If you work with categories of relations: maybe double categories of relations can help.
If you encounter $R[a] \subseteq b$: maybe Stone duality for relations is lurking in the background.
If you are interested in duality for relational structures: maybe looking at relations as “horizontal arrows” of a double category reveals a new point of view.