Towards a Big New Obstacle Theorem: Challenges in Characterizing Dualizability of Finite Algebras in Congruence Modular Varieties

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Details: Dualizability

This presentation is concerned with dualizability in the sense of Clark and Davey in [1]:
Given a finite algebra $\mathbb{A}$ and $\mathbb{D} \in \mathsf{SP}(\mathbb{A})$,

- Let $\underset{\sim}{\mathbb{A}}$ be the relational structure with underlying set $\mathbb{A}$ that is equipped with the discrete topology and each compatible relation of $\mathbb{A}$.
- Let $\mathbb{D}^*$ be the relational structure with underlying set $\mathrm{Hom}(\mathbb{D}, \mathbb{A})$ and relations and topology inherited from $\underset{\sim}{\mathbb{A}}^{\mathbb{D}}$.
- Let $\mathbb{D}^{**}$ be the subalgebra of $\mathbb{A}^{\mathbb{D}^*}$ whose universe is the set of all continuous, relation preserving maps from $\mathbb{D}^*$ into $\underset{\sim}{\mathbb{A}}$.
- For each $d \in \mathbb{D}$, let $\epsilon_d : \mathbb{D}^* \to \mathbb{A} \;:\; h \mapsto h(d)$.

We say $\mathbb{A}$ is dualizable if for each $\mathbb{D} \in \mathsf{SP}(\mathbb{A})$, the map $\epsilon : \mathbb{D} \to \mathbb{D}^{**} \;:\; d \mapsto \epsilon_d$ is a surjection.

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For example, consider the following theorem of Werner, Davey, Heindorf, and McKenzie (($\Leftarrow$) from [3] in 1980 by D & W. ($\Rightarrow$) from [2] in 1995 by D, H, & M):

Theorem (Big NU Obstacle Theorem)

Let $\mathbb{A}$ be a finite algebra and assume $\operatorname{typ}(\mathbb{A}) \subseteq \{3, 4\}$. *Then $\mathbb{A}$ is dualizable if and only if $\mathbb{A}$ has a near unanimity term operation.*

Details: Type omission and congruence-X

The following two theorems show that for a finite algebra $\mathbb{A}$, "$\mathcal{V}(\mathbb{A})$ is congruence-modular" and "$\mathcal{V}(\mathbb{A})$ is congruence-distributive" are strengthenings of "$\operatorname{typ}(\mathbb{A})$ omits 1 and 5" and "$\operatorname{typ}(\mathbb{A})$ omits 1, 2, and 5" respectively (recall that finite algebras generate locally finite varieties).

Theorem ([5] Theorem 8.5.)

A locally finite variety $\mathcal{V}$ is congruence-modular iff $\operatorname{typ}(\mathcal{V}) \cap \{1, 5\} = \emptyset$ *and for all finite* $\mathbb{B} \in \mathcal{V}$, $\alpha \prec \beta$ *in* $\operatorname{Con}\mathbb{B}$, *and* $U \in M_{\mathbb{B}}(\alpha, \beta)$, *the* $\langle \alpha, \beta \rangle$*-tail of* U *is empty.*

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Theorem ([5] Theorem 8.6.)

A locally finite variety $\mathcal{V}$ is congruence-distributive iff $\operatorname{typ}(\mathcal{V}) \cap \{1, 2, 5\} = \emptyset$ and for all finite $\mathbb{B} \in \mathcal{V}$, $\alpha \prec \beta$ in $\operatorname{Con} \mathbb{B}$, and $U \in M_{\mathbb{B}}(\alpha, \beta)$, we have $|U| = 2$.

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For this reason, we prefer to assume type omission over congruence-X wherever possible.

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- "$\mathbb{A}$ is a group" $\Rightarrow$ "$\mathcal{V}(\mathbb{A})$ is congruence-modular" $\Rightarrow$ "$\mathcal{V}(\mathbb{A})$ omits types 1 and 5"

Details: Near-unanimity terms

A term n is called a k-near-unanimity term if it satisfies all of the identities (obtained by reading row-by-row. The given matrix is k by k):

$$n \begin{pmatrix} z & y & y & & y \\ y & z & y & \cdots & y \\ y & y & z & & y \\ & \vdots & & \ddots & \\ y & y & y & \ldots & z \end{pmatrix} \approx \begin{pmatrix} y \\ y \\ y \\ \vdots \\ y \end{pmatrix}$$

Details: Parallelogram terms

A term p is called a k-parallelogram term if it satisfies all of the following identities (obtained by reading row-by-row. The given matrix is k by $3+k$):

$$p\left(\begin{array}{ccc|cccccccc} x & x & y & z & y & \cdots & y & y & \cdots & y & y \\ x & x & y & y & z & \cdots & y & y & \cdots & y & y \\ & \vdots & & \vdots & & \ddots & & & & & \vdots \\ x & x & y & y & y & \cdots & z & y & \cdots & y & y \\ \hline y & x & x & y & y & \cdots & y & z & \cdots & y & y \\ & \vdots & & \vdots & & & & & \ddots & & \vdots \\ y & x & x & y & y & \cdots & y & y & \cdots & z & y \\ y & x & x & y & y & \cdots & y & y & \cdots & y & z \end{array}\right) \approx \begin{pmatrix} y \\ y \\ \vdots \\ y \\ y \\ \vdots \\ y \\ y \end{pmatrix}$$

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Note: It is shown in [7] that if a variety has a k-parallelogram term, then for each nondegenerate choice of division of the matrix above, the variety also has a term satisfying the corresponding identities.

Extending the Big NU Obstacle Theorem

Theorem (Moore, [10])

Let $\mathbb{A}$ *be a finite algebra and suppose* $\mathrm{typ}(\mathbb{A}) \subseteq \{2, 3, 4\}$. *If* $\mathbb{A}$ *is dualizable, then* $\mathbb{A}$ *has a parallelogram term.*

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If $\mathbb{A}$ *is a finite algebra with a parallelogram term and* $\mathbb{A}$ *satisfies the split centralizer condition, then* $\mathbb{A}$ *is dualizable.*

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(Remark: If $\mathbb{A}$ is a finite algebra with a parallelogram term, then $\mathrm{typ}(\mathsf{SP}(\mathbb{A})) \subseteq \{2,3,4\}$. If it also satisfies the split centralizer condition, then $\mathcal{V}(\mathbb{A})$ is residually small).

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Remark: We actually only need to entail "critical" relations.

Details: $\models_d$ and critical relations

See chapter 2, section 4 and chapter 9, section 2 of [1] for a complete view of entailment. For an incomplete view, we can loosely view $\models_d$ as recording constructibility:

Given a set of compatible relations $R \subseteq \mathcal{R}(\mathbb{A})$ and $\rho \in \mathcal{R}(\mathbb{A})$, we say $R \models_d \rho$ if ρ can be obtained, in finitely many steps from $R \cup \{=\}$ using the admissible constructs found in chapter 2 of [1]. From this list of fifteen constructs, 4 (permutation), 6 (intersection), 7 (product), along with retractive projection, are sufficient to define $\models_d$.

A relation $\rho \in \mathcal{R}(\mathbb{A})$ is critical if it cannot be obtained in a nontrivial way from other compatible relations using only constructs 4, 6, and 7.

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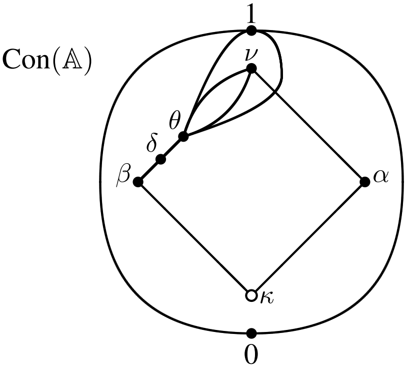
- If this interval is "high", then the above construction cannot be used directly.

The split centralizer condition



$\mathbb{A}/\kappa \in \mathcal{Q}$

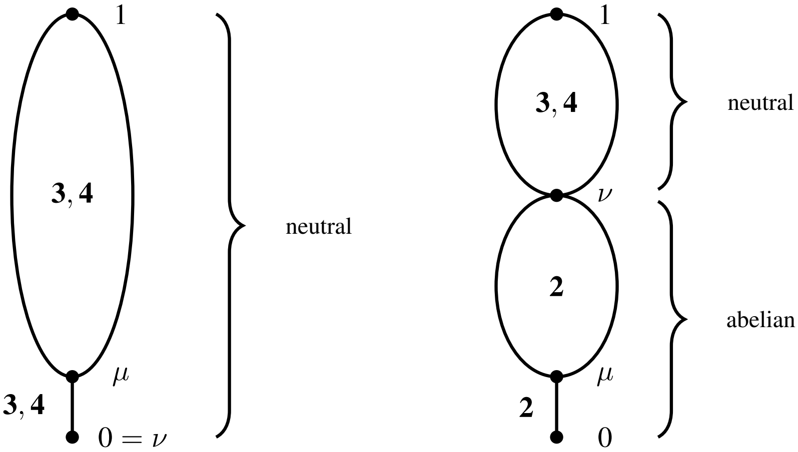
$\beta \leq \delta$

$\alpha \wedge \beta = \kappa$

$\alpha \vee \beta = \nu$

$[\alpha, \alpha] \leq \kappa$

Neutrabelian algebras



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Let $\mathbb{A}$ be a finite algebra in a congruence-modular variety. Then $\mathbb{A}$ is neutrabelian if and only if $\mathbb{A}$ has centralizers split at 0.

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If $\mathbb{A}$ is a finite algebra with a parallelogram term and $\mathbb{A}$ is hereditarily neutrabelian, then $\mathbb{A}$ is dualizable.

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Is there any sort of “Dualizable $\Rightarrow$ Neutrabelian” theorem?

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Paweł Idziak has shown the following result (using different terminology):

Theorem (Idziak [6])

Let $\mathbb{A}$ *be a finite, dualizable, endo-rigid, subdirectly irreducible algebra from a congruence permutable variety. If* μ *is the monolith of* $\mathbb{A}$, $\mathrm{typ}(0, \mu) = 2$, *and* $(0, \mu)$*-minimal sets consist of a single trace, then* $\mathbb{A}$ *is neutrabelian.*

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- Can any of the extra assumptions above be weakened or omitted?
- Can "neutrabelian" be made weaker than "has centralizers split at 0"?

Thank you!

References I

[1] Clark, D. and Davey B. A. (1998), Natural Dualities for the Working Algebraist. *Cambridge Studies in Advanced Mathematics* **57**.

[2] Davey, B. A., Heindorf, L., and McKenzie, R. (1995), Near Unanimity: An Obstacle to General Duality Theory. *Algebra Universalis* **33**, no. 3, 428–439.

[3] Davey, B. A. and Werner, H. (1980), Dualities and Equivalences for Varieties of Algebras. *Colloq. Math. Soc. János Bolyai* **33**, 101–275.

[4] Freese, R. and McKenzie, R. (1987), Commutator Theory for Congruence Modular Varieties. *London Mathematical Society Lecture Notes Series* **125**.

[5] Hobby, D. and McKenzie, R. (1988), The Structure of Finite Algebras. *Contemporary Mathematics* **76**.

[6] Idziak, P. (1994), Congruence Labelling for Dualizable Algebras. Manuscript.

[7] Kearnes, K. A. and Szendrei, Á. (2012), Clones of Algebras with Parallelogram Terms. *Internat. J. Algebra Comput.* **22**, no. 1, 1250005, 30 pp.

[8] Kearnes, K. A. and Szendrei, Á. (2016), Dualizable Algebras with Parallelogram Terms. *Algebra Universalis* **76**, no. 4, 497–539.

[9] Kearnes, K. A., Meredith, C., and Szendrei, Á. (2021), Neutrabelian Algebras. *Algebra Universalis* **82**, no. 13. https://doi.org/10.1007/s00012-020-00705-2

[10] Moore, M. (2016), Naturally Dualizable Algebras Omitting Types 1 and 5 Have a Cube Term. *Algebra Universalis* **75**, 221–230.

References III

[11] Zádori, L. (1995), Natural Duality via a Finite Set of Relations. *Bull. Austral. Math. Soc.* **51**, no. 3, 469–478.