Minimum proper extensions in lattices of subalgebras

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Let $\mathbf{W}^*$ be the class of Archimedean ℓ-groups with strong order unit. The results I'll be presenting today were obtained during our (ongoing) search for atoms in the lattice of hull operators on $\mathbf{W}^*$.

Some well known hull operators on $\mathbf{W}^*$:

- the divisible hull,
- the projectable hull,
- the maximum essential extension,
- the lateral completion,
- the Dedekind-MacNeille completion.

Birkhoff calls an element b in a lattice $(L, \leq)$ *strictly meet-irreducible* (smi) if there is $c > b$ in L such that $c \leq x$ for every $x > b$ in L.

Example

Given algebras $I \leq A$ in a class $\mathcal{A}$ of algebras, let

$$S_{\mathcal{A}}(I, A) = \{G \in \mathcal{A} \mid I \leq G \leq A\}.$$

If $S_{\mathcal{A}}(I, A)$ is closed under intersection, then $(S_{\mathcal{A}}(I, A), \leq)$ is a complete lattice. In that case we call $B < C$ a *minimum proper extension* (mpe) in $S_{\mathcal{A}}(I, A)$ if B is smi in $S_{\mathcal{A}}(I, A)$ with unique cover C. We call B the *base* and C the *co-base* of the mpe.

Question: Given $I \leq A$ in $\mathcal{A}$, what are the mpe's in $S_{\mathcal{A}}(I, A)$?

Theorem

(a) *Suppose* $B < C$ *is an mpe in* $S_{\mathcal{A}}(I, A)$*. If* $x_0 \in C - B$*, then* $C = \langle B, x_0 \rangle$ *and* B *is maximal in* $\{G \in S_{\mathcal{A}}(I, A) \mid x_0 \notin G\}$*.*

(b) *Let* $x_0 \in A - I$*. If* B *is maximal in* $\{G \in S_{\mathcal{A}}(I, A) \mid x_0 \notin G\}$*, then* $B < \langle B, x_0 \rangle$ *is an mpe in* $S_{\mathcal{A}}(I, A)$*.*

Note: If Zorn's lemma applies, then part (b) may be used to obtain mpe's in $S_{\mathcal{A}}(I, A)$.

Let $\mathcal{G}$ be the class of groups. Our study of hull operators on $\mathbf{W}^*$ led us to the question: What are the mpe's in $S_{\mathcal{G}}(\{0\}, \mathbb{R})$?

Note: If A is Abelian and $B < C$ is an mpe in $S_{\mathcal{G}}(I, A)$, then C/B is a simple Abelian group, and so C/B is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

First: What are the mpe's in $S_{\mathcal{G}}(\{0\}, \mathbb{Q})$?
Let p be a prime. For each $n \geq 0$ in $\mathbb{Z}$, let

$$T_{p,n} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \text{if } \gcd(a,b) = 1, \text{ then } p^{n+1} \nmid b \right\}.$$

For each $n < 0$ in $\mathbb{Z}$, let

$$T_{p,n} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \text{if } \gcd(a,b) = 1, \text{ then } p^{-n} \mid a \right\}.$$

Invariants for the subgroups of $(\mathbb{Q}, +)$, due to Beaumont and Zuckerman, make the following result fairly straightforward.

Theorem

For each prime p and each $n \in \mathbb{Z}$, $T_{p,n} < T_{p,n+1}$ is an mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{Q})$. Moreover, every mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{Q})$ is of that form for some p, n.

Let $T < U$ be any mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{Q})$ and choose a vector space basis H for $\mathbb{R}$ over $\mathbb{Q}$.

Example

- Fix $x_0 \in H$.
- Let D be the $\mathbb{Q}$-linear span of $H \setminus \{x_0\}$.
- Then $B < C$ is an mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{R})$, where $B = D \oplus Tx_0$ and $C = D \oplus Ux_0$. (WLOG: $T = T_{p,0}$ and $U = T_{p,1}$.)

Example

- This time fix $x_0 \neq x_1$ in H.
- Let D be the $\mathbb{Q}$-linear span of $H \setminus \{x_0, x_1\}$.
- Let F be the $\mathbb{Q}$-linear span of $\{x_0, x_1\}$, and $K = Tx_0 \oplus Tx_1$.
- If $R < R'$ is an mpe in $S_{\mathcal{G}}(K, F)$, then $B < C$ is an mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{R})$, where $B = D \oplus R$ and $C = D \oplus R'$.

Theorem

$B < C$ is an mpe in $S_{\mathcal{G}}(\{0\}, \mathbb{R})$ if and only if there are a vector space basis H for $\mathbb{R}$ over $\mathbb{Q}$, subsets $H_d, H_r \subseteq H$, and a prime p such that:

(i) *$H \subseteq B$ and $\emptyset \neq H_r = H \setminus H_d$.*

(ii) *There is an mpe $R < R'$ in $S_{\mathcal{G}}(K, F)$ such that*
- *R is reduced,*
- *$B = D + R$, and*
- *$C = D + R'$,*

where D is the $\mathbb{Q}$-linear span of H_d, F is the $\mathbb{Q}$-linear span of H_r, and $K = \bigoplus_{x \in H_r} T_{p,0}x$.

Note: The first example on the previous slide, where $H_r = \{x_0\}$, shows that the case $|H_r| = 1$ does occur.
Question: What are the possible values of $|H_r|$? In particular, does the other extreme case $H_r = H$, $H_d = \emptyset$ occur?

Theorem

Suppose H is a vector space basis for $\mathbb{R}$ over $\mathbb{Q}$. For every $\emptyset \neq H_r \subseteq H$ and every prime p, there is an mpe $R < R'$ in $S_{\mathcal{G}}(K, F)$ with R reduced, where $K = \bigoplus_{x \in H_r} T_{p,0}x$ and F is the $\mathbb{Q}$-linear span of H_r.

Sketch of proof:

- Pick $x_0 \in H_r$. We build a reduced R such that $R < \langle R, \frac{x_0}{p} \rangle$ is an mpe in $S_{\mathcal{G}}(K, F)$.
- Let $\eta \colon F \to F/K$ be the canonical epimorphism and dN be the divisible hull of the subgroup generated by $\eta(\frac{x_0}{p})$ in F/K.
- M is maximal in $\{G \in S_{\mathcal{G}}(K, F) \mid \frac{x_0}{p} \notin G\}$ if and only if $F/K = dN \oplus \eta(M)$.
- If $\phi \in \mathrm{End}(F/K)$, then $F/K = dN \oplus \theta_1(F/K)$, where $\theta_1 = \theta - \pi\phi\theta$ and θ, π are the natural projections of F/K onto $\eta(M)$, dN, respectively. Now just (!) choose ϕ carefully.

Reference:

Hager, A.W., Wynne, B.: Minimum proper extensions in some lattices of subalgebras. Algebra Universalis **83**, no. 3, (2022)