Varieties, Quasivarieties, and Maltsev Products

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Languages and Terms

Universal Algebra first-order language with no relation symbols

Term operation symbols applied to variables

Examples $f(x_1, g(x_3))$, $x \cdot (y \cdot z)$

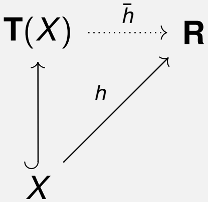
Let X be a set of variables.
$T(X)$ denotes the set of terms in the variables from X

$T(X)$ can be turned into an algebra in a strictly formal way

Example: binary operation symbol f applied to terms x_1 and $g(x_3)$ yields the term $f(x_1, g(x_3))$.

Write $\mathbf{T}(X)$ for the resulting term algebra.

$\mathbf{T}(X)$ is *absolutely free:*
For every algebra $\mathbf{R}$, and every function $h\colon X \to R$, there is a unique homomorphism $\bar{h}\colon \mathbf{T}(X) \to \mathbf{R}$ such that $\bar{h}{\restriction}_X = h$.



Identities

Identities also known as *equations* are the atomic formula in this language,
i.e., term $\approx$ term

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

$$x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)$$

Equational Class

Let Σ be a set of equations.

$$\text{Mod}(\Sigma) = \{\, \mathbf{R} : \mathbf{R} \models \Sigma \,\}$$

$\text{Mod}(\Sigma)$ is an *equational class*

Examples

The class of all semilattices

$$x \cdot x \approx x$$
$$x \cdot y \approx y \cdot x$$
$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

The class of all groups

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$
$$x \cdot e \approx x$$
$$e \cdot x \approx x$$
$$x \cdot (x^{-1}) \approx e$$
$$(x^{-1}) \cdot x \approx e$$

$\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$

Easy to verify: every equational class is closed under the formation of

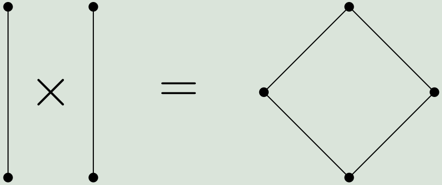
- Homomorphic Images (thus isomorphic images)
- Subalgebras
- Arbitrary Products (thus contains the trivial algebra)

A class of algebras closed under $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, is called a *variety*

NonExamples

Example

The class of chains (viewed as lattices) is closed under $\mathbb{S}$ and $\mathbb{H}$ but not $\mathbb{P}$



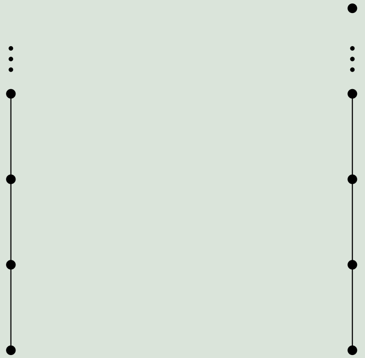
Example

The class of torsion-free abelian groups is closed under $\mathbb{S}$ and $\mathbb{P}$ but not $\mathbb{H}$

$$\mathbf{Z} \twoheadrightarrow \mathbf{Z}/6Z$$

Example

The class of lattices with an upper bound is closed under $\mathbb{H}$ and $\mathbb{P}$ but not $\mathbb{S}$



$$\omega \hookrightarrow \omega + 1$$

Birkhoff's Theorem

So every equational class is a variety. The converse is true as well.

Theorem (Birkhoff, 1935)

Every variety is an equational class.

Unfortunately, given a variety $\mathcal{A}$, it is not at all obvious how to find a set Σ of equations such that $\mathcal{A} = \mathrm{Mod}(\Sigma)$. Such a Σ is called an *equational base* for $\mathcal{A}$.

Typical questions

- Can Σ be finite?
- Can Σ be decidable?

How Can One Be "Given" a Variety?

The class of varieties (in a fixed language) is closed under arbitrary intersection

Thus it forms a *complete lattice*

Smallest variety = $\mathrm{Mod}(x \approx y)$ = {trivial algebras}
Largest variety = $\mathrm{Mod}(\varnothing)$ = {all algebras}

let $\mathcal{K}$ be a bunch of algebras.
$\mathbb{V}(\mathcal{K}) = \bigcap \{ \mathcal{A} : \mathcal{A}$ a variety and $\mathcal{K} \subseteq \mathcal{A} \}$,
the variety generated by $\mathcal{K}$

Theorem (Birkhoff, 1935)

$\mathbb{V}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$

$\mathbb{V}(\mathcal{K})$ Examples

Let **2** denote the two-element semilattice
$\mathcal{S}l$, the variety of semilattices

$\cdot$	0	1
0	0	0
1	0	1

Since every semilattice can be embedded in a power of **2**,
$\mathcal{S}l = \mathbb{V}(\mathbf{2}) = \mathbb{V}(\mathbf{S})$ for any nontrivial semilattice **S**

Proof

Let **Z** denote the (additive) group of integers and $\mathcal{AG}$ the variety of abelian groups.

Then
$\mathcal{AG} = \mathbb{V}(\mathbf{Z}) = \mathbb{V}(\{\, \mathbf{Z}/nZ : n = 2, 3, 4, \ldots \,\})$

Proof.

1. $\mathbf{Z} \leq \prod_{n>1} \mathbf{Z}/nZ \implies \mathbb{V}(\mathbf{Z}) \subseteq \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\})$
2. An algebra lies in a universal class if and only if every finitely generated subalgebra lies in the class
3. A finitely generated abelian group is a direct product of cyclic groups
4. $\therefore \mathcal{AG} = \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\})$
5. $\mathbf{Z}/nZ \in \mathbb{H}(\mathbf{Z}) \implies \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\}) \subseteq \mathbb{V}(\mathbf{Z})$

Quasiidentities

A *quasiidentity* is a universal Horn formula:

$$f_0(\mathbf{x}) \approx g_0(\mathbf{x}) \ \& \ f_1(\mathbf{x}) \approx g_1(\mathbf{x}) \ \& \ \cdots \ \& \ f_{k-1}(\mathbf{x}) \approx g_{k-1}(\mathbf{x}) \to u(\mathbf{x}) \approx v(\mathbf{x})$$

(here $\mathbf{x}$ is shorthand for $x_0, \ldots, x_{m-1}$)

If $k = 0$ we obtain the identity $u(\mathbf{x}) \approx v(\mathbf{x})$

Examples:

$x \cdot y \approx x \cdot z \to y \approx z$

$zx \approx x \ \& \ zy \approx y \ \& \ xz \approx yz \to xy \approx yx$

Quasivarieties

Let Σ be a set of quasiidentities. The class $\mathrm{Mod}(\Sigma)$ can be called a quasiequational class.

What preservation properties does $\mathrm{Mod}(\Sigma)$ have?

Answer: closure under subalgebras, products and ultraproducts (and isomorphisms)

Definition

A *quasivariety* is a class of algebras closed under $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_{\mathrm{U}}$.

Equivalently, a quasivariety is a class closed under subalgebras and reduced products

Theorem (Maltsev, 1954?)

A class of algebras is a quasivariety if and only if it is of the form $\mathrm{Mod}(\Sigma)$ *for some set* Σ *of quasiidentities.*

Examples of Quasivarieties

- Every variety is a quasivariety
- The class of cancellation semigroups $\{x(yz) \approx (xy)z, xy \approx xz \to y \approx z,\ yx \approx zx \to y \approx z\}$
- The class of torsion-free abelian groups

Torsion-free abelian groups

$$2x \approx 0 \to x \approx 0$$
$$3x \approx 0 \to x \approx 0$$
$$4x \approx 0 \to x \approx 0$$
$$\vdots$$

$\mathbf{Z}$ is torsion-free, $\mathbf{Z}/6Z$ is not
So the class of torsion-free abelian groups is not closed under $\mathbb{H}$

Exercise: Show that this class is not finitely axiomatizable

The Lattice of Quasivarieties

The class of quasivarieties is closed under arbitrary intersection. Thus it forms a complete lattice.

Smallest = Mod($x \approx y$) = {trivial algebras}
Largest = Mod($\varnothing$) = {all algebras}

$\mathbb{Q}(\mathcal{K}) = \bigcap \{ \mathcal{A} : \mathcal{A}$ a *quasi*variety and $\mathcal{K} \subseteq \mathcal{A} \}$
the quasivariety generated by $\mathcal{K}$

Theorem (Maltsev)

$\mathbb{Q}(\mathcal{K}) = \mathbb{SPP}_U(\mathcal{K}) = \mathbb{SP}_R(\mathcal{K})$

Important observations

- $\mathbb{Q}(\mathcal{K}) \subseteq \mathbb{V}(\mathcal{K})$ for any class $\mathcal{K}$
- If $\mathcal{A}$ is a quasivariety then $\mathbb{V}(\mathcal{A}) = \mathbb{H}(\mathcal{A})$
- If $\mathcal{K}$ is a finite set of finite algebras then $\mathbb{P}_{\mathsf{U}}(\mathcal{K}) = \mathcal{K}$ so $\mathbb{Q}(\mathcal{K}) = \mathbb{SP}(\mathcal{K})$
- The lattice of varieties is ***NOT*** a sublattice of the lattice of quasivarieties.
  Meets are the same, joins are different

$\mathcal{A} \vee_{\mathsf{Q}} \mathcal{B} = \mathbb{Q}(\mathcal{A} \cup \mathcal{B})$ join as quasivarieties
$\mathcal{A} \vee_{\mathsf{V}} \mathcal{B} = \mathbb{V}(\mathcal{A} \cup \mathcal{B})$ join as varieties

Examples

Example

Let $\mathbf{Z}$ denote the group of integers. Then $\mathbb{Q}(\mathbf{Z}) = \{\text{all torsion-free abelian groups}\}$

Proof.

Let $\mathcal{T}$ denote the class of torsion-free abelian groups. We have already observed that $\mathcal{T}$ forms a quasivariety (and is not a variety).

Clearly, $\mathbf{Z} \in \mathcal{T}$, hence $\mathbb{Q}(\mathbf{Z}) \subseteq \mathcal{T}$.

Conversely, let $\mathbf{G} \in \mathcal{T}$. As before, it suffices to assume $\mathbf{G}$ is finitely generated.
But every finitely generated abelian group is a product of cyclic groups. $\mathbf{G}$ torsion-free implies it is a product of copies of $\mathbf{Z}$. Hence $\mathbf{G} \in \mathbb{P}(\mathbf{Z}) \subseteq \mathbb{Q}(\mathbf{Z})$.

Let **Z** denote the (additive) group of integers and $\mathcal{AG}$ the variety of abelian groups.

Then
$\mathcal{AG} = \mathbb{V}(\mathbf{Z}) = \mathbb{V}(\{\, \mathbf{Z}/nZ : n = 2, 3, 4, \ldots \,\})$

Proof.

1. $\mathbf{Z} \leq \prod_{n>1} \mathbf{Z}/nZ \implies \mathbb{V}(\mathbf{Z}) \subseteq \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\})$
2. An algebra lies in a universal class if and only if every finitely generated subalgebra lies in the class
3. A finitely generated abelian group is a direct product of cyclic groups
4. $\therefore \mathcal{AG} = \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\})$
5. $\mathbf{Z}/nZ \in \mathbb{H}(\mathbf{Z}) \implies \mathbb{V}(\{\, \mathbf{Z}/nZ : n > 1 \,\}) \subseteq \mathbb{V}(\mathbf{Z})$

Example

Let $\mathcal{M}$ denote the variety of all monadic (i.e., S5) algebras. $\mathbf{M}_n$ denotes the simple, monadic algebra of cardinality 2^n. It is known that the subvarieties of $\mathcal{M}$ forms a chain

$$\mathbb{V}(\mathbf{M}_0) \subset \mathbb{V}(\mathbf{M}_1) \subset \mathbb{V}(\mathbf{M}_2) \subset \cdots \mathcal{M}.$$

Claim: $\mathbf{M}_2 \notin \mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1)$. Thus

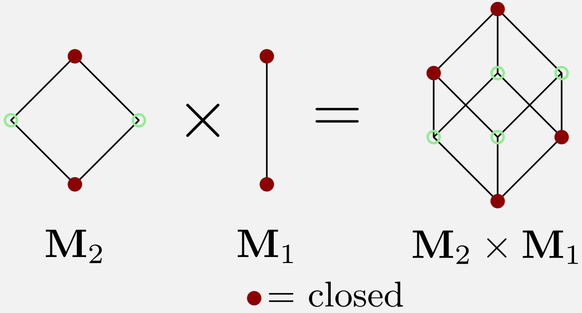
$$\mathbb{V}(\mathbf{M}_1) \subset \mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1) \subset \mathbb{V}(\mathbf{M}_2)$$

Proof.

Suppose $\mathbf{M}_2 \in \mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1) = \mathbb{SP}(\mathbf{M}_2 \times \mathbf{M}_1)$.
By simplicity

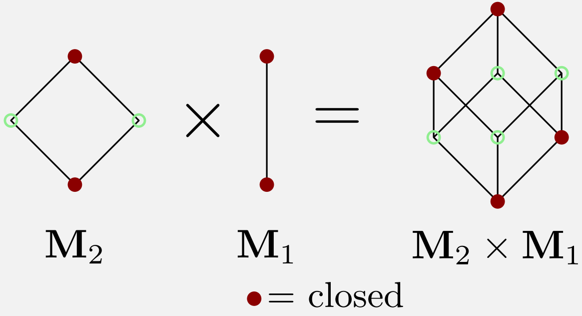
$$\mathbf{M}_2 \rightarrowtail \mathbf{M}_2 \times \mathbf{M}_1 \twoheadrightarrow \mathbf{M}_1$$

But (again, by simplicity) there is no map from $\mathbf{M}_2$ to $\mathbf{M}_1$.



$\mathbb{V}(\mathbf{M}_1) \subset \mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1) \subset \mathbb{V}(\mathbf{M}_2)$

There must be an identity separating $\mathbb{V}(\mathbf{M}_1)$ from $\mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1)$. Easy: $C(x) \approx x$



$\mathbb{V}(\mathbf{M}_1) \subset \mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1) \subset \mathbb{V}(\mathbf{M}_2)$

There must be a quasiidentity separating $\mathbb{Q}(\mathbf{M}_2 \times \mathbf{M}_1)$ from $\mathbb{V}(\mathbf{M}_2)$. $C(x) \approx C(x') \to x \approx 0$

Verbal Congruences

Let $\mathcal{A}$ be a quasivariety and $\mathbf{R}$ an algebra

$$\mathrm{Con}_{\mathcal{A}}(\mathbf{R}) = \{\, \theta \in \mathrm{Con}(\mathbf{R}) : \mathbf{R}/\theta \in \mathcal{A} \,\}$$

$\mathcal{A}$ a quasivariety $\implies$
$\mathrm{Con}_{\mathcal{A}}(\mathbf{R})$ closed under intersection $\implies$
$\mathrm{Con}_{\mathcal{A}}(\mathbf{R})$ a complete lattice

NOT a sublattice of $\mathrm{Con}(\mathbf{R})$

Congruences

Definition

$\lambda_{\mathcal{A}}^{\mathbf{R}} = \bigcap \mathrm{Con}_{\mathcal{A}}(\mathbf{R})$
the verbal congruence on **R** *induced by* $\mathcal{A}$

Thus $\mathbf{R}/\lambda_{\mathcal{A}}^{\mathbf{R}}$ is the largest quotient of **R** lying in $\mathcal{A}$

Examples

- Let **R** be a group and $\mathcal{A}$ the variety of abelian groups. Then $\lambda_{\mathcal{A}}^{\mathbf{R}} = [R, R]$ is the derived subgroup of **R**
- Let **R** be an abelian group and $\mathcal{T}$ the quasivariety of torsion-free abelian groups. Then $\lambda_{\mathcal{T}}^{\mathbf{R}} = R_{\mathrm{tors}} = \{\, r \in R : |r| < \infty \,\}$

Remark on terminology: The algebra $\mathbf{R}/\lambda_{\mathcal{A}}$ is sometimes called the $\mathcal{A}$*-replica of* **R**. (And $\lambda_{\mathcal{A}}$ is the replica congruence.)

$\lambda_{\mathcal{A}}^{\mathbf{R}} = \bigcap \mathrm{Con}_{\mathcal{A}}(\mathbf{R})$
the verbal congruence on **R** *induced by* $\mathcal{A}$

Might not be easy to compute in practice

Suppose $\mathcal{A}$ is a *variety* with equational base Σ. Then

$$\lambda_{\mathcal{A}}^{\mathbf{R}} = \mathrm{Cg}^{\mathbf{R}} \left\{ \big(p(\mathbf{r}), q(\mathbf{r})\big) : (p(\mathbf{x}) \approx q(\mathbf{x})) \in \Sigma,\ \mathbf{r} \in R^n \right\}$$

($\mathrm{Cg}^{\mathbf{R}}(\Theta)$ is the smallest congruence on **R** containing Θ)

There is a very efficient algorithm for computing Cg, at least on a finite algebra.

For example, let $\mathcal{S}l$ denote the variety of semilattices. Suppose that **R** is a binar. (I.e., one binary operation) Then $\lambda^{\mathbf{R}}_{\mathcal{S}l}$ is generated by all pairs of the form

$$
\begin{aligned}
&(a \cdot a, a),\\
&(a \cdot b, b \cdot a),\\
&(a \cdot (b \cdot c), (a \cdot b) \cdot c)
\end{aligned}
$$

with $a, b, c \in R$.

Question: Is there an efficient algorithm to compute $\lambda_{\mathcal{A}}$ when $\mathcal{A}$ is a quasivariety?

Free Algebras

Let $\mathbf{T} = \mathbf{T}(X)$ be a term algebra and $\mathcal{A}$ a quasivariety. The algebra $\mathbf{F} = \mathbf{T}/\lambda^{\mathbf{T}}_{\mathcal{A}} \in \mathcal{A}$ is *free in* $\mathcal{A}$, that is

$\mathbf{F} \in \mathcal{A}$ and for all $\mathbf{R} \in \mathcal{A}$ and $h \colon X \to R$ there is a unique homomorphism $\bar{h} \colon \mathbf{F} \to \mathbf{R}$ such that $\bar{h}{\restriction}_X = h$

- $\mathbf{F}$ is also free in the variety $\mathbb{H}(\mathcal{A})$
- An identity $f(\mathbf{x}) \approx g(\mathbf{x})$ holds in $\mathbb{H}(\mathcal{A})$ iff $(f(\mathbf{x}), g(\mathbf{x})) \in \lambda^{\mathbf{T}}_{\mathcal{A}}$.

fully invariant congs

Maltsev Products

Motivation: Schreier extension in groups.

$\mathbf{R}$ is an *extension* of $\mathbf{S}$ by $\mathbf{T}$ if $\mathbf{R}$ has a normal subgroup $\mathbf{N}$ such that $\mathbf{S} \cong \mathbf{N}$ and $\mathbf{R}/N \cong \mathbf{T}$.
Equivalently, there is a short exact sequence
$1 \to \mathbf{S} \to \mathbf{R} \to \mathbf{T} \to 1$.

Let $\mathcal{A}$ and $\mathcal{B}$ be varieties of groups. Define

$$\mathcal{A} \circ_{\mathcal{G}} \mathcal{B} = \{\, \mathbf{R} : \mathbf{R} \text{ a group and } (\exists \mathbf{S} \in \mathcal{A})\ (\exists \mathbf{T} \in \mathcal{B})\ \mathbf{R} \text{ an extension of } \mathbf{S} \text{ by } \mathbf{T} \,\}.$$

H. and B.H. Neumann (195?) observed that $\mathcal{A} \circ_{\mathcal{G}} \mathcal{B}$ is again a variety of groups. They derived some surprising properties for this "product of varieties."

Maltsev (1967) proposed a generalization to arbitrary classes of models. We shall specialize to quasivarieties of algebras.

Definitions and Notation

Let **R** be an algebra, $\theta \in \mathrm{Con}(\mathbf{R})$, $a \in R$

1. $\mathrm{Sub}(\mathbf{R})$ is the set of subuniverses of **R**
2. a is *idempotent* if $f(a, a, \ldots, a) = a$, for every basic operation.
   Equivalently, $\{a\} \in \mathrm{Sub}(\mathbf{R})$
   R is *idempotent* if every element of R is idempotent.
3. $\mathcal{B}$ is *polarized* if there is a unary term t such that $\mathcal{B} \models t(x) \approx t(y)$ & "$t(x)$ is idempotent".

   For example, groups are polarized with $t(x) = x \cdot x^{-1}$

Continued...

4. a/θ is the equivalence class of a mod θ, considered as an element of $\mathbf{R}/\theta$

   $[a]_\theta$ is the equivalence class of a mod θ, considered as a subset of R.

For any $a \in R$ and $\theta \in \mathrm{Con}(\mathbf{R})$,
if a is idempotent then $[a]_\theta \in \mathrm{Sub}(\mathbf{R})$.

More generally, if $h\colon \mathbf{R} \to \mathbf{S}$ is a homomorphism with $\theta = \ker(h)$ then

$$[r]_\theta \in \mathrm{Sub}(\mathbf{R}) \iff h(r) \text{ is idempotent in } \mathbf{S}.$$

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties. The *Maltsev Product of $\mathcal{A}$ and $\mathcal{B}$* is

$$\mathcal{A} \circ \mathcal{B} = \{ \mathbf{R} : (\exists \theta \in \mathrm{Con}(\mathbf{R}))\ \mathbf{R}/\theta \in \mathcal{B} \text{ and } (\forall r \in R)\ [r]_\theta \in \mathrm{Sub}(\mathbf{R}) \implies [r]_\theta \in \mathcal{A} \} .$$

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties. $\mathcal{A} \circ \mathcal{B}$ is always closed under subalgebras. $\mathcal{A} \circ \mathcal{B}$ will be a quasivariety if any of the following hold.

- *The language is finite*
- *$\mathcal{B}$ is idempotent*
- *$\mathcal{B}$ is polarized.*

Suppose $\mathcal{C}$ is another quasivariety. We also write

$$\mathcal{A} \circ_{\mathcal{C}} \mathcal{B} = (\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$$

the *Maltsev product of* $\mathcal{A}$ *and* $\mathcal{B}$ *relative to* $\mathcal{C}$.

Thus, in the case that $\mathcal{A}$ and $\mathcal{B}$ are varieties of groups, and $\mathcal{C}$ is the class of all groups, this is precisely the construction considered by the Neumanns. Details

Examples From the Literature

this was an active research topic in group theory in the 1960s. Some sample results:

let $\mathcal{A}_n$ denote the variety of abelian groups of exponent n. Then

$\mathcal{A}_3 \circ_{\mathcal{G}} \mathcal{A}_2 = \text{Mod}\{x^6 \approx e, [x^2, y^2] \approx e, [x, y]^3 \approx e\}$

$\mathcal{A}_2 \circ_{\mathcal{G}} \mathcal{A}_2 = \text{Mod}\{(x^2 y^2)^2 \approx e\}$

Consider these to be exercises

As early as 1940, A.H. Clifford considered Maltsev products relative to $\mathcal{SG}$ = semigroups.

Sample result. Let
$\mathcal{B}$ be the variety of idempotent semigroups
$\mathcal{S}l$ the variety of semilattices
$\mathcal{RB}$= Mod$\{x^2 \approx x, x(yz) \approx (xy)z \approx xz\}$
(rectangular bands)

Then: $\mathcal{B} = \mathcal{RB} \circ_{\mathsf{Sg}} \mathcal{S}l$ Binars

Idempotence

If $\mathcal{B}$ is idempotent and $\mathbf{R}/\theta \in \mathcal{B}$, then every θ-class is always a subalgebra. Thus the definition becomes

$$\mathcal{A} \circ \mathcal{B} = \{\, \mathbf{R} : (\exists \theta \in \mathrm{Con}(\mathbf{R}))\ \mathbf{R}/\theta \in \mathcal{B} \text{ and } (\forall r \in R)\ [r]_\theta \in \mathcal{A} \,\}.$$

Pivot Congruences

$$\mathcal{A} \circ \mathcal{B} = \{ \mathbf{R} : (\exists \theta \in \mathrm{Con}(\mathbf{R}))\ \mathbf{R}/\theta \in \mathcal{B} \text{ and } (\forall r \in R)\ [r]_\theta \in \mathrm{Sub}(\mathbf{R}) \implies [r]_\theta \in \mathcal{A} \} .$$

A congruence θ, in the above, is called a *$\mathcal{A}$,$\mathcal{B}$-pivot congruence on* **R**.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties, **R** *an algebra. Then* $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$ *iff $\lambda^{\mathbf{R}}_{\mathcal{B}}$ is an $\mathcal{A}$,$\mathcal{B}$-pivot.*

Given $\mathbf{R}$, determine whether $\mathbf{R} \in \mathcal{A} \circ \mathcal{B}$

1. Compute $\lambda = \lambda_{\mathcal{B}}^{\mathbf{R}}$
2. For each $a \in R$ check $[a]_\lambda \in \mathsf{Sub}(\mathbf{R}) \implies [a]_\lambda \in \mathcal{A}$

Often this can be done quite efficiently, especially if $\mathcal{A}$ and $\mathcal{B}$ are finitely based varieties and $\mathcal{B}$ is idempotent

Example

Let $\mathcal{S}l$ = semilattices and $\mathcal{LZ}$ = left-zero semigroups

Input: binar **R**
Query: is $\mathbf{R} \in \mathcal{LZ} \circ \mathcal{S}l$?

1. Compute $\lambda = \mathrm{Cg}^{\mathbf{R}}\{\, (a \cdot a, a), (a \cdot b, b \cdot a), ((a \cdot b) \cdot c, a \cdot (b \cdot c)) : a, b, c \in R \,\}$
2. For each $r \in R$ does $[r]_\lambda \models x \cdot y \approx x$?

Location of $\mathcal{A} \circ \mathcal{B}$ in the Lattice

Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties
$\mathcal{A} \subseteq \mathcal{A} \circ \mathcal{B}$ (0 is the pivot congruence)
$\mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$ (1 is the pivot congruence)

Therefore $\mathcal{A} \vee_Q \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$.
(Probably much smaller)

Even if $\mathcal{A}$, $\mathcal{B}$ are varieties, it is not necessarily true that
$\mathcal{A} \vee_V \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$
(Obviously it is true if $\mathcal{A} \circ \mathcal{B}$ is a variety) Example

$\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2 \implies \mathcal{A}_1 \circ \mathcal{B}_1 \subseteq \mathcal{A}_2 \circ \mathcal{B}_2$

Theorem

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be idempotent quasivarieties. Then $\mathcal{A} \circ (\mathcal{B} \circ \mathcal{C}) \subseteq (\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C}$. In general, the opposite inclusion fails.

Problem: $\mathbb{H}(\mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})) \stackrel{?}{=} \mathbb{H}((\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C})$

Identities Holding in $\mathcal{A} \circ \mathcal{B}$

Let $\mathcal{A}$ be a variety, $\mathcal{A} = \mathrm{Mod}(\Sigma)$
Let $\mathcal{B}$ be an *idempotent* variety

$\Sigma(\mathcal{B})$ is the set of all identities of the following form

$$u\big(r_1(\mathbf{x}), r_2(\mathbf{x}), \ldots, r_k(\mathbf{x})\big) \approx v\big(r_1(\mathbf{x}), r_2(\mathbf{x}), \ldots, r_k(\mathbf{x})\big)$$

in which $\big(u(y_1, \ldots, y_k) \approx v(y_1, \ldots, y_k)\big) \in \Sigma$ and
$\mathcal{B} \models r_1(\mathbf{x}) \approx r_2(\mathbf{x}) \approx \cdots \approx r_k(\mathbf{x})$.

Finally, define $\mathcal{A}\mathcal{B} = \mathrm{Mod}(\Sigma(\mathcal{B}))$. Example

Theorem

$\mathcal{A}$ and $\mathcal{B}$ as above. Then

1. *$\mathcal{A}\mathcal{B}$ is a variety*
2. *For any set X, $\mathbf{F}_{\mathcal{A}\mathcal{B}}(X) \in \mathcal{A} \circ \mathcal{B}$*
3. $\mathbb{H}(\mathcal{A} \circ \mathcal{B}) = \mathcal{A}\mathcal{B}$.

Problem: Find a set Δ of quasiidentities so that $\mathcal{A} \circ \mathcal{B} = \mathrm{Mod}\big(\Delta \cup \Sigma(\mathcal{B})\big)$

Conjecture: $\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\mathcal{C}$

Properties Preserved By Maltsev Products

Exercise

If $\mathcal{A}$ and $\mathcal{B}$ are idempotent quasivarieties, then so is $\mathcal{A} \circ \mathcal{B}$.

Unfortunately, neither commutativity nor associativity of binars are preserved in this way.

Freese and McKenzie call a property *robust* if it is preserved by the Maltsev product of *idempotent* varieties.

Freese-McKenzie Robustness

F&M prove that a number of properties are robust, including

- Having a Taylor term
- Having a near unanimity term
- $\exists k$ congruence k-permutable

They show the following are not robust:

- Being congruence k-permutable, for a fixed k (so in particular, having a Maltsev term)
- Being congruence distributive
- Being congruence modular.

Definitions

Remark: For each of the properties they consider, if it holds in a quasivariety $\mathcal{A}$, then it holds in $\mathbb{H}(\mathcal{A})$.

More Freese-McKenzie Results

Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent varieties.

- If $\mathcal{A}$ and $\mathcal{B}$ are congruence permutable then $\mathcal{A} \circ \mathcal{B}$ is congruence 3-permutable (hence is congruence modular) but not congruence permutable
- If $\mathcal{A}$ and $\mathcal{B}$ have a majority term then $\mathcal{A} \circ \mathcal{B}$ is congruence distributive

Nonrobustness of congruence permutability (CP)

F&M construct algebras $\mathbf{R}_i$, for $i = 1, 2$, with $\mathcal{A}_i = \mathbb{V}(\mathbf{R}_i)$ CP,
but $\mathbf{R}_1 \times \mathbf{R}_2$ not CP.
Hence $\mathcal{A}_1 \circ \mathcal{A}_2$ not CP.
In fact, $\mathcal{A}_1 \vee_Q \mathcal{A}_2$ not CP.

Theorem (Bergman)

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be idempotent varieties, and assume $\mathcal{A}_1 \vee_Q \mathcal{A}_2$ is CP. Then $\mathcal{A}_1 \circ \mathcal{A}_2$ is CP.

Problem: The argument for the nonrobustness of CD is similar. Is it true that $\mathcal{A}_1 \vee_Q \mathcal{A}_2$ CD $\implies$ $\mathcal{A}_1 \circ \mathcal{A}_2$ CD?
(Or maybe: majority term, k-permutable,. . .)

Is $\mathcal{A} \circ \mathcal{B}$ a variety?

Motivating Problem

Let $\mathcal{A}$ and $\mathcal{B}$ be *varieties*, with $\mathcal{B}$ idempotent. We already know that $\mathcal{A} \circ \mathcal{B}$ is a quasivariety. Under what conditions will $\mathcal{A} \circ \mathcal{B}$ be a variety?

Observe that $\mathcal{A} \circ \mathcal{B}$ is a variety iff

$$\mathcal{A} \circ \mathcal{B} = \mathcal{A}\mathcal{B} = \mathrm{Mod}(\Sigma(\mathcal{B}))$$

with $\mathcal{A} = \mathrm{Mod}(\Sigma)$

Thus, find conditions so that:

$$\mathbf{R} \models \Sigma(\mathcal{B}) \implies (\forall r \in R)\ [r]_{\lambda_{\mathcal{B}}} \models \Sigma$$

Theorem (Penza-Romanowska)

Let $\mathcal{A}$ and $\mathcal{B}$ be varieties in a language with no nullary operation symbols. Assume that

- *$\mathcal{B}$ is idempotent*

and that there are terms $f(x,y)$ and $g(x,y)$ such that

- *$\mathcal{A} \models f(x,y) \approx x \;\&\; g(x,y) \approx y$ and*
- *$\mathcal{B} \models f(x,y) \approx g(x,y)$.*

Then $\mathcal{A} \circ \mathcal{B}$ is a variety.

Examples

Example

Let $\mathcal{A}$ be an idempotent variety of binars and $\mathcal{S}l$ the variety of semilattices. Suppose $\mathcal{A} \cap \mathcal{S}l$ is trivial. Then $\mathcal{A} \circ \mathcal{S}l$ is a variety.

Proof.

From the assumptions, there is a binar term $f(x, y)$ (involving both variables) such that $\mathcal{A} \models f(x, y) \approx x$. Set $g(x, y) = f(y, x)$. In a semilattice (associative, commutative, idempotent) we must have $f(x, y) \approx f(y, x)$.

Details Special case: $\mathcal{LZ} \circ \mathcal{S}l$. Here $f(x, y) = x \cdot y$, $g(x, y) = y \cdot x$.

Example

Let $\mathcal{LZ}$ and $\mathcal{RZ}$ denote the varieties of left-zero and right-zero semigroups. Then $\mathcal{LZ} \circ \mathcal{RZ}$ is a variety.

Proof.

Take $f(x, y) = x$ and $g(x, y) = y \cdot x$.

Somewhat Harder Corollary

Let $\mathcal{A}$ be a congruence permutable variety and $\mathcal{B}$ be idempotent. Then $\mathcal{A} \circ \mathcal{B}$ is a variety.

Counterexamples

Example

$\mathcal{LZ} \circ \mathcal{LZ}$ is not a variety

Proof.

Equivalently, $\mathcal{LZ} \circ \mathcal{LZ} \neq \mathbb{H}(\mathcal{LZ} \circ \mathcal{LZ})$. They satisfy the same identities. So we must find a *quasiidentity* that separates. Here is one:

$$(q_z) \quad xy \approx yx \to x \approx y.$$

We first show $\mathcal{LZ} \circ \mathcal{LZ} \models q_z$.

. . . continued.

$$(q_z) \quad xy \approx yx \to x \approx y$$

Let $\mathbf{R} \in \mathcal{LZ} \circ \mathcal{LZ}$ with verbal congruence λ. Suppose that $a, b \in R$ and $ab = ba$. Since $\mathbf{R}/\lambda$ is left-zero,

$$(a/\lambda) \cdot (b/\lambda) = a/\lambda, \text{ i.e.,}$$

$$ab \equiv_\lambda a$$

Similarly

$$ba \equiv_\lambda b.$$

From $ab = ba$ we obtain

$$a \equiv_\lambda b.$$

Since each λ-class is left-zero, we get $ab = a$ and $ba = b$. But then $a = b$.

... continued.

Now show $\mathbb{H}(\mathcal{LZ} \circ \mathcal{LZ}) \not\models xy \approx yx \to x \approx y$

Let **R** be the 4-element binar with operation table

	0	1	2	3
0	0	0	0	0
1	1	1	1	0
2	2	2	2	2
3	3	2	3	3

$\mathbf{R} \in \mathcal{LZ} \circ \mathcal{LZ}$ with pivot congruence $\lambda = |01|23|$.
However, **R** has a congruence $\theta = |02|1|3|$ and $\mathbf{R}/\theta$ fails q_z with $x = 1/\theta$ and $y = 3/\theta$.

Thus $\mathcal{LZ} \circ \mathcal{LZ}$ is not closed under homomorphic images.

Note that $\mathcal{LZ} \cong \mathcal{RZ}$. And yet
$\mathcal{LZ} \circ \mathcal{RZ}$ is a variety but
$\mathcal{LZ} \circ \mathcal{LZ}$ is not a variety

A Few More Counterexamples

Example

Let $\mathcal{B}$ be a variety of binars containing a nontrivial semilattice. If $\mathcal{B}$ is either commutative or associative then $\mathcal{B} \circ \mathcal{LZ}$ is not a variety.

Proof.

Consider the quasiidentities

$$(q_c) \quad (yz \approx x \ \& \ zx \approx x \ \& \ zy \approx z) \rightarrow (x \approx z)$$

$$(q_a) \quad (yz \approx x \ \& \ zx \approx x \ \& \ zy \approx y \ \& \ z^2 \approx z) \rightarrow x \approx z.$$

Then $\mathcal{CB} \circ \mathcal{LZ} \models q_c$ (commutative binars)
and $\mathcal{AB} \circ \mathcal{LZ} \models q_a$ (associative binars, i.e., semigroups).

. . . continued.

Let **R** be the binar with table

	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	2	2	2	2
3	2	3	2	3

$\lambda_{\mathcal{LZ}} = |01|23| \implies \mathbf{R} \in \mathcal{S}l \circ \mathcal{LZ} \subseteq \mathcal{B} \circ \mathcal{LZ}$. Let $\theta = |02|1|3| \in \mathrm{Con}(\mathbf{R})$ and let $\mathbf{S} = \mathbf{R}/\theta$. Then $\mathbf{S} \not\models q_c \ \&\ q_a$.

So $\mathcal{LZ} \circ \mathcal{S}l$ is a variety but $\mathcal{S}l \circ \mathcal{LZ}$ is not.

Problems

1. Is $\mathcal{CB} \circ \mathcal{LZ}$ axiomatized, relative to $(\mathcal{CB})(\mathcal{LZ})$, by q_c?
2. Does there exist any proper variety, $\mathcal{B}$, of binars with $\mathcal{S}l \subseteq \mathcal{B}$ such that $\mathcal{B} \circ \mathcal{LZ}$ is a variety?

(Observe that if we take $\mathcal{B}$ to be the variety of all binars then $\mathcal{B} \circ \mathcal{LZ} = \mathcal{B}$ which is obviously a variety.)

In a similar vein, $\mathcal{CB} \circ \mathcal{S}I \models q_s$
with q_s the quasiidentity

$$(q_s) \qquad (zx = x \;\&\; zy = y \;\&\; xz = yz) \to (xy = yx)$$

but there is a 7-element binar $\mathbf{R}$ with $\mathbf{R} \in \mathcal{S}I \circ \mathcal{S}I$ while $\mathbf{R}/\theta \not\models q_s$.

Thus if $\mathcal{S}I \subseteq \mathcal{B} \subseteq \mathcal{CB}$ then $\mathcal{B} \circ \mathcal{S}I$ is not a variety.
So in particular $\mathcal{S}I \circ \mathcal{S}I$ is not a variety.

Thank you

Part I

Additional Details

A Note on Congruences

Let **R** and **S** be algebras and $h\colon \mathbf{R} \to \mathbf{S}$ a homomorphism. Define

$$\ker(h) = \{\, (x, y) \in R^2 : h(x) = h(y) \,\}$$

$\ker(h)$ is clearly an equivalence relation on R.
$\ker(h)$ also has the *substitution property:*

$$(x_1, y_1), \ldots, (x_n, y_n) \in \ker(h) \implies \big(f(x_1, \ldots, x_n),\ f(y_1, \ldots, y_n)\big) \in \ker(h)$$

as f ranges over all the operation symbols in the language.

A *congruence relation on* **R** is any equivalence relation on R with the substitution property.

$\mathrm{Con}(\mathbf{R})$ is the set of congruences on **R**. Forms a complete lattice under inclusion.

Let θ be a congruence relation, $r \in R$. r/θ is the equivalence class of r modulo θ.
The *quotient algebra* is $\mathbf{R}/\theta = \{\, r/\theta : r \in R \,\}$.

The "canonical homomorphism" $q\colon \mathbf{R} \to \mathbf{R}/\theta$; $q(r) = r/\theta$.
$\theta = \ker(q)$.

This use of ‘kernel’ is closely related to that of “mainstream” algebra.

Let **R** be a group. There is a lattice isomorphism

$$\{\text{normal subgroups of } \mathbf{R}\} \leftrightarrows \mathrm{Con}(\mathbf{R})$$
$$N \mapsto \{\, (x, y) : xy^{-1} \in N \,\}$$
$$e/\theta \leftarrow\!\shortmid \theta$$

Similarly, let **R** be a boolean algebra. There is a lattice isomorphism

$$\{\text{ideals of } \mathbf{R}\} \leftrightarrows \mathrm{Con}(\mathbf{R})$$
$$I \mapsto \{\, (x, y) : (x \wedge y') \vee (x' \wedge y) \in I \,\}$$
$$0/\theta \leftarrow\!\shortmid \theta$$

Return

A Note On Fully Invariant Congruences

Let $\mathbf{R}$ be an algebra. $\mathrm{End}(\mathbf{R}) = \{\text{endomorphisms of } \mathbf{R}\}$
$\theta \in \mathrm{Con}(\mathbf{R})$ is *fully invariant* if
$(x, y) \in \theta \implies (f(x), f(y)) \in \theta$, for all $f \in \mathrm{End}(\mathbf{R})$

Exercise: every verbal congruence is fully invariant.
The converse is false in general, however it is true in an important special case.

Theorem

Let $\mathcal{A}$ be a variety and $\mathbf{F}$ a free algebra in $\mathcal{A}$. Then every fully invariant congruence on $\mathbf{F}$ is verbal.

Return

Some Familiar Classes of Algebras

Binar A model in a language with one binary operation symbol, and no others

$\mathcal{S}g$ *Semigroups* are binars satisfying $x(yz) \approx (xy)z$

$\mathcal{LZ}$ *Left-zero semigroups* are binars satisfying $xy \approx x$

$\mathcal{RZ}$ *Right-zero semigroups* are binars satisfying $xy \approx y$

$\mathcal{S}l$ *Semilattices* are binars satisfying $x(yz) \approx (xy)z$, $xy \approx yx$, $x^2 \approx x$

Return

Facts about Semilattices

Let $\mathbf{S}$ be a semilattice. For $x, y \in S$ define
$x \leq y \iff xy = x$.
This yields a partial order on S with $xy = \inf\{x, y\}$.

Write $\mathbf{2} = \{0, 1\}$ with $0 \cdot 1 = 0$. For $s \in S$ define $h_s \colon \mathbf{S} \to \mathbf{2}$ by

$$h_s(x) = \begin{cases} 1 & x \geq s \\ 0 & \text{otherwise} \end{cases}$$

This family induces an injective homomorphism $\mathbf{S} \rightarrowtail \prod_{s \in S} \mathbf{2}$. Thus $\mathcal{S}l = \mathbb{SP}(\mathbf{2}) = \mathbb{Q}(\mathbf{2})$.

Return

Manipulating Verbal Congruences

Notation: Let θ be a congruence on **R**.
For **S** $\leq$ **R** write $\theta\restriction_S = \theta \cap S^2$
For h: **S** $\twoheadrightarrow$ **R** a surjective homomorphism write
$\overleftarrow{h}(\theta) = \{\, (x, y) \in S^2 : \big(h(x), h(y)\big) \in \theta \,\}$

- Both $\theta\restriction_S$ and $\overleftarrow{h}(\theta)$ are congruences on **S**
- Both transformations are order-preserving
- $\overleftarrow{h}(\theta) \supseteq \ker(h)$

Manipulating Verbal Congruences

Notation: Let θ be a congruence on **R**.
For **S** $\leq$ **R** write $\theta{\restriction}_S = \theta \cap S^2$
For $h\colon \mathbf{S} \twoheadrightarrow \mathbf{R}$ a surjective homomorphism write
$\overleftarrow{h}(\theta) = \{\, (x, y) \in S^2 : (h(x), h(y)) \in \theta \,\}$

Theorem

*Let $\mathcal{A}$ be a quasivariety, **R**, **S** algebras.*

1. *Let $\mathbf{S} \leq \mathbf{R}$. Then $\lambda_{\mathcal{A}}^{\mathbf{S}} \subseteq \lambda_{\mathcal{A}}^{\mathbf{R}}{\restriction}_S$*
2. *Let $h\colon \mathbf{S} \twoheadrightarrow \mathbf{R}$ be a surjective homomorphism. Then $\overleftarrow{h}(\lambda_{\mathcal{A}}^{\mathbf{R}}) = \lambda_{\mathcal{A}}^{\mathbf{S}} \vee \ker(h)$*

Group Extensions

Let $\mathcal{G}p$ denote the variety of groups. This variety is polarized. For example use the unary term $e(x) = x \cdot x^{-1}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be varieties of groups and $\mathbf{R} \in \mathcal{A} \circ_{\mathcal{G}p} \mathcal{B}$.
Suppose $\theta \in \mathrm{Con}(\mathbf{R})$.
In a polarized variety, exactly one θ-class is a subalgebra, namely $[e(x)]_\theta$. Thus $N = [e(x)]_\theta$ is a normal subgroup, $\mathbf{N} \in \mathcal{A}$ and $\mathbf{R}/N \in \mathcal{B}$.

Put another way, $\mathbf{R}$ is an extension of $\mathbf{N}$ by $\mathbf{R}/N$.

Return

$\mathcal{S}l \vee_V \mathcal{LZ} \not\subseteq \mathcal{S}l \circ \mathcal{LZ}$

Let **R** denote this three-element binar
$\mathcal{S}l$, the variety of semilattices
$\mathcal{LZ}$, the variety of left-zero semigroups

$\cdot$	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

$\mathrm{Con}(\mathbf{R}) = \{0, \lambda_{\mathcal{S}l}, 1\}$.
Thus $\mathbf{R} \in \mathcal{LZ} \circ \mathcal{S}l$ but $\mathbf{R} \notin \mathcal{S}l \circ \mathcal{LZ}$ since $\lambda_{\mathcal{LZ}} = 1$.

On the other hand, $\mathbf{R} \in \mathbb{H}(\mathbf{2} \times \mathbf{L}_2) \subseteq \mathcal{S}l \vee_V \mathcal{LZ}$
($\mathbf{L}_2$ the 2-element left zero semigroup)

Therefore $\mathcal{S}l \vee_V \mathcal{LZ} \not\subseteq \mathcal{S}l \circ \mathcal{LZ}$.

◂ Return

Example of $\Sigma(\mathcal{B})$

Let $\mathcal{CB} = \mathrm{Mod}(\Sigma)$ with $\Sigma = \{xy \approx yx\}$.
Let $\mathcal{LZ} = \mathrm{Mod}\big(\{xy \approx x\}\big)$,
but $\mathcal{LZ}$ satisfies many more identities, such as

$$(xy)z \approx x$$
$$(xy)(zw) \approx x(zwy)$$
$$(x(yz))(yy) \approx x$$

Among the identities in $\Sigma(\mathcal{LZ})$ will be

$$(xy)x \approx x(xy)$$
$$((xy)z)x \approx x((xy)z)$$
$$((xy)(zw))(x(zwy)) \approx (x(zwy))((xy)(zw))$$

But *not* $xy \approx yx$ or $xy \approx x$.

Return

Advanced Definitions

Let $\mathcal{A}$ be a variety.

1. A ternary term $f(x, y, z)$ is a *majority term for* $\mathcal{A}$ if $f(x, x, y) \approx f(x, y, x) \approx f(y, x, x)$ holds in $\mathcal{A}$.
2. More generally, a k-ary term $f(x_1, \ldots, x_k)$ is a *near-unanimity term for* $\mathcal{A}$ if $\mathcal{A} \models f(x, x, \ldots, x, y) \approx f(x, x, \ldots, y, x) \approx \cdots \approx f(y, x, \ldots, x, x)$.
3. $\mathcal{A}$ is *congruence distributive* if every $\mathbf{R} \in \mathcal{A}$ has a distributive congruence lattice.

majority $\Longrightarrow$ near-unanimity $\Longrightarrow$ cong distributive

4. Given two binary relations, θ and ψ on a set U, we define

$$\theta \circ \psi = \{ (x, z) : (\exists y \in U)\ (x, y) \in \theta \ \&\ (y, z) \in \psi \}$$

Recursively, define

$$\theta \circ_1 \psi = \theta \text{ and } \theta \circ_{k+1} \psi = \theta \circ (\psi \circ_k \theta)$$

The variety $\mathcal{A}$ is *congruence k-permutable* if, for every $\mathbf{R} \in \mathcal{A}$ and $\theta, \psi \in \mathrm{Con}(\mathbf{R})$ we have $\theta \circ_k \psi = \psi \circ_k \theta$.

Congruence permutable = congruence 2-permutable

Return

Regular Identities

Since semilattices are associative, commutative and idempotent, it is easy to see that an identity holds in $\mathcal{S}l$ iff exactly the same variables appear on both sides. (Or use the fact that $\mathcal{S}l = \mathbb{V}(\mathbf{2})$.) Such an identity is called *regular.* Suppose $\mathcal{A}$ is an idempotent variety of binars and $\mathcal{A} \cap \mathcal{S}l$ is trivial. Then $\mathcal{A}$ must satisfy some irregular identity, say $f(x_1, \dots, x_k, y_1, \dots, y_l) \approx d(x_1, \dots, x_k, z_1, \dots, z_m)$. Set every x_i and every z_i to x and every y_i to y, yielding $f(x, y) \approx d(x, x)$. Since $\mathcal{A}$ is idempotent, $d(x, x) \approx x$ holds in $\mathcal{A}$. Hence $\mathcal{A} \models f(x, y) \approx x$.

◂ Return