MV-topologies and MV-uniformities

Luz Victoria De La Pava
joint work with Marby Bolaños

Departamento de Matemáticas
Universidad del Valle, Colombia

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Definition

An MV-algebra $\langle A, \oplus, ^*, \mathbf{0} \rangle$ is an algebra of type (2,1,0) such that

- $\langle A, \oplus, \mathbf{0} \rangle$ is a commutative monoid
- $(x^*)^* = x$
- $x \oplus \mathbf{0}^* = \mathbf{0}^*$
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

On each MV-algebra is defined the constant $\mathbf{1}$, the operations $\odot$ y $\ominus$, and the order $\leq$ so:

- $\mathbf{1} := \mathbf{0}^*$
- $x \odot y := (x^* \oplus y^*)^*$
- $x \ominus y := x \odot y^* = (x^* \oplus y)^*$
- $x \leq y$ iff $x \ominus y = 0$ (lattice order)
- $x \wedge y := x \odot (x^* \oplus y)$
- $x \vee y := (x \ominus y) \oplus y$

Example

$\langle [0,1], \oplus, ^*, 0 \rangle$, with $x \oplus y := \min\{x+y, 1\}$ and $x^* := 1-x$, is an MV-algebra, called standard.

i) $x \odot y = \max(0, x+y-1)$

ii) $x \ominus y = \max(0, x-y)$

iii) $x \wedge y = \min(x, y)$

iv) $x \vee y = \max(x, y)$

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Example

Let X be a set. The set $[0,1]^X$ of the functions from X to $[0,1]$ is an MV-algebra.

Remark

Note that $[0,1]^X$ is also a set of fuzzy subsets of X.

MV-Topologies [Russo, 2016]

Let X be a not empty set and $\tau \subseteq [0,1]^X$. We say that (X, τ) is an MV-topological space if τ (the MV-topology) satisfies:

1. $\mathbf{0}, \mathbf{1} \in \tau$
2. For each family $\{\alpha_i\}_{i \in I}$ of elements of τ, $\bigvee_{i \in I} \alpha_i \in \tau$.

and, for all $\alpha, \beta \in \tau$

3. $\alpha \wedge \beta \in \tau$

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3. $\alpha \wedge \beta \in \tau$
4. $\alpha \oplus \beta \in \tau$
5. $\alpha \odot \beta \in \tau$.

The elements of τ are the open MV-subsets of X, while $\tau^* = \{\alpha^* : \alpha \in \tau\}$ is the set of closed MV-subsets of X.

Definition

Let (X, τ_X) and (Y, τ_Y) be two MV-topological spaces. We say that a map $f : X \longrightarrow Y$ is:

- MV-continuous or continuous if $f^{\leftsquigarrow}[\tau_Y] \subseteq \tau_X$ $(f^{\leftsquigarrow}(\alpha) = \alpha \circ f)$,
- an MV-homeomorphism if it is bijective and both f and f^{-1} are continuous.

Now, we can define the category of MV-spaces, denoted by ${}^{\mathcal{MV}}\mathcal{T}\mathrm{op}$, whose objects are the MV-spaces and the morphisms the continuous functions according to the previous definition.

Compactness

An MV-space (X, τ) is said to be

- compact if any open covering of X contains an additive covering, i.e., for any $\Gamma \subseteq \tau$ such that $\bigvee \Gamma = \mathbf{1}$, there exists a finite family $\{\alpha_i\}_{i=1}^{n}$ of elements of Γ, $n < \omega$, such that $\alpha_1 \oplus \cdots \oplus \alpha_n = \mathbf{1}$.
- strongly compact if any open covering of X contains a finite covering.

Separation

An MV-space (X, τ) is Hausdorff if for $x \neq y \in X$, there exist $\alpha_x, \alpha_y \in \tau$ such that:

1. $\alpha_x(x) = \alpha_y(y) = 1$,
2. $\alpha_x \wedge \alpha_y = \mathbf{0}$.

Definition

A Stone MV-space is an MV-space which is compact, separated and zero-dimensional.

Remark

The category ${}^{\mathcal{MV}}\mathcal{S}$tone of Stone MV-spaces, with fuzzy continuous maps as morphisms, is a full subcategory of ${}^{\mathcal{MV}}\mathcal{T}$op.

A (proper) extension of Stone duality

Theorem – Russo(2016)

1. The mappings

$$\begin{array}{llllllll} \text{Clop}: & \langle X, \tau\rangle & \in & {}^{\mathcal{MV}}\mathcal{T}\text{op} & \longmapsto & \text{Clop}(\langle X, \tau\rangle) & \in & \mathcal{MV}^{\text{ss}} \\ \text{Max}: & A & \in & \mathcal{MV}^{\text{ss}} & \longmapsto & \langle \text{Max}\, A, \tau_A\rangle & \in & {}^{\mathcal{MV}}\mathcal{T}\text{op} \end{array}$$

   define two contravariant functors.

2. The restriction of Clop and Max to the subcategories $\mathcal{MV}^{\text{lcc}}$ and ${}^{\mathcal{MV}}\mathcal{S}\text{tone}$ is a duality.
3. The restriction of such a duality to Boolean algebras and Stone spaces coincide with the classical Stone duality.

Graphically

$$
\begin{array}{ccc}
\mathcal{MV}^{\mathsf{lcc}} & \underset{\mathsf{Clop}}{\overset{\mathsf{Max}}{\rightleftarrows}} & {}^{\mathcal{MV}}\mathcal{S}\mathsf{tone}^{\mathsf{op}} \\
\mathsf{B}\downarrow\uparrow\cup\!\mid & & \cup\!\mid\uparrow\downarrow\mathsf{Sk} \\
\mathcal{B}\mathsf{oole} & \underset{\mathsf{Max}_{\restriction}}{\overset{\mathsf{Clop}_{\restriction}}{\leftrightarrows}} & \mathcal{S}\mathsf{tone}^{\mathsf{op}}
\end{array}
$$

Horizontal arrows: equivalences
Vertical arrows: inclusions of full subcategories and their left-inverses

Some Results about MV-spaces that extend classical results

- A Tychonoff-Type Theorem
- Stone-Čech Compactification (categorical)
- Normality and Urysohn's Lemma
- MV-sheaves

A Tychonoff-Type Theorem

Theorem (Tychonoff-type, De La Pava - Russo, 2020)

If $\{(X_i, \tau_i)\}_{i \in I}$ is a family of compact MV-topological spaces, then so is their product space (X, τ).

Lemma (Analogous of Alexander Subbase Theorem)

Let (X, τ) be an MV-topological space and S a large subbase for τ. If every collection of sets from S that cover X has an additive subcover, then X is compact.

Stone-Čech Compactification

Theorem (De La Pava - Russo, 2020)

The functor $i : {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \longrightarrow {}^{\mathcal{MV}}\mathcal{T}\mathrm{op}$ *has a left adjoint. We denote by* $\widehat{\beta} : {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} \longrightarrow {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ *the left adjoint functor of* i.

It is a good extension

- The functors β and $\iota\widehat{\beta}\omega$ are naturally isomorphic.
- The initial topology of an MV-space (X, τ) determines the initial topology of the *MV-compactification* of X
- For each topologically generated MV-space X, the compactification $\widehat{\beta}(X)$ is completely determined by its initial topology.

Normality and Urysohn's Lemma

Definition [Hutton, 1975]

A fuzzy topological space (X, τ) is normal if for every closed set $\alpha \in \tau^*$ and open set $\beta \in \tau$ such that $\alpha \leq \beta$, there exists $\gamma \in \tau$ such that

$$\alpha \leq \gamma \leq \overline{\gamma} \leq \beta.$$

Proposition

An MV-topological space (X, τ) is normal if and only if for each pair of closed fuzzy sets α and β such that $\alpha \odot \beta = \mathbf{0}$ there are $\gamma, \delta \in \tau$ such that $\alpha \leq \gamma$, $\beta \leq \delta$ and $\gamma \odot \delta = \mathbf{0}$.

The fuzzy unit interval (Hutton, 1975)

Let $Z_I(\mathbb{R})$ be the set of monotonic decreasing functions $f \in [0,1]^{\mathbb{R}}$ such that:

1. $f(x) = 1$ for all $x \in (-\infty, 0)$ and
2. $f(x) = 0$ for all $x \in (1, \infty)$.

On $Z_I(\mathbb{R})$ Hutton introduced the following equivalence relation:

$$f_1 \sim f_2 \text{ iff } f_1(x+) = f_2(x+)(\text{ iff } f_1(x-) = f_2(x-)) \text{ for all } x \in \mathbb{R}$$

where $f(x+) := \bigvee_{t>x} f(t)$ and $f(x-) := \bigwedge_{t<x} f(t)$.

Definition (Hutton, 1975)

The fuzzy unit interval $\mathfrak{F}(I)$ is the set of all monotonic decreasing maps $f \in Z_I(\mathbb{R})$ after the identification by the relation $\sim$. That is, $\mathfrak{F}(I) = Z_I(\mathbb{R})/\sim$.

Normality and Urysohn's Lemma

We define an MV-topology σ on $\mathfrak{F}(I)$ by taking as a subbase the family of fuzzy sets $\{L_t, R_t : t \in \mathbb{R}\}$ where $L_t, R_t : \mathfrak{F}(I) \longrightarrow [0,1]$ are such that

$$L_t(f) = f(t-)^* \text{ and } R_t(f) = f(t+) \text{ for all } t \in \mathbb{R}$$

Theorem (Urysohn-type Lemma)

An MV-topological space (X, τ) is normal if and only if for every closed $\beta, \alpha \in \tau^*$ such that $\beta \odot \alpha = \mathbf{0}$ ("disjoint"), there exists a continuous function $f : X \longrightarrow \mathfrak{F}(I)$ such that for every $x \in X$,

$$\beta(x) \leq f(x)(1-) \leq f(x)(0+) \leq \alpha(x).$$

MV-uniformities

These results are inspired in [3][Hutton, 1977]

Let $\mathbb{B}$ denote the family of all the functions $f : [0,1]^X \longrightarrow [0,1]^X$ with the following properties:

1. $f(\mathbf{0}) = \mathbf{0}$
2. $\alpha \leq f(\alpha)$ for each $\alpha \in [0,1]^X$
3. $f(\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I} f(\alpha_i)$ for each family $\{\alpha_i\} \subseteq [0,1]^X$,
4. $f(\alpha \oplus \beta) \leq f(\alpha) \oplus f(\beta)$ for each $\alpha, \beta \in [0,1]^X$.

MV-uniformities

Definition

An MV-uniformity on a set X is a subset $\mathfrak{D}$ of $\mathbb{B}$ such that:

1. $\mathfrak{D} \neq \emptyset$
2. $f \in \mathfrak{D}$ and $f \leq g$, with $g \in \mathbb{B}$, implies $g \in \mathfrak{D}$
3. $f \in \mathfrak{D}$ and $g \in \mathfrak{D}$ implies $f \odot g \in \mathfrak{D}$
4. $f \in \mathfrak{D}$ implies there exists $g \in \mathfrak{D}$ such that $g \circ g \leq f$
5. $f \in \mathfrak{D}$ implies $f^{-1} \in \mathfrak{D}$ (where $f^{-1}(\alpha) = \bigwedge\{\beta : f(\beta^*) \leq \alpha^*\}$

The pair $(X, \mathfrak{D})$ is called an MV-uniform space.

MV-uniformities

Proposition

Let $(X, \mathfrak{D})$ be an MV-uniform space. The operator Int : $[0,1]^X \to [0,1]^X$ *defined by*

$$\mathrm{Int}(\beta) = \bigvee \{\alpha \in [0,1]^X : f(\alpha) \leq \beta \text{ for some } f \in \mathfrak{D}\}$$

is an MV-interior operator.

Definition

The MV-topology generated by an MV-uniformity $\mathfrak{D}$ is the MV-topology generated by the MV-interior operator Int of the previous proposition.

Definition (Hutton, 1977)

For $\epsilon > 0$, we define

$$B_\epsilon : [0,1]^{\mathfrak{F}(I)} \longrightarrow [0,1]^{\mathfrak{F}(I)}$$

by $B_\epsilon(\alpha) = \bigwedge\{R_{s-\epsilon} : \alpha \leq L_s^*\}$.

Theorem

The set $\{B_\epsilon, B_\epsilon^{-1} : \varepsilon > 0\}$ is a sub-basis for an MV-uniformity on $\mathfrak{F}(I)$. The MV-topology generated by the MV-uniformity is the usual MV-topology. This MV-uniformity is called the usual MV-uniformity for the usual MV-topology on $\mathfrak{F}(I)$.

MV-uniformities and Complete Regularity

Definition

An MV-topological space (X, τ) is completely regular if for each $\alpha \in \tau$ there are a family of fuzzy sets $\{\gamma_i : i \in I\}$ and a family of maps $\{f_i : X \longrightarrow \mathfrak{F}(I) \mid i \in I\}$ such that $\bigvee_{i \in I} \gamma_i = \alpha$ and

$$\gamma_i(x) \leq f_i(x)(1-) \leq f_i(x)(0+) \leq \alpha(x)$$

for all $i \in I$ and $x \in X$.

Theorem

The MV-topological space generated by an MV-uniform space is completely regular. The converse is true, if (X, τ) is a completely regular MV-space, then is MV-uniformizable.

Related topics for research

- De La Pava L., *Normality and Urysohn-type Lemma*. (preprint).
- De La Pava L. and Bolaños, Marby. *MV-systems and MV-uniformities* (In preparation).
- Convergence Theory.
- "Pointfree" MV-topology or MV-frames.
- Study the MV-fuzzy interval.
- Algebraic MV-topology?

Main References

De La Pava, L.V. and Russo, C., Compactness in MV-topologies: Tychonoff Theorem and Stone–Čech Compactification. *Archive for Mathematical Logic*, **59** (2020) 57–79.

Hutton, B.; Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50** (1975), 74–79.

Hutton, B.; Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.* **58** (1977), 559–571.

Russo, C., An extension of Stone duality to fuzzy topologies and MV-algebras. *Fuzzy Sets and Systems*, **303** (2016), 80–96.

Thank you!
¡Gracias!