Satisfiability degree for BCK-algebras

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Overview

Theorem (Gustafson 1973)

Let G be a finite non-Abelian group. Choose $x, y \in G$ uniformly randomly with replacement. The probability that $xy = yx$ is $\leq \frac{5}{8}$.

Said another way: In any finite group, the probability that two randomly chosen elements satisfy $xy = yx$ is either

- 1 (if G is Abelian), or
- $\leq \frac{5}{8}$.

There is a *gap* in the possible probabilities.

There is a substantial literature about commuting probabilities in finite groups. There is a good survey by Das, Nath, and Pournaki (2013).

There are several natural ways to generalize this. For example:

- What can we say about commuting probabilities in other algebraic systems?
- What is the probability that an algebraic system satisfies some specified first-order formula?

MacHale investigated commuting probability for finite rings in 1976. While less has been written about commuting probabilities in the case of rings, there is some recent work by

Buckley and MacHale (2013)

Buckley, MacHale, and Ní Shé (2013)

Basnet, Dutta, and Nath (2017)

Probabilities for some commutator-like equations in groups were considered by Lescot (1995), and more recently by Delizia, Jezernik, Moravec, et al (2020).

Kocsis (2020) studied probabilities for some equations generalizing the commutativity equation in groups.

The primary inspiration for the present work is a recent paper of Bumpus and Kocsis (2022) which considers probability questions for the class of Heyting algebras.

Definition

Given a first-order language $\mathcal{L}$, a finite $\mathcal{L}$-structure M, and an $\mathcal{L}$-formula $\varphi(x_1, x_2, \dots x_n)$ in n variables, the quantity

$$\mathrm{ds}(\varphi, M) = \frac{|\{ (a_1, a_2, \dots, a_n) \in M^n \mid \varphi(a_1, a_2, \dots a_n) \}|}{|M|^n}$$

is the *degree of satisfiability* of the formula φ in the structure M.

Definition

Let T be a theory over a first-order language $\mathcal{L}$ and φ an $\mathcal{L}$-formula. We say that φ has *finite satisfiability gap* ε in T if there is a constant $\varepsilon > 0$ such that, for every finite model M of T, either $\mathrm{ds}(\varphi, M) = 1$ or $\mathrm{ds}(\varphi, M) \leq 1 - \varepsilon$.

Example

Gustafson's result can be rephrased as saying the equation $xy = yx$ has finite satisfiability gap $\frac{3}{8}$ in the language of groups.

Theorem (Bumpus and Kocsis 2022)

In the language of Heyting algebras,

- the equations $x = \top$ and $\neg x = \top$ have satisfiability gap $\frac{1}{2}$,
- the equation $x \vee \neg x = \top$ has satisfiability gap $\frac{1}{3}$.

Further, these are the only formulas in one variable with finite satisfiability gap.

Definition

A *BCK-algebra* is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of type $(2, 0)$ such that

1. $\big[(x \cdot y) \cdot (x \cdot z)\big] \cdot (z \cdot y) = 0$
2. $\big[x \cdot (x \cdot y)\big] \cdot y = 0$
3. $x \cdot x = 0$
4. $0 \cdot x = 0$
5. $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

for all $x, y, z \in A$.

These algebras are partially ordered by: $x \leq y$ iff $x \cdot y = 0$.

The element $x \wedge y := x \cdot (x \cdot y)$ is a lower bound for x and y. If $x \wedge y = y \wedge x$ for all $x, y \in \mathbf{A}$, we say $\mathbf{A}$ is *commutative*.

A *bounded BCK-algebra* is an algebra $\mathbf{A} = \langle A; \cdot, 0, 1\rangle$ of type (2,0,0) satisfying (1-5) and $x \cdot 1 = 0$ for all $x \in A$.

If **A** is bounded, we define the term operation $\neg x := 1 \cdot x$.

If **A** is bounded and commutative, we define the term operation

$$x \vee y := \neg(\neg x \wedge \neg y)\,.$$

For the class `bBCK` of bounded BCK-algebras, consider the equation

(DN) $\neg\neg x = x$

For the class `bcBCK` of bounded commutative BCK-algebras, consider the equation

(EM) $x \vee \neg x = 1$

Let $\mathbf{A}$ be a bounded BCK-algebra of order n and put

$$D(\mathbf{A}) = \{\, x \in \mathbf{A} \mid \neg\neg x = x \,\}\,.$$

Define the *double negation degree of* $\mathbf{A}$ to be the degree of satisfiability of the equation $\neg\neg x = x$:

$$\mathrm{dnd}(\mathbf{A}) = \mathrm{ds}(\mathrm{DN}, \mathbf{A}) = \frac{|D(\mathbf{A})|}{n}\,.$$

This is just the probability that a randomly chosen element is fixed under double negation.

It is easy to check that $\{0, 1\} \subseteq D(\mathbf{A})$, so $|D(\mathbf{A})| \geq 2$.

If $\mathbf{A}$ is commutative, then $D(\mathbf{A}) = \mathbf{A}$ and so $\mathrm{dnd}(\mathbf{A}) = 1$.

Thus, for a bounded non-commutative BCK-algebras $\mathbf{A}$, we have

$$\frac{2}{n} \leq \operatorname{dnd}(\mathbf{A}) \leq \frac{n-1}{n}.$$

Theorem

For each $n \geq 3$, there are bounded non-commutative BCK-algebras realizing these bounds.

Proof of lower bound.

Let $\mathbf{A}$ be any BCK-algebra of order $n-1$. Define a one-element extension $\mathbf{A} \oplus \top$ with new top element $\top$ as follows:

$$x \cdot \top = 0$$
$$\top \cdot \top = 0$$
$$\top \cdot x = \top$$

for all $x \in \mathbf{A}$. This is known as Iséki's extension of $\mathbf{A}$.

Thus, for a bounded non-commutative BCK-algebras $\mathbf{A}$, we have

$$\frac{2}{n} \leq \mathrm{dnd}(\mathbf{A}) \leq \frac{n-1}{n}\,.$$

Theorem

For each $n \geq 3$, there are bounded non-commutative BCK-algebras realizing these bounds.

Proof of lower bound cont'd.

Iséki's extension always yields a bounded non-commutative BCK-algebra. Note that

$$\neg\neg x = \top \cdot (\top \cdot x) = \begin{cases} 0 & \text{if } x \in \mathbf{A} \\ \top & \text{if } x = \top \end{cases},$$

so only 0 and $\top$ are fixed by double negation. Hence, $\mathrm{dnd}(\mathbf{A} \oplus \top) = \frac{2}{n}$. □

Thus, for a bounded non-commutative BCK-algebras $\mathbf{A}$, we have

$$\frac{2}{n} \leq \mathrm{dnd}(\mathbf{A}) \leq \frac{n-1}{n}\,.$$

Theorem

For each $n \geq 3$, there are bounded non-commutative BCK-algebras realizing these bounds.

Corollary

In the language of bBCK, the equation $\neg\neg x = x$ has no finite satisfiability gap.

Let $\mathbf{A} \in \texttt{bcBCK}$ with order n and put

$$E(\mathbf{A}) = \{\, x \in \mathbf{A} \mid x \vee \neg x = 1 \,\}\,.$$

Define the *excluded middle degree of* $\mathbf{A}$ to be

$$\mathrm{emd}(\mathbf{A}) = \frac{|E(\mathbf{A})|}{n}\,.$$

This is just the probability that a randomly chosen element satisfies the law of the excluded middle.

It is easy to check that $\{0, 1\} \subseteq E(\mathbf{A})$, so $|E(\mathbf{A})| \geq 2$.

Theorem

In the language of bcBCK, the equation $x \vee \neg x = 1$ has satisfiability gap $\frac{1}{3}$.

For $n \geq 2$, let $C_n = \{0, 1, 2, \ldots, n-1\}$. This becomes a (commutative) BCK-algebra $\mathbf{C}_n$ when equipped with the operation $x \cdot y = \max\{x - y, 0\}$.

Lemma

$\text{emd}(\mathbf{C}_n) = \frac{2}{n}$

Lemma

$\text{emd}(\mathbf{A} \times \mathbf{B}) = \text{emd}(\mathbf{A}) \cdot \text{emd}(\mathbf{B})$

Theorem (Romanowska and Traczyk, 1980)

Every finite commutative BCK-algebra is a product of chains.

Theorem

In the language of bcBCK, the equation $x \vee \neg x = 1$ has satisfiability gap $\frac{1}{3}$.

Proof.

From the first lemma, we have $\mathrm{emd}(\mathbf{C}_2) = 1$ and $\mathrm{emd}(\mathbf{C}_3) = \frac{2}{3}$. Let $\mathbf{A} \in$ bcBCK be finite with order n. Then by R & T's theorem,

$$\mathbf{A} \cong \mathbf{C}_{j_1} \times \mathbf{C}_{j_2} \times \cdots \times \mathbf{C}_{j_k},$$

where $n = j_1 j_2 \cdots j_k$. Since emd is multiplicative, we have

$$\mathrm{emd}(\mathbf{A}) = \prod_{i=1}^{k} \mathrm{emd}(\mathbf{C}_{j_i}) = \frac{2}{j_1} \cdot \frac{2}{j_2} \cdots \frac{2}{j_k} = \frac{2^k}{n}.$$

If all $j_i = 2$, then $\mathrm{emd}(\mathbf{A}) = 1$. Suppose o/w. WLOG, suppose $j_1 > 2$. Then $\mathrm{emd}(\mathbf{A}) = \frac{2^k}{n} \le \frac{2}{3}$.

□

Let **A** be a BCK-algebra of order n and put

$$C(\mathbf{A}) = \{\,(x,y) \in A^2 \mid x \wedge y = y \wedge x\,\}$$
$$P(\mathbf{A}) = \{\,(x,y) \in A^2 \mid (x \cdot y) \cdot y = x \cdot y\,\}$$
$$I(\mathbf{A}) = \{\,(x,y) \in A^2 \mid x \cdot (y \cdot x) = x\,\}$$

We define

- the *commuting degree* of **A** to be
$$\mathrm{cd}(\mathbf{A}) = \frac{|C(\mathbf{A})|}{n^2}\,.$$
- the *positive implicative degree* of **A** to be
$$\mathrm{pid}(\mathbf{A}) = \frac{|P(\mathbf{A})|}{n^2}\,.$$
- the *implicative degree* of **A** to be
$$\mathrm{id}(\mathbf{A}) = \frac{|I(\mathbf{A})|}{n^2}\,.$$

Theorem

The following bounds are sharp:

- $\frac{3n-2}{n^2} \leq \mathrm{cd}(\mathbf{A}) \leq \frac{n^2-2}{n^2}$
- $\frac{4n-4}{n^2} \leq \mathrm{pid}(\mathbf{A}) \leq \frac{n^2-1}{n^2}$
- $\frac{4n-4}{n^2} \leq \mathrm{id}(\mathbf{A}) \leq \frac{n^2-1}{n^2}$.

Consequently, none of the equations

$$x \wedge y = y \wedge x \,,$$
$$(x \cdot y) \cdot y = x \cdot y \,, \text{ or}$$
$$x \cdot (y \cdot x) = x$$

has a finite satisfiability gap in BCK.

$\cdot$	0	1
0	0	0
1	1	0

$\cdot$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

$\cdot$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Table: The algebras $\mathbf{2}$, $\mathbf{3}^{\mathrm{p}}$, and $\mathbf{3}^{\mathrm{c}}$

$\mathbf{2}$ is the unique BCK-algebra of order 2, and it is implicative

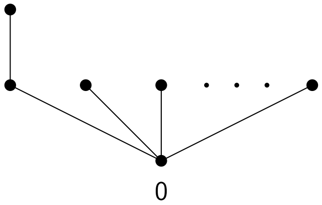
$\mathbf{3}^{\mathrm{p}}$ is positive implicative but not commutative

$\mathbf{3}^{\mathrm{c}}$ is commutative but not positive implicative

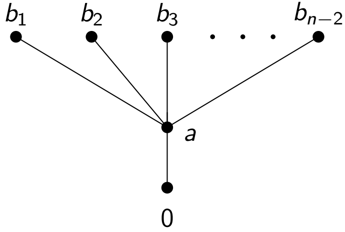
All three are linear.

Min. value of cd is obtained by $\mathbf{3}^{\mathrm{p}} \oplus_{i=1}^{n} \top$

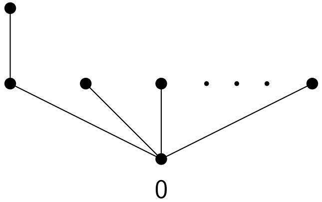
Max. value of cd is obtained by $\mathbf{3}^{\mathrm{p}} \sqcup \left(\bigsqcup_{i=1}^{n-1} \mathbf{2} \right)$



Min. value of pid/id is obtained by



Max. value of pid/id is obtained by $\mathbf{3}^{\mathrm{c}} \sqcup \left(\bigsqcup_{i=1}^{n-1} \mathbf{2} \right)$



For $n = 3$, 4, and 5, every possible value of dnd, cd, pid, and id is obtained by some algebra.

Conjecture

The above is true for all $n \geq 3$.

Questions

- Which equations in one variable have finite satisfiability degree?
- Are there any equations in two variables with finite satisfiability degree?
- Is there a way to do this with infinite algebras?

Thanks!

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