Universal models for classes of abelian groups for purity

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Outline

Basic notions: Abstract class

Abstract class (Grossberg)

An *abstract class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where:

- K is class of $\tau(\mathbf{K})$ structures.
- $\leq_{\mathbf{K}}$ is a partial order on K.
- If $M \leq_{\mathbf{K}} N$, then M is a substructure of N.
- $(K, \leq_{\mathbf{K}})$ is closed under isomorphisms.

$\mathbf{K}$-embeddings

$f : M \to N$ is a **K**-embedding if $f : M \cong f[M] \leq_{\mathbf{K}} N$.

Notation: $\mathbf{K}_\lambda = \{M \in K : M \text{ has cardinality } \lambda\}$

Basic notions: Universal models

Universal model

- $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$.
- We say that $\mathbf{K}$ has a universal model of cardinality λ if there is a universal model in $\mathbf{K}_\lambda$.

Example

($\mathbb{Q}$-vector spaces, $\subseteq$): For every λ, $\mathbb{Q}^{(\lambda)}$ is a universal model of size λ.

Basic notions: Abelian groups

All groups are abelian groups.

p-groups

G is a p-group if every element of G has order p^n for some some n.

Example

$\mathbb{Z}(p), \mathbb{Z}(p^2), \mathbb{Z}(p^\infty)$.

Torsion-free

G is torsion-free if every element different from zero has infinite order.

Basic notions: Pure subgroup

Example

- (p-groups, $\subseteq$): For every λ, $\mathbb{Z}(p^\infty)^{(\lambda)}$ is a universal model of size λ.
- (torsion-free, $\subseteq$): For every λ, $\mathbb{Q}^{(\lambda)}$ is a universal model of size λ.

Pure subgroup (Prüfer)

$G \leq_p H$ if for every $n \in \mathbb{N}$, $nH \cap G = nG$.

Example

$G \leq_p G \oplus H$, $t(G) \leq_p G$, $\mathbb{Z} \not\leq_p \mathbb{Q}$.

Outline

Main Problem

Main Problem (*Abelian groups* by L. Fuchs)

For which cardinals λ, does $(p\text{-groups}, \leq_p)$ has a universal model of cardinality λ? The same question for torsion-free abelian groups with pure embeddings.

$\mathbb{Z}(p^\infty)^{(\lambda)}$ is not a universal group in $(p\text{-groups}, \leq_p)_\lambda$.

Outline

1. Basic notions
2. Main problem
3. **Existence results**
4. Non-existence results
5. Conclusions

Existence results: Abstract elementary classes

- Abstract elementary classes were introduced by Shelah in the 70's.
- Shelah's eventual categoricity conjecture.
- The abstract theory has developed rapidly.

Abstract elementary class (Shelah)

An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where:

1. $\mathbf{K} = (K, \leq_{\mathbf{K}})$ is an abstract class.
2. Tarski-Vaught axioms.
3. Löwenheim-Skolem-Tarski axiom ($LS(\mathbf{K})$).

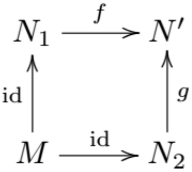
Existence results: Abstract elementary classes

- (p-groups, $\leq$) and (p-groups, $\leq_p$)
- (torsion-free, $\leq$) (torsion-free, $\leq_p$).
- ($\mathrm{Mod}(T), \preceq$) for T a complete first-order theory.
- ($\mathbb{R}$-VS, $\subseteq$).
- $(Ab, \leq)$ and $(Ab, \leq_p)$.
- $(RTF, \leq_p)$.
- ($\aleph_1$-free, $\leq_p$).
- (Tor, $\leq_p$)
- (R-Mod, $\leq$) and (R-Mod, $\leq_p$).
- (R-Flat, $\leq_p$).
- (R-Absp, $\leq_p$).
- (R-l-inj, $\leq_p$) and (R-l-pi, $\leq_p$).

Existence results: Some properties

Amalgamation property (AP)

Every $M \leq_{\mathbf{K}} N_1, N_2$ can be completed to a commutative square in **K**.



Some properties

1. **K** has JEP: if every $M, N \in K$ can be **K**-embedded to a model in **K**.
2. **K** has NMM: if every $M \in K$ can be properly extended in **K**.

Existence results: Some properties

(p-groups, $\leq_p$) is an AEC such that $LS(\mathbf{K}) = \aleph_0$ with AP, JEP, NMM.

Answer under GCH ($2^\lambda = \lambda^+$)

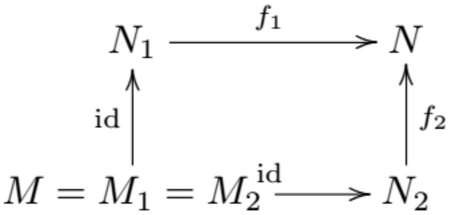
There is a universal model for every uncountable cardinal for both (p-groups, $\leq_p$).

Existence results: Galois-types and stability

We assume amalgamation for simplicity.

Pre-types

1. $\mathbf{K}^3 = \{(a, M, N) : M, N \in \mathbf{K}, M \leq_{\mathbf{K}} N \text{ and } a \in N\}$.
2. For $(a_1, M_1, N_1), (a_2, M_2, N_2) \in \mathbf{K}^3$, we say $(a_1, M_1, N_1)E(a_2, M_2, N_2)$ if:



$$f_1(a_1) = f_2(a_2)$$

Existence results: Galois-types and stability

Galois-types

1. For $(a, M, N) \in \mathbf{K}^3$, let $\mathbf{gtp}_{\mathbf{K}}(a/M; N) := [(a, M, N)]_E$.
2. $\mathbf{gS}(M)$ is the set of Galois-types over M.

T is a complete first-order theory: $(Mod(T), \preceq)$

$\mathbf{gtp}_{\mathbf{K}}(a/M; N_1) = \mathbf{gtp}_{\mathbf{K}}(b/M; N_2)$ if and only if $tp(a/M, N) = tp(b/M, N)$

Existence results: Galois-types and stability

pp-formulas and *pp*-types

- ϕ is a positive primitive formula (*pp*-formula) if it is an existentially quantified finite system of linear equations.
- $pp(\bar{b}/A, N)$ are the *pp*-formulas satisfied by $\bar{b}$ in N with parameters in A.

Lemma (M.)

Let $M, N_1, N_2 \in p$-groups, $M \leq_p N_1, N_2$, $b_1 \in N_1$ and $b_2 \in N_2$. Then:

$$\mathbf{gtp}(b_1/M; N_1) = \mathbf{gtp}(b_2/M; N_2) \text{ iff } \mathrm{pp}(b_1/M, N_1) = \mathrm{pp}(b_2/M, N_2).$$

Existence results: Galois-types and stability

Stable

- **K** is λ-stable if $|\mathbf{gS}(M)| \leq \lambda$ for all $M \in \mathbf{K}$ of cardinality λ.
- **K** is stable if there is a λ such that **K** is λ-stable.

Lemma (M.)

$\lambda^{\aleph_0} = \lambda$ if and only if (p-groups, $\leq_p$) is λ-stable.

Question

Let R be an associative ring with unity.
If $(K, \leq_p)$ is an AEC of modules, is **K** stable? Is this true if $R = \mathbb{Z}$? Under what conditions on R is this true?

Existence results

Theorem (M.)

If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$, then $(p\text{-groups}, \leq_p)$ has a universal model of cardinality λ.

Proof sketch.

- If $\lambda^{\aleph_0} = \lambda$, then $(p\text{-groups}, \leq_p)$ is λ-stable.
- $(p\text{-groups}, \leq_p)$ has AP, JEP and NMM.
- (with Kucera) $(p\text{-groups}, \leq_p)$ has a universal model of cardinality λ.

Lemma (Kucera-M.)

Let $\mathbf{K}$ be an AEC with AP, JEP and NMM. Assume there is a κ such that if $\theta^\kappa = \theta$, then $\mathbf{K}$ is θ-stable.
If $\lambda^\kappa = \lambda$ or $\forall \mu < \lambda (\mu^\kappa < \lambda)$, then $\mathbf{K}$ has a universal model of size λ.

Existence results: $\aleph_0$ (joint work with Ivo Herzog)

$q(\bar{x})$ is a *complete p-torsion pp-type* if there is a p-group M and $\bar{a} \in M^n$ with $q(\bar{x}) = pp(\bar{a}/\emptyset, M)$.

Lemma (Herzog-M.)

The number of complete p-torsion pp-types of any finite length is countable.

Theorem (Herzog-M.)

(p-groups, $\leq_p$) has a universal model of cardinality $\aleph_0$.

Proof sketch.

- $T = Th(\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\aleph_0)} \oplus \mathbb{Z}(p^\infty)^{(\aleph_0)})$.
- Build U a countable saturated model for p-torsion pp-types.
- U is universal for countable models of T.
- The results follows form the fact that every p-group can be purely embedded into a model of T of the same size.

Existence results: $\aleph_0$ (joint work with Ivo Herzog)

Theorem (Herzog-M.)

$\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\aleph_0)} \oplus \mathbb{Z}(p^{\infty})^{(\aleph_0)}$ is a universal model of cardinality $\aleph_0$ in $(p\text{-groups}, \leq_p)_{\aleph_0}$.

Remark

- (Kojman-Shelah) The class of reduced p-groups with pure embeddings and (torsion-free, $\leq_p$) do not have a universal model of cardinality $\aleph_0$.
- Even the countable groups of infinite length can be purely embedded into $\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\aleph_0)} \oplus \mathbb{Z}(p^{\infty})^{(\aleph_0)}$.

Outline

Non-existence results: Club guessing sequences

Kojman and Shelah, *Universal abelian groups*.
Mirna Džamonja, *Club guessing and the universal models*.

Club guessing sequences

Let λ be a regular cardinal and S be a stationary subset of λ.
$\bar{A} = \{A_\alpha : \alpha \in S\}$ is a club guessing sequence for S if:

1. For every $\alpha \in S$, $A_\alpha \subseteq \alpha$ and A_α is unbounded.
2. For every $D \subseteq \lambda$, if D is a club then $\{\alpha \in S : A_\alpha \subseteq D\}$ is not empty.

Non-existence results

Lemma (M.)

Let λ and μ be regular cardinals. If $\mu^+ < \lambda < \mu^{\aleph_0}$, then ($p$-groups, $\leq_p$) does not have a universal model of cardinality λ.

Proof sketch.

- (M.) Assume for the sake of contradiction that there is G a universal p-group.
- (Shelah) There is a club guessing sequence for S^λ_μ using that $\mu^+ < \lambda$.
- (Kojman-Shelah) Introduce an invariant $P_\alpha(\bar{G}, \bar{A})$ which is preserved under pure embeddings.
- (K.-S.) There is a p-group of size λ that cannot be purely embedded into G because of the invariant using that $\lambda < \mu^{\aleph_0}$.

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Conclusions: An answer below $\aleph_\omega$

Theorem (M.)

For $n \geq 2$, $(p\text{-groups}, \leq_p)$ has a universal model of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.

Proof sketch.

- $\Leftarrow$: $\aleph_n^{\aleph_0} = \aleph_n$.
- $\Rightarrow$: $\aleph_0^+ < \aleph_n < \aleph_n^{\aleph_0}$.

Lemma (M.)

- If $2^{\aleph_0} = \aleph_1$, then $(p\text{-groups}, \leq_p)$ has a universal model of cardinality $\aleph_1$.
- If $2^{\aleph_0} > \aleph_1$ and $\clubsuit$, then $(p\text{-groups}, \leq_p)$ does not have a universal model of cardinality $\aleph_1$.

$\clubsuit$ holds if there is a club guessing sequence in $\aleph_1$.

Conclusions: Torsion-free groups

Theorem (Kojman-Shelah, Shelah, Kucera-M.)

If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then $(\text{torsion-free}, \leq_p)$ has a universal model of cardinality λ.

Lemma (Kojman-Shelah)

If $\lambda < 2^{\aleph_0}$, then $(\text{torsion-free}, \leq_p)$ does not have a universal model of cardinality λ.

An answer below $\aleph_\omega$

- There is no universal model of cardinality $\aleph_0$ for $(\text{torsion-free}, \leq_p)$.
- For $n \geq 1$, $(\text{torsion-free}, \leq_p)$ has a universal model of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.

Conclusions: Some questions

Questions

- Does (p-groups, $\leq_p$) have a universal model of cardinality $\aleph_\omega$?
- What about other singular cardinals? For example for λ singular when $\mu^+ < \lambda < \mu^{\aleph_0}$.

References

- Marcos Mazari-Armida, *A model theoretic solution to a problem of László Fuchs*, Journal of Algebra **567** (2021), 196–209.
- Ivo Herzog and Marcos Mazari-Armida, *A countable universal torsion abelian group for purity*, in preparation.

Thank you!