# Mal'tsev products of varieties

**Anna Romanowska** and Tomasz Penza
Warsaw University of Technology
email: Anna.Romanowska@pw.edu.pl

## OUTLINE

- Mal'tsev products in general
- Mal'tsev products of varieties of algebras
- The role of idempotent elements
- Identities true in Mal'tsev products of varieties
- When is a Mal'tsev product of two varieties a variety?
- Some sufficient conditions and their consequences

# MAL'TSEV PRODUCTS

Fix a type $\tau\colon \Omega \to \mathbb{N}$ of $\Omega$-algebras (without nullary operations), and the class $\mathcal{T}$ of all $\Omega$-algebras.

The **Mal'tsev product** of two subclasses $\mathcal{Q}$ and $\mathcal{R}$ of $\mathcal{T}$ is the class

$$\mathcal{Q} \circ \mathcal{R} := \{A \in \mathcal{T} \mid \exists \theta \in CgA \text{ such that } A/\theta \in \mathcal{R} \text{ and } (\forall a \in A)\ (a/\theta \leq A \Rightarrow a/\theta \in \mathcal{Q})\}$$

## MAL'TSEV PRODUCTS OF QUASIVARIETIES

• If $\mathcal{Q}$ and $\mathcal{R}$ are quasivariety and the type is finite (or $\mathcal{R}$ is idempotent), then $\mathcal{Q} \circ \mathcal{R}$ is a quasivariety (Mal'tsev 1967).

In this case the congruence $\theta$ may be taken to be the $\mathcal{R}$-**replica congruence** $\varrho$ of $A$, the smallest congruence of $A$ whose induced quotient falls into the quasivariety $\mathcal{R}$.

$$\begin{aligned}\mathcal{Q} \circ \mathcal{R} = &\{A \in \mathcal{T} \mid A/\varrho \in \mathcal{R} \text{ and} \\ &(\forall a \in A)\ (a/\varrho \leq A \Rightarrow a/\varrho \in \mathcal{Q})\}.\end{aligned}$$

# MAL'TSEV PRODUCTS OF VARIETIES

Let $\mathcal{V}$ and $\mathcal{W}$ be subvarieties of $\mathcal{T}$.

Then each $\Omega$-algebra $A$ has its largest quotient $A/\varrho$ in $\mathcal{W}$, the $\mathcal{W}$-**replica of** $A$, by the $\mathcal{W}$-**replica congruence** $\varrho$.

$\mathcal{V} \circ \mathcal{W} = \{A \in \mathcal{T} \mid (\forall a \in A)\,(a/\varrho \leq A \Rightarrow a/\varrho \in \mathcal{V})\}$.

- $\mathcal{V} \circ \mathcal{W}$ is a quasivariety, not necessarily a variety (**Mal'tsev, 1967**).

## A PROBLEM

**Problem: When is the quasivariety $\mathcal{V} \circ \mathcal{W}$ a variety?**

**Note** that:

- $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V} \vee_Q \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}$,

and if $\mathcal{V} \circ \mathcal{W}$ is a variety, then

- $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V} \vee_Q \mathcal{W} \subseteq \mathcal{V} \vee_V \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}$.

## SOME MOTIVATION

It is possible that $\mathcal{V} \circ \mathcal{W}$ is not a variety, but the intersection $(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$ with another subvariety $\mathcal{U}$ of $\mathcal{T}$ is.

But if $\mathcal{V} \circ \mathcal{W}$ is a variety, then $(\mathcal{V} \circ \mathcal{W}) \cap \mathcal{U}$ is necessarily a variety.

**Theorem** [McLean, 1954]: Each band (idempotent semigroup) is a semilattice of rectangular bands.

Thus $\mathcal{B} = (\mathcal{RB} \circ \mathcal{S}) \cap \mathcal{SG}$, where
$\mathcal{RB}$ - the variety of rectangular band;
$\mathcal{S}$ - the variety of semilattices;
$\mathcal{B}$ - the variety of bands (idempotent semigroups);
$\mathcal{SG}$ - the variety of semigroups.

However, it was not clear if $\mathcal{RB} \circ \mathcal{S}$ is a variety.

## THE ROLE OF IDEMPOTENTS

For a congruence $\theta$ of an algebra $A$, and $a \in A$,
$a/\theta \leq A$ if and only if $a/\theta$ is an idempotent element of $A/\theta$.

• If no non-trivial $\mathcal{W}$-algebra has idempotent elements, then $\mathcal{V} \circ \mathcal{W}$ contains all $\Omega$-algebras.
(Each of them has a $\mathcal{W}$-replica congruence $\varrho$, and no corresponding congruence classes are subalgebras.)

• If $\mathcal{W}$ is idempotent, then each $a/\varrho$ is a subalgebra, and

$$\mathcal{V} \circ \mathcal{W} = \{A \mid (\forall a \in A)\ (a/\varrho \in \mathcal{V})\}.$$

We say that $A$ is a $\mathcal{W}$-*sum of* $\mathcal{V}$-*algebras*.

• Finally it may happen that, though $\mathcal{W}$ is not idempotent, all $\mathcal{W}$-algebras have some idempotent elements. (Examples are provided by inverse semigroups.)

# IDENTITIES

$X\Omega$ - the set of terms of type $\tau$ over countably infinite set $X$.

**Definition**: Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$,
let $\Sigma$ be an equational base for $\mathcal{V}$.
For any $u, v, r_i \in X\Omega$ define:

$$\begin{aligned}\Sigma^{\mathcal{W}} := \{u(r_1, \ldots, r_n) = v(r_1, \ldots, r_n) \mid \\ (u = v) \in \Sigma, \\ \forall\, i = 1, \ldots, n-1, \quad \mathcal{W} \models r_i = r_{i+1}, \\ \forall\, \omega \in \Omega, \quad \mathcal{W} \models \omega(r_i, \ldots, r_i) = r_i\}.\end{aligned}$$

## ...IDENTITIES

• An $\Omega$-term $t$ such that $\mathcal{W}$ satisfies

$$\omega(t,\ldots,t) = t,$$

for each $\omega \in \Omega$,is called a **term idempotent** of $\mathcal{W}$.

• Every identity in $\Sigma^{\mathcal{W}}$ is obtained from some identity of $\Sigma$ by substituting for its variables pairwise $\mathcal{W}$-equivalent term idempotents of $\mathcal{W}$.

# EXAMPLES

**Example** 1: Take $\mathcal{V} = \mathcal{LZ}$, the variety of left-zero bands defined by $xy = x$.

• Let $\mathcal{W} = \mathcal{S}$, the variety of semilattices. Then
$r = s$ is an identity true in $\mathcal{S}$ precisely if the terms $r$ and $s$ have the same sets of variables.

• Then $\Sigma^{\mathcal{W}} = \{r(x_1, \ldots, x_m) \cdot s(x_1, \ldots, x_m) = r(x_1, \ldots, x_m) \mid m = 1, 2, \ldots\}$.

## ...EXAMPLES

**Example** 2: Let $\mathcal{V} = \mathcal{LZ}$ and let $\mathcal{W} = \mathcal{CS}$, the variety of constant semigroups defined by $x \cdot y = z \cdot t$.

• $r = s$ is a (non-trivial) identity true in $\mathcal{CS}$ precisely if neither $r$ nor $s$ is a variable.

• For $r = r_1 \cdot r_2$, $\mathcal{CS} \models r \cdot r = r_1 \cdot r_2 = r$.

• $\Sigma^{\mathcal{CS}} = \{(r_1 \cdot r_2) \cdot (s_1 \cdot s_2) = r_1 \cdot r_2\} = \{r \cdot s = r \mid \mathcal{CS} \models r = s = c\}$, where $r_1, r_2, s_1, s_2$ are any groupoid terms, and $c := x \cdot x$.

## ...IDENTITIES

**Lemma**: For any $\Sigma \subseteq \mathrm{Id}(\mathcal{V})$, the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ satisfies the identities $\Sigma^{\mathcal{W}}$.

**Theorem** [PR21]: Let $\Sigma$ be an equational base for $\mathcal{V}$.
Then the variety $\mathsf{H}(\mathcal{V} \circ \mathcal{W})$, generated by $\mathcal{V} \circ \mathcal{W}$, is defined by $\Sigma^{\mathcal{W}}$.

**Corollary**: If $\Sigma$ is an equational base for $\mathcal{V}$ and $\mathcal{V} \circ \mathcal{W}$ is a variety, then $\Sigma^{\mathcal{W}}$ is an equational base for $\mathcal{V} \circ \mathcal{W}$.

# EXAMPLES

**Example** 1 **(cont.)**:
$\Sigma^{\mathcal{S}}$ of Example 1 is an equational base for $\mathsf{H}(\mathcal{LZ} \circ \mathcal{S})$.

**Example** 2 **(cont.)**:
$\Sigma^{\mathcal{CS}}$ of Example 2 is an equational base for $\mathsf{H}(\mathcal{LZ} \circ \mathcal{CS})$.

# A SUFFICIENT CONDITION

**Theorem** 1 [PR21]: Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, let $\mathcal{W}$ be an idempotent variety.

If there exist terms $f(x,y)$ and $g(x,y)$ such that

(a) $\mathcal{V} \models f(x,y) = x$ and $\mathcal{V} \models g(x,y) = y$,
(b) $\mathcal{W} \models f(x,y) = g(x,y)$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

## REGULAR and IRREGULAR VARIETIES

• An identity is called **regular** if both its sides contain precisely the same variables; otherwise it is **irregular**.

• A **strongly irregular variety** is a variety $\mathcal{V}_t$ defined by a set of regular identities and a **strongly irregular** identity $t(x,y) = x$.

• For any plural type $\tau$, the variety $\mathcal{S}_\tau$ of $\tau$-semilattices is the variety satisfying precisely the regular identities of type $\tau$.

The variety $\mathcal{S}_\tau$ and the variety $\mathcal{S}$ of semilattices are (definitionally) equivalent.

# SOME CONSEQUENCES

As a corollary of Theorem 1 one obtains the following theorem.

**Theorem** 2 [BPR20]: If $\mathcal{V}_t$ is a strongly irregular variety of a plural type $\tau$ and $\mathcal{S}_\tau$ is the variety of the same type $\tau$ equivalent to the variety of semilattices, then $\mathcal{V}_t \circ \mathcal{S}_\tau$ is a variety.
If $\Sigma$ is an equational base for $\mathcal{V}$, then $\Sigma^{\mathcal{W}}$ is an equational base for the Mal'tsev product $\mathcal{V}_t \circ \mathcal{S}_\tau$.

**Example** 4: The Mal'tsev product $\mathcal{LZ} \circ \mathcal{S}$ of Example 1 is a variety and is defined by $\Sigma^{\mathcal{S}}$.

# TERM IDEMPOTENT VARIETIES

Recall that a term $t$ is a term idempotent of $\mathcal{W}$, if $\mathcal{W}$ satisfies the identities

$$\omega(t, \ldots, t) = t, \tag{1}$$

for all $\omega \in \Omega$.

• A nontrivial identity $p = q$ satisfied in a variety $\mathcal{W}$ is called **term idempotent**, if both $p$ and $q$ are term idempotents of $\mathcal{W}$. A variety $\mathcal{W}$ is called **term idempotent**, if every nontrivial identitity it satisfies is term idempotent.

Note that every idempotent variety is term idempotent.

# EXAMPLE

Let $\mathcal{CS}$ be the variety of constant semigroups (defined by the identity $xy = zt$). A nontrivial identity $p = q$ is satisfied in $\mathcal{CS}$ precisely if neither $p$ nor $q$ is a variable. In particular $\mathcal{CS}$ satisfies $p \cdot p = p$ for every term $p$ different from a variable. Consequently, all such terms are term idempotents of $\mathcal{CS}$, and so $\mathcal{CS}$ is a term idempotent variety.

# THE EXTENDED SUFFICIENT CONDITION

**Theorem** 3 [PR22]: Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same type, and let $\mathcal{W}$ be term idempotent. If there exist terms $f(x, y, z)$ and $g(x, y, z)$ such that

(a) $\mathcal{V} \models f(x, y, y) = x$ and $\mathcal{V} \models g(x, x, y) = y$,

(b) $\mathcal{W} \models f(x, x, y) = g(x, x, y)$,

(c) $f(x, x, y)$ is a term idempotent of $\mathcal{W}$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

## POSSIBLE GENERALIZATIONS?

The assumption concerning strong irregularity of $\mathcal{V}$ in all sufficient conditions is essential.

**Proposition** [BPR20]: The Mal'tsev product $\mathcal{S} \circ \mathcal{S}$ fails to be a variety.

**Proposition** [PR22]: The Mal'tsev product $\mathcal{CS} \circ \mathcal{S}$ fails to be a variety.

## SOME COROLLARIES

**Corollary**: Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$, let $\mathcal{W}$ be a term idempotent variety.

If there exist terms $f(x, y)$ and $g(x, y)$ such that

(a) $\mathcal{V} \models f(x, y) = x$ and $\mathcal{V} \models g(x, y) = y$,
(b) $\mathcal{W} \models f(x, y) = g(x, y)$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

**Corollary**: Let $\mathcal{V}$ and $\mathcal{W}$ be nontrivial varieties, and let $\mathcal{W}$ be term idempotent. If there exists a term $f(x,y)$ such that

(a) $\mathcal{V} \models f(x,y) = x$,
(b) $\mathcal{W} \models f(x,y) = y$,

then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

**Corollary**: Let $\mathcal{V}$ be a congruence permutable variety and $\mathcal{W}$ be an idempotent variety. Then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

This generalizes the following.

**Theorem** [C. Bergman, 2020]: Let $\mathcal{V}$ and $\mathcal{W}$ be idempotent subvarieties of a congruence permutable variety. Then $\mathcal{V} \circ \mathcal{W}$ is a variety.

# POLARIZED VARIETIES

A variety $\mathcal{V}$ that has a **polar term** $p(x)$ (a constant unary term idempotent) is called **polarized**.

A variety is called **purely polarized**, if it is both polarized and term idempotent.

Typical examples of purely polarized varieties are the variety $\mathcal{CS}$ of constant semigroups and, more generally, the varieties $\mathcal{C}_\tau$ of constant algebras of type $\tau$.

**Theorem** [PR22]: Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of type $\tau$. If $\mathcal{W}$ is a purely polarized variety, then the Mal'tsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

**Corollary**: Let $\mathcal{V}$ be a variety of type $\tau$. Then the Mal'tsev product $\mathcal{V} \circ \mathcal{C}_\tau$ is a variety.

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