Central extensions, presentations and Hochschild-Serre in varieties with a difference term

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Hochschild-Serre Sequence

Cartan-Leray spectral sequences (homological algebra of abelian group chains) applied to particular chains (in low dimension related to group extensions): **M** is **G**-module

- $0 \to H^m(\mathbf{Q}, \mathbf{M}^K) \to H^m(\mathbf{G}, \mathbf{M}) \to H^m(\mathbf{K}, \mathbf{M})^{\mathbf{G}} \to H^{m+1}(\mathbf{Q}, \mathbf{M}^K) \to H^{m+1}(\mathbf{G}, \mathbf{M})$

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- Low-dimensional version ($m = 1$) with trivial action (central extensions):

$$0 \to \mathrm{Hom}(\mathbf{Q}, \mathbf{M}) \stackrel{\check{\sigma}}{\to} \mathrm{Hom}(\mathbf{G}, \mathbf{M}) \stackrel{\check{r}}{\to} \mathrm{Hom}(\mathbf{K}, \mathbf{M}) \stackrel{\delta}{\to} H^2(\mathbf{Q}, \mathbf{M}) \stackrel{\check{\sigma}}{\to} H^2(\mathbf{G}, \mathbf{M})$$

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- abelian **A** is $\mathbb{Z}$-module, $|\mathbf{A}| < \omega \Rightarrow \mathrm{Hom}(A,\mathbb{C}^\times) \approx A$
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Q: all nice commutator properties and abelian algebras + constants are **R**-modules
Q: Interpretation of spectral sequences with **R**-module coefficients

Extensions Realizing Affine Datum I

$$\mathbf{A}(\alpha) = \{(a,b) \in A^2 : (a,b) \in \alpha\}, \ \Delta_{\alpha\beta} = \mathrm{Cg}^{\mathbf{A}(\alpha)}\left(\left\{ \left\langle \begin{bmatrix} u \\ u \end{bmatrix}, \begin{bmatrix} v \\ v \end{bmatrix} \right\rangle : (u,v) \in \beta \right\} \right)$$

- $\mathbf{A}(\alpha)/\Delta_{\alpha\alpha}$ - (groups) $\mathbf{K} \leq \mathbf{H} \Rightarrow \mathbf{G}(\alpha_K)/\Delta_{\alpha_K \alpha_H} \approx \mathbf{K}/[\mathbf{K},\mathbf{H}] \rtimes_\phi \mathbf{G}/\mathbf{H}$

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$\delta : A \to A(\alpha)/\Delta_{\alpha\alpha}$ by $\delta(u) = \begin{bmatrix} u \\ u \end{bmatrix} /\Delta_{\alpha\alpha}$

Definition

Fix signature τ and ternary term symbol m. Define $\mathbf{A}^{\alpha,\tau} = \left(\mathbf{A}, \alpha, \{f^\Delta : f \in \tau\}\right)$ where

- $\mathbf{A} = \langle A, m \rangle$ is an algebra in the single operation symbol m;
- $\alpha \in \mathrm{Con}\,\mathbf{A}$;
- $\{f^\Delta : f \in \tau\}$ is a sequence of operations $f^\Delta : A(\alpha)/\Delta_{\alpha\alpha} \times \delta(A)^{n_f - 1} \to A(\alpha)/\Delta_{\alpha\alpha}$

affine if $\alpha \in \mathrm{Con}\,\mathbf{A}$ is abelian and $m(x,y,z) = x - y + z$ on α-blocks

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- actions: sequence of pairings $\{a(f,i) : a(f,i) : Q^i \times A^{n_f - i} \to A\}_{f \in \tau}$.
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$$\mathbf{Q} * \mathbf{A}^{\alpha,\tau} \vDash_* t(\bar{x}) = s(\bar{x}) \Leftrightarrow t^*(\bar{c}) = s^*(\bar{c}), \bar{c} \in C_t \cap C_s$$

Action $\mathbf{Q} * \mathbf{A}^{\alpha,\tau}$ *compatible* with $\mathcal{U}$ if $\mathbf{Q} * \mathbf{A}^{\alpha,\tau} \vDash_* \mathrm{Id}\,\mathcal{U}$

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- realization algebras $A_T : F_f = f^\Delta +_I \sum a(f,i) +_I T_f$ (universe $A(\alpha)/\Delta_{\alpha\alpha}$)

Extensions Realizing Affine Datum II

- 2^{nd}-cohomology group relative to $\mathcal{U}$ is

$\mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) := Z^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *)/B^2(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) = \mathcal{U}$-compatible 2-cocycles/2-coboundaries

- characterize extensions: $[\mathbf{A}_{T+T'}] = [\mathbf{A}_T] + [\mathbf{A}_{T'}]$
- $\mathbf{A} \approx \mathbf{A}(\alpha)/\Delta_{\alpha\alpha} \Leftrightarrow$ retraction $r : \mathbf{A} \to \mathbf{A}, \ker r = \alpha \Leftrightarrow$ hom section $s : \mathbf{A}/\alpha \to \mathbf{A}$
- cohomology groups parametrized by varieties containing datum:

$\mathcal{L}(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) =$ lattice of varieties containing datum
$H^2(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) =$ abelian group generated by some $\mathcal{U}$-compatible T

  - Galois connection $(-, \Psi) : \operatorname{Sub} \mathbf{H}^2(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) \to \mathcal{L}(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *)$ with

$\Psi(\mathcal{U}) := \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *)$

- Abelian group of derivations $Z^1(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) \approx \mathrm{Stab}(\pi : \mathbf{A} \to \mathbf{Q})$ automorphisms stabilizing extensions
- There is a notion of principal stabilizing automorphism which defines

$$\mathbf{H}^1(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) = \text{ derivations / principal stabilizing automorphisms}$$

- Central extensions $\Rightarrow \mathbf{H}^1(\mathbf{Q}, \mathbf{A}^{\alpha,\tau}, *) \approx \mathrm{Hom}(\mathbf{Q}, \mathbf{A}(\alpha)/\Delta_{\alpha\alpha})$
- trivial actions "$\equiv$" module terms ($x^q = qxq^{-1} = x$ – $\mathbb{Z}$-module)
  - in varieties with difference term, trivial actions characterize central extensions of affine datum

central extensions with difference term

- algebras $\mathbf{B}$ and $\mathbf{Q}$ in the same signature τ
- binary operation on B denoted by $x + y$
- operation symbol $f \in \tau$ ($n = \text{arity} f$) we have an operation $T_f : Q^n \to B$

Define a new algebra $\mathbf{B} \otimes^T \mathbf{Q}$ over the universe of the direct product $B \times Q$ where each operation symbol $f \in \tau$ is interpreted by the rule

$$F_f\left((b_1, q_1), \ldots, (b_n, q_n)\right) := \left(f(b_1, \ldots, b_n) + T_f(q_1, \ldots, q_n), f(q_1, \ldots, q_n)\right).$$

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Proposition

(basically Freese + McKenzie, CM book, Ch 7) Let $\mathcal{V}$ be a variety with a difference term and $\mathbf{A} \in \mathcal{V}$. If $\alpha \in \operatorname{Con} \mathbf{A}$ is abelian, then

$$\mathbf{A}/[\alpha, 1_A] \approx \mathbf{A}(\alpha)/\Delta_{\alpha 1} \otimes^T \mathbf{A}/\alpha.$$

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• $\mathcal{V}$ difference term: central extensions are reconstructed as $\mathbf{A} \approx \mathbf{B}^\tau \otimes^T \mathbf{Q}$ from datum $(\mathbf{B}^\tau, \mathbf{Q})$ where

- $\mathbf{B}^\tau$ is term-affine interpretable: $f^{\mathbf{B}}(x_1,\ldots,x_n) = a_1 \cdot x_1 + \cdots + a_n \cdot x_n \quad (a_i \in \mathbf{R})$
- $m(xyz)$ is idempotent

action is folded into module terms, 2-cocycle equations come from composition of T with module terms

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• *Inflation* maps:

$$\check{\sigma} : \mathrm{Hom}(\mathbf{Q}, \mathbf{E}^\tau) \longrightarrow \mathrm{Hom}(\mathbf{A}, \mathbf{E}^\tau)$$
$$\check{\sigma} : \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau) \longrightarrow \mathbf{H}^2_{\mathcal{U}}(\mathbf{A}, \mathbf{E}^\tau)$$

defined by

- $\check{\sigma}(\phi) := \phi \circ \pi$
- $\check{\sigma}([T]) := [T \circ \pi]$ where $(T \circ \pi)_f(a_1, \ldots, a_n) = T_f(\pi(a_1), \ldots, \pi(a_n))$ for $f \in \tau, (a_1, \ldots, a_n) \in A^n$

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- *Restriction* map $\check{r} : \text{Hom}(\mathbf{A},\mathbf{E}^\tau) \longrightarrow \text{Hom}(\mathbf{B}^\tau,\mathbf{E}^\tau)$ defined by

$$\check{r}(\phi)(b) := \phi(b,q) - \phi(0,q) \quad \text{for any choice of q } \in Q$$

Hochschild-Serre Sequence with difference term I

$$0 \to \mathrm{Hom}(\mathbf{Q},\mathbf{M}) \xrightarrow{\check{\sigma}} \mathrm{Hom}(\mathbf{G},\mathbf{M}) \xrightarrow{\check{r}} \mathrm{Hom}(\mathbf{K},\mathbf{M}) \xrightarrow{\delta} H^2(\mathbf{Q},\mathbf{M}) \xrightarrow{\check{\sigma}} H^2(\mathbf{G},\mathbf{M})$$

Fix $\pi : \mathbf{A} \approx \mathbf{B}^\tau \otimes^T \mathbf{Q} \to \mathbf{Q} \quad (\mathbf{A}(\alpha)/\Delta_{\alpha 1} \approx \mathbf{B}^\tau)$.

- *Inflation* maps:

$$\check{\sigma} : \mathrm{Hom}(\mathbf{Q},\mathbf{E}^\tau) \longrightarrow \mathrm{Hom}(\mathbf{A},\mathbf{E}^\tau)$$

$$\check{\sigma} : \mathbf{H}^2_{\mathfrak{U}}(\mathbf{Q},\mathbf{E}^\tau) \longrightarrow \mathbf{H}^2_{\mathfrak{U}}(\mathbf{A},\mathbf{E}^\tau)$$

defined by

- $\check{\sigma}(\phi) := \phi \circ \pi$
- $\check{\sigma}([T]) := [T \circ \pi]$ where $(T \circ \pi)_f(a_1,\dots,a_n) = T_f(\pi(a_1),\dots,\pi(a_n))$ for $f \in \tau, (a_1,\dots,a_n) \in A^n$

- *Restriction* map $\check{r} : \mathrm{Hom}(\mathbf{A},\mathbf{E}^\tau) \longrightarrow \mathrm{Hom}(\mathbf{B}^\tau,\mathbf{E}^\tau)$ defined by

$$\check{r}(\phi)(b) := \phi(b,q) - \phi(0,q) \quad \text{for any choice of q } \in Q$$

- *Transgression* homomorphism

$$\delta : \mathrm{Hom}(\mathbf{B}^\tau,\mathbf{E}^\tau) \times H^2_{\mathfrak{U}}(\mathbf{Q},\mathbf{B}^\tau) \longrightarrow H^2_{\mathfrak{U}}(\mathbf{Q},\mathbf{E}^\tau)$$

defined by $\delta(\phi,[T]) := [\phi \circ T]$.

- well-defined and respects equational compatibility
- transgression is bilinear

Hochschild-Serre Sequence with difference term II

- *Transgression* homomorphism

$$\delta : \mathrm{Hom}(\mathbf{B}^\tau, \mathbf{E}^\tau) \times H^2_{\mathfrak{U}}(\mathbf{Q}, \mathbf{B}^\tau) \longrightarrow H^2_{\mathfrak{U}}(\mathbf{Q}, \mathbf{E}^\tau)$$

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Hochschild-Serre Sequence with difference term II

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Theorem

Let $(\mathbf{B}^\tau, \mathbf{Q})$ and $(\mathbf{E}^\tau, \mathbf{Q})$ be central datum. Let $\mathbf{A} \in \mathcal{U}$ a variety with a difference term and $\pi : \mathbf{A} \to \mathbf{Q}$ a central extension realizing $(\mathbf{B}^\tau, \mathbf{Q})$. Then

$$0 \to \mathrm{Hom}(\mathbf{Q}, \mathbf{E}^\tau) \xrightarrow{\check{\sigma}} \mathrm{Hom}(\mathbf{A}, \mathbf{E}^\tau) \xrightarrow{\check{r}} \mathrm{Hom}(\mathbf{B}^\tau, \mathbf{E}^\tau) \xrightarrow{\delta} \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau) \xrightarrow{\check{\sigma}} \mathbf{H}^2_{\mathcal{U}}(\mathbf{A}, \mathbf{E}^\tau)$$

is a finite complex of abelian groups which is exact at the first three groups. If $\mathbf{A}$ has idempotent element, then the sequence is exact.

Hochschild-Serre Sequence with difference term III

The transgression is bilinear.

Theorem

Let $\mathbf{Q} \in \mathcal{U}$ *a variety with a difference term in the signature* τ*. Let* $(\mathbf{B}^\tau, \mathbf{Q})$ *and* $(\mathbf{E}^\tau, \mathbf{Q})$ *be central datum. Then the sequence*

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is exact. The chain map induced by evaluation $i : \mathbf{B}^\tau \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{B}^\tau, \mathbf{E}^\tau), \mathbf{E}^\tau)$ *is null-homotopic.*

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$$\mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{B}^\tau) \approx \mathbf{Ext}_{\mathcal{U}}(\mathbf{Q}/[1,1], \mathbf{B}^\tau) \times \mathrm{Hom}\left(\mathrm{Hom}(\mathbf{B}^\tau, \mathbf{E}^\tau), \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau)\right)$$

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    - finiteness $\Rightarrow \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{B}^\tau) \approx \mathbf{B}^\tau \otimes \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau)$

Regular datum and covers

regular datum: $(\mathbf{E}^\tau, \mathbf{Q})$

- (injective) $\mathbf{B}^\tau \leq \mathbf{C}^\tau$ and $\mathbf{B}^\tau \stackrel{\phi}{\to} \mathbf{E}^\tau$, there exists $\mathbf{C}^\tau \stackrel{\psi}{\to} \mathbf{E}^\tau$ with $\psi|_B = \phi$
- (separation) $(a,b) \notin \alpha \in \mathrm{Con}\,\mathbf{B}^\tau$, exists $\phi : \mathbf{B}^\tau \to \mathbf{E}^\tau$ such that $(a,b) \notin \ker\phi$ and $\alpha \leq \ker\phi$

• Fact: regular datum exists.........but usually for a larger variety you want to consider !

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• Fact: regular datum exists.........but usually for a larger variety you want to consider !

Theorem

Let $(\mathbf{E}^\tau, \mathbf{Q})$ be regular in $\mathcal{U}$ a variety with a difference term and an idempotent element. There is a cover $\pi : \mathbf{A} \to \mathbf{Q}$; that is, the transgression

$$\delta : \operatorname{Hom}(\mathbf{A}(\ker \pi)\Delta_{\ker \pi 1}, \mathbf{E}^\tau) \to \mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau)$$

is bijective. Consequently, $(\mathbf{H}^2_{\mathcal{U}}(\mathbf{Q}, \mathbf{E}^\tau), \mathbf{E}^\tau)$ is a reflexive pair.

• δ injective $\Leftrightarrow \ker \pi \leq [1,1]$
• δ surjective $\Leftrightarrow$ for any $\mathbf{C} \to \mathbf{P} \leftarrow \mathbf{Q}$, there is $\mathbf{A} \to \mathbf{C}$ complete the square

Schur and presentations

Extension of the classical characterization of the Schur multiplier:
$\kappa_{\alpha\beta} : \mathbf{A}(\alpha) \to \mathbf{A}(\alpha)/\Delta_{\alpha\beta}$

Theorem

Let $\mathcal{U}$ a variety with a difference term in the signature τ and $(\mathbf{E}^\tau, \mathbf{Q})$ regular datum in $\mathcal{U}$. Let $\mathbf{F}_\mathcal{U}/\theta \approx \mathbf{Q}$ be a free presentation. Assume $\mathbf{F}_\mathcal{U}$ has an idempotent element. Then

$$\mathrm{Hom}\left(\kappa_{\theta/[\theta,1]1}\left(\mathbf{F}_\mathcal{U}/[\theta,1](\theta \wedge [1,1]/[\theta,1])\right), \mathbf{E}^\tau\right) \approx \mathbf{H}^2_\mathcal{U}(\mathbf{Q}, \mathbf{E}^\tau).$$

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Derive Schur's result for groups: free presentation $\mathbf{F}/\mathbf{R} \approx \mathbf{Q}$ for finite $\mathbf{Q}$.

- $\mathbf{F}/[\alpha_R, 1]\left(\alpha_R/[\alpha_R,1]\right)/\Delta_{\alpha_R/[\alpha_R,1]1} \approx \mathbf{R}/[\mathbf{R},\mathbf{F}]/\left[\mathbf{R}/[\mathbf{R},\mathbf{F}], \mathbf{F}/[\mathbf{R},\mathbf{F}]\right] \approx \mathbf{R}/[\mathbf{R},\mathbf{F}]$ and so
$$\kappa_{\alpha_R/[\alpha_R,1]1}\left(\mathbf{F}/[\alpha_R,1]\left(\alpha_R \wedge [1,1]/[\alpha_R,1]\right)\right) \approx \mathbf{R} \wedge [\mathbf{F},\mathbf{F}]/[\mathbf{R},\mathbf{F}]$$
- $\mathbf{K} \wedge [\mathbf{G},\mathbf{G}]$ finite for any central extension $\mathbf{K} \hookrightarrow \mathbf{G} \to \mathbf{Q}$ (Schur).
    - central extension $\mathbf{R}/[\mathbf{R},\mathbf{F}] \hookrightarrow \mathbf{F}/[\mathbf{R},\mathbf{F}] \to \mathbf{Q} \Rightarrow \mathbf{R} \wedge [\mathbf{F},\mathbf{F}]/[\mathbf{R},\mathbf{F}]$ finite
- $\mathbb{C}^{\times}$ is regular for abelian groups
- finite abelian groups are self-dual:
$$\begin{aligned}\mathbf{R} \wedge [\mathbf{F},\mathbf{F}]/[\mathbf{R},\mathbf{F}] &\approx \mathrm{Hom}\left(\mathbf{R} \wedge [\mathbf{F},\mathbf{F}]/[\mathbf{R},\mathbf{F}], \mathbb{C}^{\times}\right) \\ &\approx \mathrm{Hom}(\kappa_{\alpha_R/[\alpha_R,1]1}\left(\mathbf{F}/[\alpha_R,1]\left(\alpha_R \wedge [1,1]/[\alpha_R,1]\right)\right), \mathbb{C}^{\times}) \\ &\approx \mathbf{H}^2(\mathbf{Q}, \mathbb{C}^{\times})\end{aligned}$$

Varieties with a difference term

Recover many recent results (by specialization): abelian groups expanded by linear and bilinear operations (just Mal'cev varieties)

- algebras of Loday: Lie (Batten '93), Leibniz (Elyse '19, Mainellis '21, '22), diassociative (Mainellis '21), dendriform (Das '19),
- Rota-Baxter-type algebras (Das '22):

- Example:
  - diassociative $\langle L, +, \mathbb{F}, \dashv, \vdash \rangle$: $L(\alpha_R)/\Delta_{\alpha_R 1} \approx R/(R \diamond L + L \diamond R)$, $R \diamond L = R \dashv L + L \vdash R$
  - Leibniz $\langle L, +, \mathbb{F}, \cdot \rangle$: $L(\alpha_R)/\Delta(\alpha_R 1) \approx R/(RL + LR)$
  - presentation $\mathbf{F}/\mathbf{R} \approx \mathbf{L}$

$$\mathbf{H}^2(\mathbf{Q}, \mathbb{F}) \approx R/(RF + FR) \qquad \mathbf{H}^2(\mathbf{Q}, \mathbb{F}) \approx R/(R \diamond F + F \diamond R)$$

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• Cohomology parametrized by equational theories - proceed by specialization in each case (unified approach)

- evaluate special form of generators of commutator ($[R, L] = R \diamond L + L \diamond R$, etc.)
- determine "nice" regular datum ($\mathbb{C}^\times, \mathbb{Z}(p^\infty), \mathbb{F}$, etc.)
- utilize any self-duality or reflexivity ! (simplify nested tensors and homsets)

Now what ?

- local calculations with regular datum - "regular enough" for a class of datum
- higher cohomology realizing affine datum in general varieties
- unrestricted Hochschild-Serre
- commutators for expanded semilattices, commutators for SD($\vee$), etc.
- What is an abelian lattice ?