Generalized ultraproducts for positive logic

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(Prime) filter products

Definition

1. A wellfounded forest is a poset where every principal down-set is a wellorder.
2. Given:
   - $(I, \leq)$ a wellfounded forest
   - F a filter in $\mathrm{Up}(I, \leq)$,
   - $(h_{ij} : M_i \to M_j \mid i \leq j \in I)$ a "direct system" of homs

   define

$$\prod_F A_i := \{a \in \prod_{i \in I} M_i : \exists I' \in F \, \forall i \leq j \in I' \, h_{ij}(a(i)) = a(j)\}.$$

$(a \equiv_F b \overset{\text{def}}{\iff} [\![a = b]\!] \in F)$ is a congruence on $\prod_F M_i$.
$\prod_F M_i / \equiv_F$ is a filter product of $\{M_i : i \in I\}$
It is a prime (filter) product, when F is a prime filter of $\mathrm{Up}(I, \leq)$.

Examples

1. Every ultraproduct is a prime product $((I,\leq) = (I,\Delta_I))$.
2. Every direct limit along a wellorder I is a filter product $(F := \{\mathrm{Up}(I)\})$
3. In the language of unital rings:
    - $(I,\leq) := (\omega,\leq)$
    - F consists of all nonempty up-sets
    - K a field
    - $M_i := K[X]$ $(i < \omega)$
    - h_{ij} is $f \mapsto f(a)$ for some fixed $a \in K$

   Then $\prod_F M_i$ is a prime product ("prime power") isomorphic to K.

Filter products vs. ultraproducts

Remark

An ultraproduct $\prod_D M_i$ is the stalk at D of a sheaf obtained by composing the Čech-Stone compactification $I \to \beta I$ with the sheaf

$$I'(\subseteq_{\text{fin}} I) \mapsto \prod_{i \in I'} M_i.$$

A prime filter product $\prod_F M_i$ is the stalk at F of a sheaf obtained by composing the Nachbin compactification $(I, \leq) \to \beta_o I$ with the sheaf

$$I' \mapsto \{a \in \prod_{i \in I'} M_i \mid \forall i \leq j \in I' \; h_{ij}(a(i)) = a(j)\}.$$

Positive logic

Definition (Ben Yaakov and Poizat (2007))

1. A positive existential (or $\exists^+$) formula is a first-order formula built from atomic formulae (including $\bot$) by $\exists, \wedge, \vee$ (but not $\forall$).
2. A basic h-inductive formula is a first-order formula obtained by universally quantifying, finitely many times, a conditional between $\exists^+$ formulae. An h-inductive (or $\forall_2^+$) formula is a conjunction of basic h-inductive formulae.

Example

Each of the field axioms are h-inductive in the language of unital rings:

Not a zero ring $0 = 1 \to \bot$

No zero divisors $\forall x \forall y [xy = 0 \to x = 0 \vee y = 0]$

Multiplicative inverses $\forall x [x = 0 \vee \exists y\, xy = 1]$

etc.

Łoś

A $\exists^+$ formula is p.p. if it does not contain any nontrivial disjunctions.

Theorem

Given:

- *a wellfounded forest* I
- *a direct system* $(h_{ij} : M_i \to M_j \mid i \leq j \in I)$
- *a filter* F *in* $\mathrm{Up}(I)$,

for every p.p. formula $\phi(\overline{x})$ *and a tuple* $\overline{a}$ *in* $\prod_F M_i$,

$$\prod_F M_i \models \phi(\overline{a}) \iff [\![\phi(\overline{a})]\!] \in F.$$

If F *is prime, the displayed biconditional is true of all* $\exists^+$ *formulae.*

Wellfoundedness is used in the proof of the second claim.

Łoś-Suszko

Theorem

More generally, if ϕ is merely $\forall_2^+$, and the other assumptions are the same, then

$$\prod_F M_i \models \phi(\bar{a}) \implies [\![\phi(\bar{a})]\!] \in F.$$

if F is prime.

Theorem

A class K of structures is axiomatized by $\forall_2^+$ sentences if and only if K is closed under ultraroots and prime products.

Keisler-Shelah: Theorem

Theorem

Let K *a quasivariety of finite type, with a finite nontrivial member. Then TFAE:*

1. *The nontrivial members of* K *satisfy the same* $\exists^+$ *sentences.*
2. *Any two nontrivial members of* K *have isomorphic prime powers.*

This follows from

Theorem (Moraschini, Raftery, and Wannenburg (2022))

Let K *a quasivariety of finite type, with a finite nontrivial member. Then TFAE:*

1. *The nontrivial members of* K *satisfy the same* $\exists^+$ *sentences.*
2. *Any two nontrivial members of* K *have a common retract.*

Keisler-Shelah: Conjecture

Conjecture

Let T be a $\forall_2^+$ theory. The following are equivalent.

1. "Big" models of T have the same $\exists^+$ theory.
2. "Big" models of T have isomorphic prime powers.

Potentially useful fact: Feferman-Vaught

Theorem

Let $\prod_F M_i$ be a prime product. For a $\exists^+$ formula $\phi(\overline{x})$, there exists:

- *$\exists^+$ formulas $\theta_1(\overline{y}), \ldots, \theta_k(\overline{y})$*
- *a $\exists^+$ formula ϕ^* in the language of Heyting algebras*

such that

$$\forall \bar{a} \in \prod_F M_i \left[\prod_F M_i \models \phi(\bar{a}) \iff \mathrm{Up}(I, \leq) \models \phi^*([\![\theta_1(\bar{a})]\!], \ldots, [\![\theta_k(\bar{a})]\!]) \right]$$

References

Ben Yaakov, I. and Poizat, B. (2007). Fondements de la Logique Positive. The Journal of Symbolic Logic, 72 (**4**), 1141–1162.

Moraschini, T., Raftery, J.G. and Wannenburg, J.J. (2020), Singly generated quasivarieties and residuated structures. Math. Log. Quart., 66: 150–172.