Directed Partial Orders and Birkhoff-Pierce Problem

The Story

DPO with $1 > 0$

DPO with $1 \not> 0$

Doubly convex set

Schwartz-Yang Orders

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A simple question: Is $1 > 0$?

- Is $1 > 0$ in $\mathbb{C}$?
- Is always $1 > 0$ in $\mathbb{R}$?
- General question: Birkhoff-Pierce problem.
- More general question: Fuchs' problem.

Birkhoff-Pierce problem: Orders over $\mathbb{C}$?

- 1956, Birkhoff-Pierce problem:
  Does $\mathbb{C}$ admit any lattice orders?
- 1963, L. Fuchs:
  Do $\mathbb{R}, \mathbb{H}$ admit any lattice orders besides the usual one?
- $\mathbb{R}$ admits infinitely many lattice orders.
- Still open over $\mathbb{C}$ and $\mathbb{H}$.

Positivity of 1 in $\mathbb{C}$ or $\mathbb{R}$?

- Is 1 always positive as a complex number?
- Very easy: No.
- Is 1 always positive in $\mathbb{R}$ when lattice-ordered?
- Answer: No. (Wilson, 1976.)

Directed partial Orders (DPO)

- DPO = Directed partial Order: Every two elements a, b enjoy an upper bound and a lower bound.
- A lattice order is a DPO.

DPO – the "BEST" orders on $\mathbb{C}$?

- A positive cone P of a po-ring R is a DPO if and only if it is a partial order and $P - P = R$.
- Theorem(R. DeMarr, A. Steger, 1972). [1]
  $\mathbb{C}$ admits no DPO as an $\mathbb{R}$ algebra.

[1] R.DeMarr, A.Steger, PAMS 31(1972).

The "BEST" orders on $\mathbb{C}$

- Schwarz-Yang Theorem, 2011.[2]
  The field $\mathbb{C}$ admits at least one DPO which is not a lattice order.
- Existence proof. None DPO on $\mathbb{C}$ is constructed so far.

[2] N. Schwartz, Y. Yang, *Fields with directed partial orders*, J. Algebra **336**(2011)342-348. (2011) 342-348.

Bounded semigroup

- Let F be a non-archimedean o-field. A convex additive semigroup S of F^+ containing 0, 1 is called "bounded" if there exists $z \in F^+$ such that $s \ll z$ for all $s \in S$, written as $S \ll z$.
- Let P be a DPO with $1 > 0$ (i.e. $1 \in P$). Define

  $$s(P) = \{x \in F^+ \,|\, x \leq a, \forall a \in F^+ \textit{ with } a + i \in P\}.$$

  Then $s(P)$ is a bounded semigroup of F^+.
- Let S be a bounded semigroup of F^+. Define

  $$p(S) = \{a + bi \mid a \in F^+, 0 \leq sb \leq a \text{ for all } s \in S\}.$$

  Then $p(S)$ is a DPO on C with $1 > 0$.

Classification of DPOs with $1 > 0$

- Theorem[3]. Let F be a non-archimedean o-field and $C = F(i)$. Then there is a bijection between the set of DPOs with $1 > 0$ over $\mathbb{C}$ and the set of bounded semigroups of F.

[3] J. Ma, L. Wu, Y. Zhang, *Directed Partial Orders on $F(i)$ with $1 > 0$*, Order, 35(2018)3, 461-466.

Special convex subsets

Definition

Let $\emptyset \neq V \subseteq F^+$. Then V is **special convex** if
(1) V is convex in F^+;
(2) $1 \ll V$, namely $1 \ll v$ for any $v \in V$;
(3) If $a \sim b, a < b \in V$, then $a \in V$.

Correspondence

- Let V be a special convex subset of F^+. Define

  $$P_V = F^+\{a + i \,|\, a \in V\} = \{r(a + i) \,|\, \forall r \in F^+, \forall a \in V\}.$$

  Then P_V is a DPO of C such that $1 \notin P_V$,
- Let P be a DPO on C such that $1 \notin P$. Define

  $$L(P) = \{a \in F^+ \,|\, a + i \in P\}.$$

  Then $L(P)$ is a special convex subset of F^+.

DPOs on $C = F(i)$ with $1 \not> 0$

- Theorem[4] Let F be a non-archimedean o-field and $C = F(i)$. Then there is a bijection between the set of DPOs with $1 \not> 0$ over $\mathbb{C}$ and the set of special convex sets of F.

[4] J. Ma, L. Wu, Y. Zhang, *Directed partial orders on the field of generalized complex numbers with* $1 \not> 0$, Positivity, 2019(24)3,1001-1007.

Doubly convex set of non-archimedean o-field

Definition

Let $\emptyset \neq V \subseteq F$. Then V is **doubly convex** if
(1) $V \neq \{0\}$.
(2) $|V| \ll 1$, namely $\forall v \in V, |v| \ll 1$.
(3) $\forall a, b \in V, \forall r, s \in F^+$ and $1 \leq r + s \leq 2$, $ra + sb \in V$.

- Remark. A doubly convex set of F is convex, and in general the inverse is not true.

DPO and doubly convex sets over $F(i)$

- Theorem[5] Let F be a non-archimedean o-field and $C = F(i)$. Then there is a bijection between the set of all DPOs over $\mathbb{C}$ and the set of doubly convex sets of F.
- Corollary. Suppose V is a doubly convex set of F. Then
  1. If $0 \in V$, then P_V is a DPO on C with $1 > 0$.
  2. If $0 \notin V$, then P_V is a DPO on C with $1 \not> 0$.

[5] Zhipeng XU, Yuehui Zhang, Directed Partial Orders over Non-Archimedean *o*-Fields, Positivity, 24(2)2020, 1279 – 1291.

DPO over $H = F + Fi + Fj + Fk$

Definition

A subset $\emptyset \neq V \subseteq F^3$ is **doubly convex** provided
(1) V contains at least 2 linearly independent elements.
(2) $|V| \ll 1$, that is $\forall \alpha = (a_1, a_2, a_3) \in V, |\alpha| \ll 1$.
(3) V is convex, i.e. $\forall \alpha, \beta \in V$, $\forall r, s \in F^+, r + s = 1$, then $r\alpha + s\beta \in V$.
(4) $\forall \alpha, \beta \in V, \dfrac{\alpha + \beta + \alpha \times \beta}{1 - \alpha \cdot \beta} \in V, \dfrac{\alpha + \beta - \alpha \times \beta}{1 - \alpha \cdot \beta} \in V$.

DPO and Doubly convex sets over $H = F + Fi + Fj + Fk$, Xu-Z, 2020

Theorem

Let $\mathcal{V}$ be the set of all doubly convex sets of F^3. Let $\mathcal{P}$ be the set of all directed partial orders on H. Then there is a bijection between $\mathcal{V}$ and $\mathcal{P}$.

Schwartz-Yang orders

- Let F be a non-archimedean o-field, K a directed extension field of F with $F^+ = K^+ \bigcap F$.
- Let $\sigma \in K, \sigma \geq 1$. Define

$$K[x]^+_\sigma = \{\sum_{i=0}^{n} a_i x^i \mid \textit{if } a_i > 0, i \geq 1, \textit{then } \sigma a_i \ll a_{i-1}\}.$$

- Schwartz-Yang's Theorem[6].
  $K[x]^+_\sigma$ defines a DPO over $K[x]$.

[6] N. Schwartz, Y. Yang, *Fields with directed partial orders*, J. Algebra **336**(2011)342-348. (2011) 342-348

$\mathbb{C}$ has a DPO

- Lemma. Let $F \subseteq K$ be fields as above. If $K \subseteq L$ is an algebraic extension then there is a DPO L^+ on L such that $L^+ \bigcap K = K^+$.
- Corollary. If L is any field that has transcendence degree at least 1 over $\mathbb{Q}$ then L has a DPO.
- Schwartz-Yang's Theorem[7].
  The field $\mathbb{C}$ of complex numbers has a DPO.

[7] N. Schwartz, Y. Yang, *Fields with directed partial orders*, J. Algebra **336**(2011)342-348. (2011) 342-348

Schwartz-Yang orders and doubly convex sets

- Let $D_\sigma = \{s \in F | 1 + sx \in F[x]_\sigma^+\}$.
- Theorem (Dong-Xu-Zhang, 2022). D_σ is a doubly convex set of F.
- Schwartz-Yang orders can be realized by doubly convex sets.

References

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3. N.Schwartz, Y.Yang, JA 336(2011).
4. Y.Yang, JA, 2006
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6. MWZ, DPO on NA Fields, Order, 2017
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Thank you for your attention!