Axiomatising the Class of All Set-theoretic Multiverses

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BLAST2022 - Chapman University

12th August 2022

Outline

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The multiverse *vs* universe debate

The standard position: *universism*

There is just one single set-theoretic universe, namely the cumulative hierarchy V.

An alternative, pluralist position: the *set theoretic multiverse*

There are several set-theoretic universes, each one instantiating a different conception of set and membership.

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Types of set-theoretic multiverses

Broad multiverse The collection of all possible set theories and their models (Hamkins);

Set-generic multiverse The collection of all extensions of a ground universe generated through set-generic forcing (Steel, Venturi, Woodin);

Hyperuniverse The collection of all countable, transitive models of V, defined using the infinitary V-logic (Friedman);

Parallel Universes The collection of all possible instantiation of different conception of the power set operation (Väänänen).

My plan: a common environment for all multiverses

- Each one of these multiverses has its own characterisation;
- In most cases, it is very difficult to compare them and assess their differences;
- My plan is to define a common environment for all of them:
  - Step 1 Define the most basic characterisation of the multiverses;
  - Step 2 Define the multiverse operator;
  - Step 3 Define the structure of all multiverses.

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General features of a multiverse

Ground Universe

The starting point of the multiverse: all the other universes are generated from this universe using a particular generating method.

- E.g. V, constructible universes, etc.

Generating method

The method (or methods!) that, applied to the ground universe, generates the whole multiverse.

- E.g. forcing, axioms, strong logics, etc.

Definition of the multiverse operator

Definition (Multiverse Operator)

Let V_κ be a universe of set theory, and i a generating method. Then the multiverse operator Mlt maps V_κ to the set-theoretic multiverse MV generated through the method i applied to V_κ:

$$Mlt_i : V_\kappa \mapsto MV.$$

We write this $Mlt_i(V_\kappa) = MV$.

Defining multiverses with the multiverse operator

Broad multiverse $Mlt_{broad}(V_\kappa) = MV_{broad}$;

Set-generic multiverse $Mlt_{genericmv}(V_\kappa) = MV_{steel}$;

Hyperuniverse Let M be a countable transitive model of ZFC,

$$Mlt_{vlogic}(M) = HP;$$

Parallel Universes $Mlt_{ts}(V_\kappa) = MV_{parallel}$.

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Defining the structure of all set-theoretic multiverses

- The triple (Mlt, i, V_κ) define each single multiverse;
- However, each one of them is still its own, separate, entity;
- In particular, there is no common feature between all the triples (Mlt, i, V_κ)!

Question

Is it possible to actually define a single structure $\langle Mlt_i, V_\kappa \rangle$ that encompasses all set-theoretic multiverses?

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Axioms for the multiverse operator

1. A multiverse contains its ground universe: $V_\kappa \preceq Mlt_i(V_\kappa)$;
2. The multiverses are in the same relation as their ground universes: $V_\kappa \preceq U_\lambda \Rightarrow Mlt_i(V_\kappa) \preceq Mlt_i(U_\lambda)$;
3. Iterating the generation of the multiverse doesn't escape the original multiverse: $Mlt_i(Mlt_i(V_\kappa)) \preceq Mlt_i(V_\kappa)$.

An example: Steel's set-generic multiverse satisfies the axioms

1. The ground universe is contained in its extensions;
2. If a ground universe is contained in another universe, then the extensions of the smaller ones are also contained in the extensions of the bigger one;
3. The iteration of the set-generic multiverse is still in the set-generic multiverse.

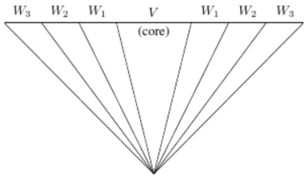

Figure: The set-generic multiverse

The structure of all set-theoretic multiverses is a Tarski structure

Theorem

The structure $\langle Mlt, V_\kappa \rangle$ is a Tarski Structure. That is, for any Mlt_i, V_κ, the following axioms hold:

1. $V_\kappa \preceq Mlt_i(V_\kappa)$;
2. $V_\kappa \preceq U_\lambda \Rightarrow Mlt_i(V_\kappa) \preceq Mlt_i(U_\lambda)$;
3. $Mlt_i(Mlt_i(V_\kappa)) \preceq Mlt_i(V_\kappa)$

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Further steps: categorising multiverses in subclasses

- Now that we have a unified structure of all multiverses, we can start comparing and categorising its elements;
- For example, we can define different classes of multiverses using the multiverse operator:
  - Open *vs* closed multiverses;
  - Linear *vs* branching multiverse;
- Open questions:
  - Are there multiverses that don't satisfy the axioms given? [Claim: no];
  - Can we use the multiverse operator to characterise the Single universe [Claim: yes, but it doesn't satisfy the axioms];
  - Can we use the multiverse operator to define the actualist *vs* potentialist distinction [Claim: yes!]

Final summary

- There is a huge variety of set-theoretic multiverse, each one with its one mathematical characterisation;
- They are all generated from a ground universe, using a finite set of possible methods;
- Using this feature, it is possible to define the Multiverse Operator, mapping a universe of set theory to its corresponding multiverse (given a generating method);
- With the multiverse operator it is then possible to define the structure of all multiverses, $\langle Mlt, V_\kappa \rangle$;
- Finally, this structure is a Tarski Structure.

Thank you very much for your attention!

Outline

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Claim

Theorem

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Sketch of the proof

Sketch.

1. To prove the theorem, we need to show that all possible triples (Mlt, i, V_κ) satisfy the axioms;
2. The proof proceeds by cases, one for each possible triple;
3. However, it is possible to consider less cases, going through only one type of multiverse (then variants work exactly the same);
4. This gives us a finite number of cases to check.

□

Are there set-theoretic multiverses that don't satisfy the axioms?

- I claim that this is not the case;
- All the multiverses can be described using the multiverse operator;
- There is a finite amount of meaningful combination we can use to generate a multiverse;
- All these combinations satisfy the axiom given.

Is it possible to define the Single Universe with the multiverse operator?

- Yes it is!
- $Mlt_i(V_\kappa) = M[G]$, where $M[G]$ is the extension of a countable transitive model $M \in V_\kappa$;
- However, it doesn't satisfy axiom (3), consequently it is not part of the structure $\langle Mlt, V_\kappa \rangle$.

Is it possible to characterise the actualism - potentialism distinction with the multiverse operator?

Actualism This is the Single Universe already defined;

Height potentialism It's possible to add new ordinals to V: $Mlt_i(V_\kappa) = V_{\kappa+1}$;

Width potentialism It's possible to add new subsets to V: $Mlt_i(V_\kappa) = V_\kappa[G]$;

Radical potentialism It's possible to add both new subsets and ordinals to V: $Mlt_i(V_\kappa) = V_{\kappa+1}[G]$