Fixed-point theory in the varieties D_n

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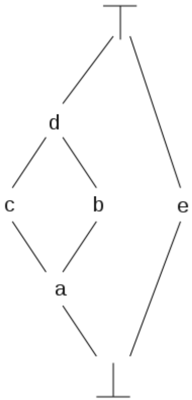
Outline

Lattices $(L, \bot, \top, \wedge, \vee)$



Let L be an ordered set s.t. :
$\forall x, y \in L \;\; \exists u, v \in L$ s.t.

$$u = x \vee y = \text{ the least upper bound}$$
$$= \text{ supremum}$$
$$v = x \wedge y = \text{ the greatest lower bound}$$
$$= \text{ infimum.}$$

$\bot$: the **smallest element** of L.
$\top$: the **largest element** of L.

Fixed points and lattices

Let

- $(L, \leqslant)$ be a lattice,
- $f : L \to L$ *increasing*.
- $Fix(f) = \{\, x \in L \mid f(x) = x \,\}$

Let's note :

$$\mu_x.f(x) = \bigwedge Fix(f)\,, \quad \nu_x.f(x) = \bigvee Fix(f)\,.$$

Theorem (Tarski '55)

If L is a complete lattice and f is increasing, then

$$\mu_x.f(x) = min\ Fix(f) = \textit{the least fixed point of } f$$

$$\nu_x.f(x) = max\ Fix(f) = \textit{the greatest fixed point of } f$$

Algorithm to calculate fixed points

L complete lattice,
$\perp$ the least element, $\top$ the largest element,
$f : L \to L$ increasing.

- The least fixed point :

$$\perp \leqslant f(\perp) \leqslant f^2(\perp) \leqslant ... \leqslant f^k(\perp) \leqslant ... \leqslant f^n(\perp) \ = \ f^{n+1}(\perp)$$

Here we have : $f^n(\perp) = \mu_x.f(x)$.

- The greatest fixed point :

$$\top \geqslant f(\top) \geqslant f^2(\top) \geqslant ... \geqslant f^k(\top) \geqslant ... \geqslant f^n(\top) \ = \ f^{n+1}(\top)$$

Here we have : $f^n(\top) = \nu_x.f(x)$.

μ-calculus

Lattices μ-calculus :

$$\phi \quad = \quad x \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \quad \mu_x \phi(x) \mid \nu_x \phi(x)$$

- $\mu_x \nu_y \phi(x, y)$: difficult to calculate

$$\psi = \mu_{x_d}.\nu_{y_d}.\mu_{x_{d-1}}.\nu_{y_{d-1}}. \ \ldots \ .\mu_{x_1}.\nu_{y_1}.\varphi(x_1, y_1, x_2, y_2, \ldots, x_d, y_d)$$

with φ containing neither μ nor ν, $\mathrm{complexity}(\psi) = d$.

Complexity of a formula = number of blocks $\mu\nu$.

expressiveness : the alternation hierarchy of μ-calculus

The hierarchy

... is strict :

for all d there exists ψ with $complexity(\psi) = d$ such that
if $complexity(\phi) < d$ then $\phi \not\equiv \psi$.

... is degenerate :

there exists d such that if ψ verifies $complexity(\psi) > d$
then there exists ϕ with $complexity(\phi) \leq d$ and $\phi \equiv \psi$.

$$\psi = \mu_{x_d}.\nu_{y_d}.\mu_{x_{d-1}}.\nu_{y_{d-1}}. \ \ldots \ .\mu_{x_1}.\nu_{y_1}.\varphi(x_1, y_1, x_2, y_2, \ldots, x_d, y_d)$$

Can we simplify ψ ?

Motivations

Lattices μ-calculus :

$\phi \quad = \quad x \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \quad \mu_x \phi(x) \mid \nu_x \phi(x)$

$\psi = \mu_{x_d}.\nu_{y_d}.\mu_{x_{d-1}}.\nu_{y_{d-1}}. \ \dots \ .\mu_{x_1}.\nu_{y_1}.\varphi(x_1, y_1, x_2, y_2, \dots, x_d, y_d)$

The alternation hierarchy of μ-calculus :

- strict: lattices [San02]
- degenerate: distributive lattices

$$\mu_x \phi(x) = \phi(\bot) \text{ and } \nu_x \phi(x) = \phi(\top)$$

- Varieties of lattices $\mathcal{D}_n$, with $n \in \mathbb{N}$ [Nat90], [Sem05].

  Examples :

  - $\mathcal{D}_0$ = distributive lattices
  - lattices of permutations: $S_n \in \mathcal{D}_{n-2}$

Characterization of the varieties $\mathcal{D}_n$

$\mathcal{D}_n$: defined via equations of a weaker version of distributivity

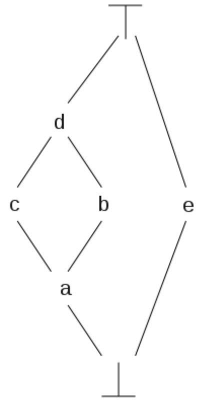
Lattices in $\mathcal{D}_n$ are locally finite

$\mathcal{D}_n \cap$ finite lattices: combinatorial characterization

OD-graph of a finite lattice L**:** $G(L) := \langle J(L), \leq, \mathcal{M} \rangle$

1. $J(L)$: join-irreducible elements ($j = a \vee b$ iff $j = a$ or $j = b$)
2. $\leq$: order restricted to $J(L)$
3. $\mathcal{M} : J(L) \longrightarrow \mathcal{PP}J(L)$: minimal covers

The OD-graph of a lattice



$\langle J(L), \leq, \mathcal{M} \rangle$ with $\mathcal{M} : J(L) \to \mathcal{P}(\mathcal{P}(J(L)))$

1. **C is a cover of j : $j \leq \bigvee C$**
2. **order $\ll \subseteq \mathcal{P}L \times \mathcal{P}L$:**
   $A \ll B$ iff $\downarrow A \subseteq \downarrow B$.
3. C is a minimal cover of j if
   - C is a cover of j,
   - C is a $\leq$-antichain,
   - for any $\leq$-antichain $D \subseteq L$,
     ($j \leq \bigvee D$ and $D \ll C$) imply $D = C$.
4. $\mathcal{M}(j) =$ minimal covers of j

1. $b \leq \bigvee\{b\}$ trivial cover
2. $b \leq \bigvee\{d\} = \bigvee\{c, b\}$ minimal covers are subsets of $J(L)$
3. $b \leq \bigvee\{c, e\}$ minimal

Finite lattices and their OD-graphs

L a finite lattice and $G(L) := \langle J(L), \leq, \mathcal{M} \rangle$ its OD-graph.

$$(L, \leqslant) \rightsquigarrow G(L) = \langle J(L), \leqslant_{J(L)}, \mathcal{M} \rangle \rightsquigarrow (\mathfrak{L}(G(L)), \subseteq)$$

$G(L)$ is similar to a neighborhood frame.

- Language on L: $\phi := \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi$
- Logic on the frame $G(L)$: $\phi := \bot \mid \top \mid \phi \wedge \phi \mid (\exists\forall)(\phi \vee \phi)$
  $\rightsquigarrow$ monotone modal logic,
  let $a \in L$ and v an assignment, let the valuation v' be as follows: $v'(j) = {\downarrow} j$, we have:

$$\begin{aligned} a \leq v(\phi) \quad &\text{iff} \quad \forall j \leq a, \quad G(L), j \vdash_{v'} \tau(\phi) \\ G(L), j \vdash_{v'} (\exists\forall)(\phi \vee \psi) \quad &\text{iff} \quad \exists C \in \mathcal{M}(j), \forall c \in C, \\ &\qquad G(L), c \vdash_{v'} \phi \text{ or } G(L), c \vdash_{v'} \psi \end{aligned}$$

Finite lattices in $\mathcal{D}_n$

L a finite lattice in $\mathcal{D}_n$ and $G(L) := \langle J(L), \leq, \mathcal{M} \rangle$ its OD-graph.

The relation D

Let $j, k \in J(L)$, jDk if $j \neq k$ and $\exists C \in \mathcal{M}(j)$ s.t. $k \in C$

the class of finite lattices in $\mathcal{D}_n$

A finite lattice L ($\langle J(L), \leq, \mathcal{M} \rangle$) belongs to **the class** $\mathcal{D}_n$ iff any path $j_0 D j_1 D \ldots D j_k$ has length at most n.

Results

for each variety $\mathcal{D}_n$ with $n \in \mathbb{N}$:

1. **Upper bound on the approximations chain** :
   The μ-calculus hierarchy on $\mathcal{D}_n$ is degenerate

   $$\mathcal{D}_n \vDash \mu_x.\phi(x) = \phi^{n+1}(\bot) \text{ and } \mathcal{D}_n \vDash \nu_x.\phi(x) = \phi^{n+1}(\top)$$

2. **Lower bound** :
   On the lattices in $\mathcal{D}_n$ the value $n+1$ is optimal.

3. Lower bound :
   On the atomistic lattices in $\mathcal{D}_n$ the value $n+1$ is optimal.

4. Lower bound :
   On the lattices in $\mathcal{D}_n \cap \mathcal{D}_n^{op}$ the value $n+1$ is optimal.

Upper bound for the operator ν on the varieties $\mathcal{D}_n$

Upper bound $= n+1$

For the variety $\mathcal{D}_n$ with $n \in \mathbb{N}$, the hierarchy of the μ-calculus is degenerated (upper bound) :
$\mathcal{D}_n \vDash \mu_x.\phi(x) = \phi^{n+1}(\bot)$ and $\mathcal{D}_n \vDash \nu_x.\phi(x) = \phi^{n+1}(\top)$

Sketch of proof: $\mathcal{D}_n \vDash \nu_x.\phi(x) = \phi^{n+1}(\top)$
$\Leftrightarrow \mathcal{D}_n \cap \textit{finite} \vDash \nu_x.\phi(x) = \phi^{n+1}(\top)$ (Nation '90 : locally finite)

$\Leftrightarrow \mathcal{D}_n \cap \textit{finite} \vDash \phi^{n+1}(\top) = \phi^{n+2}(\top)$

$\Leftrightarrow \mathcal{D}_n \cap \textit{finite} \vDash \phi^{n+2}(\top) \leqslant \phi^{n+1}(\top)$ and $\phi^{n+1}(\top) \leqslant \phi^{n+2}(\top)$

$\Leftrightarrow \mathcal{D}_n \cap \textit{finite} \vDash \phi^{n+1}(\top) \leqslant \phi^{n+2}(\top)$

Tool: game semantic on the OD-graph

Game semantic

$\mathcal{D}_n \cap \mathit{finite} \vDash \phi^{n+1}(\top) \leqslant \phi^{n+2}(\top)$

$\Leftrightarrow$ for any finite lattice L in $\mathcal{D}_n$, $L \vDash \phi^{n+1}(\top) \leqslant \phi^{n+2}(\top)$

$\Leftrightarrow$ for any finite lattice L in $\mathcal{D}_n$, for any closed valuation v, for any $j \in J(L)$,

$$G(L), j \vDash_v \tau(\phi^{n+1}(\top)) \text{ implies } G(L), j \vDash_v \tau(\phi^{n+2}(\top))$$

we define a finite 2 player game such that:
player A has a winning strategy from the position (j, ψ) iff $G(L), j \vDash \psi$.

Results

For variety $\mathcal{D}_n$ with $n \in \mathbb{N}$:

1. The hierarchy of the μ-calculus is degenerated (upper bound) :
   $\mathcal{D}_n \vDash \mu_x.\phi(x) = \phi^{n+1}(\bot)$ and $\mathcal{D}_n \vDash \nu_x.\phi(x) = \phi^{n+1}(\top)$
2. Optimality :
   $\mathcal{D}_n \nvDash \mu_x.\phi(x) = \phi^{n}(\bot)$ and $\mathcal{D}_n \nvDash \nu_x.\phi(x) = \phi^{n}(\top)$

open problems and outlook

- $\exists$? a term t_ϕ "simpler" than $\phi^{n+1}(\bot)$ s.t. $\mathcal{D}_n \vDash \mu_x \phi(x) = t_\phi$
- links between lattice theory and modal logic ?
- similar results on fixed points for modal logic ?

Références I

J. B. Nation.
An approach to lattice varieties of finite height.
Algebra Universalis, 27(4):521–543, 1990.

Luigi Santocanale.
The alternation hierarchy for the theory of μ-lattices.
Theory Appl. Categ., 9:166–197, 2001/02.
CT2000 Conference (Como).

M. V. Semënova.
On lattices that are embeddable into lattices of suborders.
Algebra Logika, 44(4):483–511, 514, 2005.