Relation Lifting

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30. 4. 2014

Overview

Topic: Some elements of **the category theory of relations**

1. Introduction. (Why functors instead of signatures?)
2. Relation lifting over Set, universal property of Rel
3. Monotone relations
4. Many-valued (fuzzy) relations
5. Application: Bisimulation
6. Application: Simulation
7. Application: Coalgebraic logic
8. Conclusion

Why functors instead of signatures?

Algebras for a functor

One can replace signatures by functors, for example a constant

$$1 \to X$$

and a binary operation

$$X \times X \to X$$

can be assembled into

$$1 + X \times X \to X$$

or,

$$FX \to X$$

An algebra morphism is simply a function $f : X \to X'$ such that

$$\begin{array}{ccc} FX & \longrightarrow & X \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FX' & \longrightarrow & X' \end{array}$$

Are functors more general than signatures?

Example: The *powerset functor* $\mathcal{P}$ takes $f : X \to X'$ to

$$\mathcal{P}f : \mathcal{P}X \to \mathcal{P}X' \qquad X \supseteq a \mapsto f[a]$$

$\mathcal{P}_\omega$ takes finite subsets only

What are powerset algebras?

$$\mathsf{List}(X) \longrightarrow \mathcal{P}_\omega X \longrightarrow X$$

More generally, for every functor $T : \mathsf{Set} \to \mathsf{Set}$ there is a signature Σ such that a T-algebra $TX \to X$ is an F_Σ algebra $F_\Sigma \to TX \to X$, where F_Σ is the functor corresponding to the signature Σ.

Coalgebras for a functor

A **coalgebra** is an arrow

$$X \to TX$$

mapping a state to its set of successors. **Morphisms** are given as

$$\begin{array}{ccc} X & \longrightarrow & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ X' & \longrightarrow & TX' \end{array}$$

and induce a notion of bisimilarity or **behavioural equivalence**

$$x \simeq x' \quad \Leftrightarrow \quad \exists f, f' \,.\, f(x) = f(x')$$

The definition of coalgebra is parametric in the category and the functor

Example: Tranisition systems

$$\begin{array}{ccc} X & \xrightarrow{\ \xi\ } & \mathcal{P}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ X' & \xrightarrow[\ \xi'\ ]{} & \mathcal{P}X' \end{array}$$

$$\forall y \in \xi(x)\,.\,\exists y' \in \xi'(x')\,.\,f(y) = y'$$

$$\forall y' \in \xi'(x')\,.\,\exists y \in \xi(x)\,.\,f(y) = y'$$

Initial algebras and final coalgebras

TX	initial algebra	final coalgebra
$\mathcal{P}$	(well-founded) sets	non-well-founded sets
$1 + X$	$\mathbb{N}$	$\mathbb{N} \cup \{\infty\}$
$A \times X$	$\emptyset$	streams (=infinite lists)
$1 + A \times X$	lists over A	finite and infinite lists over A
$1 + X \times X$	finite binary trees	non-well founded binary trees
$\ldots$		

For every functor $T : \mathsf{Set} \to \mathsf{Set}$ the final coalgebra exists and consists of states up to behavioural equivalence

Relation lifting

Aim: Extend (or lift) a functor $T : \mathsf{Set} \to \mathsf{Set}$ to a functor $\bar{T} : \mathsf{Rel} \to \mathsf{Rel}$

(Notation "bar T" in honour of the construction's inventer Michael Barr)

The category Rel

Given relations

$$A \xrightarrow{R} B \quad \text{and} \quad B \xrightarrow{S} C$$

there is the composition

$$A \xrightarrow{S \cdot R} C$$

which is defined by having the graph

$$\mathcal{G}(S \cdot R) = \{(a, c) \mid \exists b \in B \,.\, (a, b) \in R \;\wedge\; (b, c) \in S\}.$$

Relations form a Pos-category: They are partially ordered by inclusion $\subseteq$

Fact 1: Recognise Set inside Rel

$f : A \to B$ gives rise to two relations

$$A \xrightarrow{f_*} B$$

has the graph $\{(a, f(a) \mid a \in A\}$.

$$B \xrightarrow{f^*} A$$

has the graph $\{(f(a), a) \mid a \in A\}$.

A relation R is of the form f_* for some map f iff R is left-adjoint:

$$Id \subseteq S \cdot R \qquad R \cdot S \subseteq Id$$

Moreover, $S = f^*$.

Fact 2: Recover relations from maps

Every relation $R : A \longrightarrow B$ can be 'tabulated as a span' of maps

$$\begin{array}{ccc} & \mathcal{G}R & \\ {}^{dR}\swarrow & & \searrow^{cR} \\ A & & B \end{array}$$

and one recovers the relation from the maps:

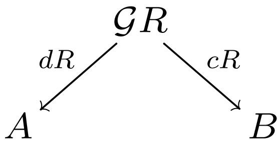
$$R = cR_* \cdot dR^*$$

$$(-)_* : \mathsf{Set} \to \mathsf{Rel}$$

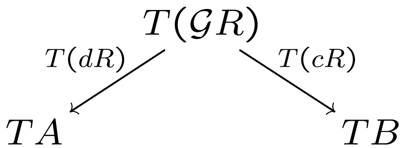
together with Facts 1 and 2 provides a very tight relationship between maps and relations

Lifting a relation $A \to B$ to a relation $TA \to TB$

Tabulate R as



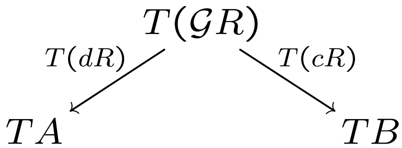
and apply T



and then reconstruct a relation using $(-)_*$ and $(-)^*$ in order to obtain

$$\bar{T}R = T(cR)_* \cdot T(dR)^*$$

Example



$$\bar{T}R = T(cR)_* \cdot T(dR)^*$$

leads to the explicit formula

$$\mathcal{G}\bar{T}R = \{(t,s) \in TA \times TB \mid \exists w \in T\mathcal{G}R\,.\,T\pi_1(w) = t, T\pi_2(w) = s\},$$

which can be used to calculate concrete examples:

$$a\bar{\mathcal{P}}b \;\Leftrightarrow\; \exists w \in \mathcal{P}(\mathcal{G}R)\,.\,\pi_1[w] = a \;\&\; \pi_2[w] = b,$$

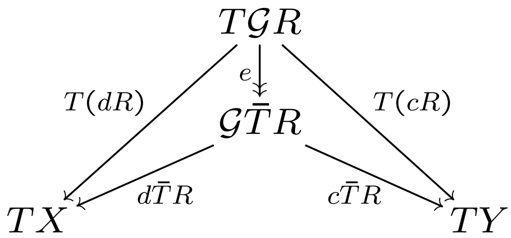
$$a\bar{\mathcal{P}}b \;\Leftrightarrow\; (\forall x \in a\,.\,\exists y \in b\,.\,xRy) \;\&$$
$$(\forall y \in b\,.\,\exists x \in a\,.\,xRy)$$

Relation lifting does not need to preserve graphs

Important: The last example does not satisfy

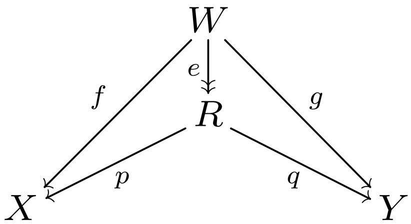
$$\mathcal{G}(\bar{T}R) \cong T(\mathcal{G}R)$$

Solution: Factor spans through an epi e and a mono-span:



Fact 3: Lifting is independent of choice of span

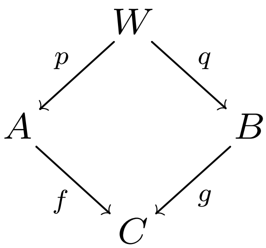
A relation can be represented by different spans.



$q_* \cdot p^* = g_* \cdot f^*$ iff e epi iff $e_* \cdot e^* = Id$.

$$\begin{aligned}(Tg)_* \cdot (Tf)^* &= (Tq \circ Te)_* \cdot (Tp \circ Te)^* \\ &= (Tq)_* \cdot (Te)_* \cdot (Te)^* \cdot (Tp)^* \\ &= (Tq)_* \cdot (Tp)^*\end{aligned}$$

Fact 4: Preservation of exact squares/weak pullbacks

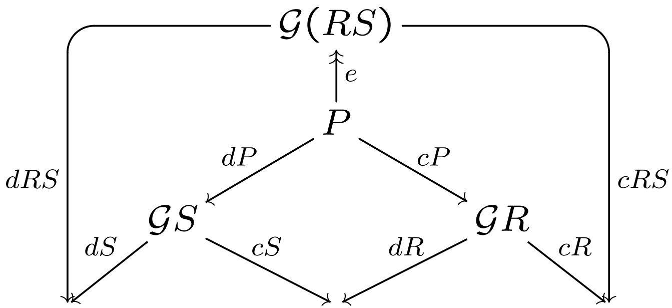


(1)

is exact iff $q_* \cdot p^* = g^* \cdot f_*$

$q_* \cdot p^* \subseteq g^* \cdot f_*$ iff (1) commutes.
$q_* \cdot p^* = g^* \cdot f_*$ iff (1) is a weak pullback.

Relation lifting $\bar{T}$ preserves composition



Thm (Barr, Trnkova, Carboni-Kelly-Wood, Hermida)
The relation lifting $\bar{T}$ satisfies

$$\bar{T}(R \cdot S) \subseteq (\bar{T}R) \cdot (\bar{T}S)$$

and

$$\bar{T}(R \cdot S) = (\bar{T}R) \cdot (\bar{T}S)$$

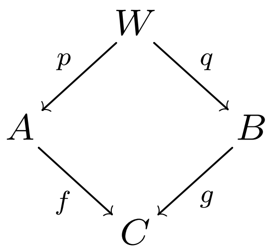
if and only if T preserves weak pullbacks.

The facts summarized

The functor $(-)_* : \mathsf{Set} \to \mathsf{Rel}$ has the following three properties:

1. $(-)_*$ preserves maps, that is, every f_* has a right-adjoint (denoted f^*).

2. For every weak pullback (=exact square)



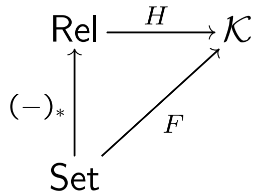
(2)

   it holds $q_* \cdot p^* = g^* \cdot f_*$

3. It holds $e_* \cdot e^* = Id$ for every epi e.

Universal property of Rel

The functor $(-)_*$ is universal:



If $\mathcal{K}$ is any Pos-category, to give a locally monotone functor H : Rel $\to \mathcal{K}$ is the same as to give a functor F : Set $\to \mathcal{K}$ with the following three properties:

1. Every Ff has a right adjoint, denoted by $(Ff)^r$.
2. For every weak pullback (2) the equality $Fq \cdot (Fp)^r = (Fg)^r \cdot Ff$ holds.
3. For every epi e it holds $Fe \cdot (Fe)^r = Id$.

Intuitively, Rel is obtained from Set by freely adding adjoints of maps.
Works not only for Set but for all regular categories.

Monotone relations

Monotone relations

Relations $R : X \to Y$ are monotone functions

$$X^{\text{op}} \times Y \to \mathbb{2}$$

where X, Y are posets (or preorders) and $\mathbb{2}$ is the two-chain.

Examples: Any relation $\Vdash$ that is *weakening-closed*:

$$\frac{x' \leq x \quad x \Vdash y \quad y \leq y'}{x' \Vdash y'}$$

the order of a lattice, the turnstile in a sequent calculus, ...

Rel(Pos): Composition and Identity

$$A \xrightarrow{R} B \quad \text{and} \quad B \xrightarrow{S} C$$

gives

$$A \xrightarrow{S \cdot R} C$$

defined by

$$S \cdot R(a, c) = \bigvee_b R(a, b) \wedge S(b, c)$$

where $\bigvee$ and $\wedge$ are taken in the lattice $\mathbb{2}$.

The identity on a poset (preorder) A is given by the order relation and written variously as

$$Id(a, a') \;=\; A(a, a') \;=\; a \leq_A a'$$

$(-)_* : \mathsf{Pos} \to \mathsf{Rel}(\mathsf{Pos})$

$$
\begin{aligned}
(-)_* : \mathsf{Pos} &\to \mathsf{Rel}(\mathsf{Pos})^{\mathsf{co}} \\
A &\mapsto B \\
f : A \to B &\mapsto \lambda a, b \,.\, B(fa, b) : A \to B \\
f \leq g &\mapsto g_* \leq f_*
\end{aligned}
$$

where the $^{\mathsf{co}}$ indicates that the order between relations gets reversed and

$$
\begin{aligned}
(-)^* : \mathsf{Pos} &\to \mathsf{Rel}(\mathsf{Pos})^{\mathsf{op}} \\
A &\mapsto B \\
f : A \to B &\mapsto \lambda a, b \,.\, B(b, fa) : B \to A \\
f \leq g &\mapsto f^* \leq g^*
\end{aligned}
$$

where the $^{\mathsf{op}}$ indicates that the order of the relations gets reversed.

Maps are adjoints

We want to show that

$$A(a, a') \leq \bigvee_b R(a, b) \wedge S(b, a')$$

and

$$\bigvee_a S(b, a) \wedge R(a, b') \leq B(b, b')$$

only if

$$R(a, b) = B(fa, b)$$

for some $f : A \to B$.

First consider the special case where A is the one element set. Then R is an upset, S is a downset, and the two inequalities ensure that there is $f \in B$ such that $S = {\downarrow}f$ and $R = {\uparrow}f$, or, in our notation, $S(b, a) = B(b, f)$ and $R(a, b) = B(f, b)$. In the general case, the same reasoning gives an fa for each $a \in A$ with $S(b, a) = B(b, fa)$ and $R(a, b) = B(fa, b)$.

Theorem

The relation lifting $\bar{T}$ satisfies

$$\bar{T}(R \cdot S) \subseteq (\bar{T}R) \cdot (\bar{T}S)$$

if T preserves epis and is functorial

$$\bar{T}(R \cdot S) = (\bar{T}R) \cdot (\bar{T}S)$$

if and only if T preserves exact squares.

The proof follows the same lines as for the discrete case (although some of the details are more intricate). The universal property is also stated and proved similarly.

Many-valued (fuzzy) relations

Generalise from $\mathbb{2}$ to a lattice $\mathcal{V}$ of truth values

Interesting example (Lawvere)

(lattice of truth values) = (lattice of distances)$^{\text{op}}$

$\mathcal{V} = (([0, \infty], \geq_{\mathbb{R}}), +, 0)$

0 is top, ∞ is bottom, join is inf, meet is sup, implication is truncated minus $-$

Instead of Pos or Preord one obtains generalised metric spaces (gms)

A gms is a metric space, but distances need not be symmetric.

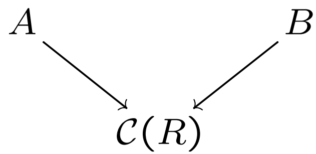
A gms comes equipped with order:

$$x \leq y \Leftrightarrow X(x, y) = 0$$

Example: Finite and infinite words with metric and prefix order in one structure

Main idea

Tabulate relations $R : A \to B$ as cospans



where the 'collage' $\mathcal{C}(R)$ is defined as $A + B$ with homs

$$\mathcal{C}(R)(a, a') = A(a, a')$$

$$\mathcal{C}(R)(b, b') = B(b, b')$$

$$\mathcal{C}(R)(a, b) = R(a, b)$$

$$\mathcal{C}(R)(b, a) = \bot$$

Application: Bisimulation

Bisimulation

Aczel-Mendler

$$\begin{array}{ccccc} X & \longleftarrow & R & \longrightarrow & X' \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \xi'} \\ TX & \longleftarrow & TR & \longrightarrow & TX' \end{array}$$

Relation lifting

$$\begin{array}{ccc} X & \xleftarrow{\;R\;} & X' \\ \downarrow{\scriptstyle \xi} & \subseteq & \downarrow{\scriptstyle \xi'} \\ TX & \xleftarrow{\;\bar{T}R\;} & TX' \end{array}$$

Post-fixed point $\quad R \subseteq (\xi \times \xi')^{-1}\bar{T}(R)$

Bisimulation

Coinduction theorem:
If x, x' are related by some bisimulation, then $x = x'$ in the semantics.
The converse holds if T preserves weak pullbacks.

Example: $T = \mathcal{P}$

$$R \subseteq (\xi \times \xi')^{-1} \cdot \bar{T}(R)$$

$$xRx' \;\Rightarrow\; \xi(x)\, \bar{T}R\, \xi(x')$$

$$xRx' \;\Rightarrow\; \forall y \in \xi(x) \,.\, \exists y' \in \xi(x') \,.\, yRy' \;\&\; \forall y' \in \xi(x') \,.\, \exists y \in \xi(x) \,.\, yRy'$$

Application: Simulation

Bisimulation and Coinduction

[”Change the category, not the definition”]

Let T be a functor $\mathsf{Pos} \to \mathsf{Pos}$ or $\mathsf{Preord} \to \mathsf{Preord}$.

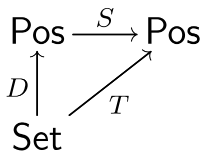
Coinduction theorem:

If xRx' for some bisimulation, then $x \leq x'$ in the semantics.

The converse holds if T preserves embeddings.

Example: Extensions of Set-functors

Theorem: There is a one-to-one correspondence between 'nerve-preserving' locally monotone functors $S : \mathsf{Pos} \to \mathsf{Pos}$ and functors $T : \mathsf{Set} \to \mathsf{Pos}$



Example: If $T = \mathcal{P}$, then S is the convex powerset functor.
The order on $S(X, \leq)$ is given by

$$a \leq_{SX} b \quad \Leftrightarrow \quad \begin{array}{l} \forall x \in a \,.\, \exists y \in b \,.\, x \leq y \\ \forall y \in b \,.\, \exists x \in a \,.\, x \leq y \end{array}$$

aka: Egli-Milner order, Plotkin powerdomain

Remark: A set-functor and its Pos-extension have the same final coalgebra.

Example: Simulations

Example: For a preorder X, let TX be the set of subsets ordered by

$$a \leq_H b \quad \Leftrightarrow \quad \forall x \in a \,.\, \exists y \in b \,.\, x \leq y$$

On posets this functor takes downsets ordered by inclusion (aka Hoare powerdomain).

Remark: TX has the same underlying set as $\mathcal{P}X$. It follows that the final T-coalgebra has the same carrier and structure as the final $\mathcal{P}$-coalgebra, but equipped with a preorder.

Metric coinduction

[“Change the lattice of truth values, not the category”]

Let T be a functor $\mathcal{V}\text{-cat} \to \mathcal{V}\text{-cat}$ and
denote by d the distance in the final coalgebra

Coinduction theorem (Rutten, Worrell):

$$d(x, x') = \mathsf{inf}\{R(x, x') \mid R \text{ bisimulation}\}$$

Note the approximating character of bisimulation.

A number of sophisticated examples from domain theory can be found in Worrells phd thesis.

Application: Coalgebraic logic

Moss's cover modality ∇

$\mathcal{V} = \mathbb{2}$ (or any commutative quantale)

$T : \mathcal{V}\text{-cat} \to \mathcal{V}\text{-cat}$

$x \in X$ and $(X, \xi) \in \mathsf{Coalg}(T)$, that is, $X \xrightarrow{\xi} TX$

$\gamma \in \mathcal{L}$ and $\mathcal{L}$ a set (or $\mathcal{V}$-cat) of 'formulas'

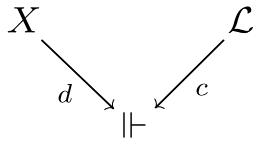
Def (Moss): For $\alpha \in T\mathcal{L}$ we define the relation $\Vdash : X \to \mathcal{L}$

$$x \Vdash \nabla\alpha \quad \Leftrightarrow \quad \xi(x)\ \bar{T}(\Vdash)\ \alpha$$

Thm (Moss): For $T : \mathsf{Set} \to \mathsf{Set}$: The logic characterises behavioural equivalence if T preserves weak pullbacks.

Example

Take $TX = \mathcal{D}own X = \mathcal{D}X = [X^{\mathrm{op}}, \mathcal{V}]$ (Hoare powerdomain)



$$
\begin{aligned}
(Tc)^* \cdot (Td)_* \, (\xi(x), \alpha) &= [\Vdash^{\mathrm{op}}, \mathcal{V}](\mathcal{D}(d)(\xi(x)), \mathcal{D}(c)(\alpha)) \\
&= [\Vdash^{\mathrm{op}}, \mathcal{V}](\xi(x)), [d^{\mathrm{op}}, \mathcal{V}] \cdot \mathcal{D}(c)(\alpha)) \\
&= \bigwedge\nolimits_{y \in \xi(x)} (\xi(x)(y) \triangleright \bigvee\nolimits_{\phi \in \mathcal{L}} \alpha(\phi) \, \otimes \, \Vdash(y, \phi)) \\
&= \mathsf{sup}_{y \in \xi(x)} \, (\, \mathsf{inf}_{\phi \in \mathcal{L}} \, (\, \alpha(\phi) \, + \, y \Vdash \phi \,) - \xi(x)(y) \,)
\end{aligned}
$$

If $\alpha = \{\phi\}$ and ϕ is crisp and X is discrete:

$$(Tc)^* \cdot (Td)_* \, (\xi(x), \alpha) \;=\; \forall y \in \xi(x) \, . \, y \Vdash \phi$$ that is $\nabla = \Box$

Note that the functor-specific part (in brown) is purely algebraic.

Conclusion

Changing the category $\mathcal{V}$ of truth values: There is category theory parametric in $\mathcal{V}$ (enriched categories), with substantial simplifications if $\mathcal{V}$ is a poset.

It should be interesting to develop the metric aspects further.

Applications to quantitative aspects of verification?

What is the relation algebra of monotone relations?

What is the universal algebra of mixed variance?

Connections with parametricity a la Reynolds?

Some References

Michael Barr

Claudio Hermida

Bart Jacobs

Larry Moss

Jan Rutten

James Worrell

Kupke-Kurz-Venema, Bilkova-Kurz-Petrisan-Velebil, Balan-Kurz-Velebil

Appendix

Proposition: If in the situation

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{R} & \mathcal{B} \\ F\uparrow & \cong & \uparrow G \\ \mathcal{A} & \xrightarrow[R]{} & \mathcal{B} \end{array}$$

R is a right-adjoint, then R lifts to a right-adjoint

$$\tilde{R} : \mathsf{Coalg}(F) \to \mathsf{Coalg}(G)$$

Proof: Let L be the left-adjoint of R. One defines

$$\tilde{R}(A \to FA) = RA \to RFA \cong GRA$$

Moreover, $GR \to RF$ has a ‘mate’ $LG \to FL$ which allows us to define

$$\tilde{L}(B \to GB) = LB \to LGB \to FLB$$