A point-free relation-algebraic approach to general topology

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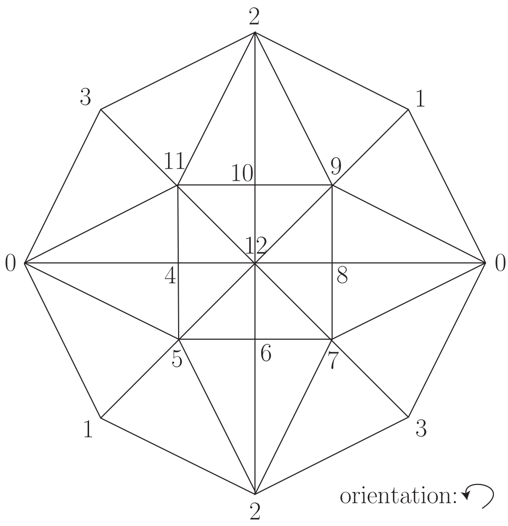
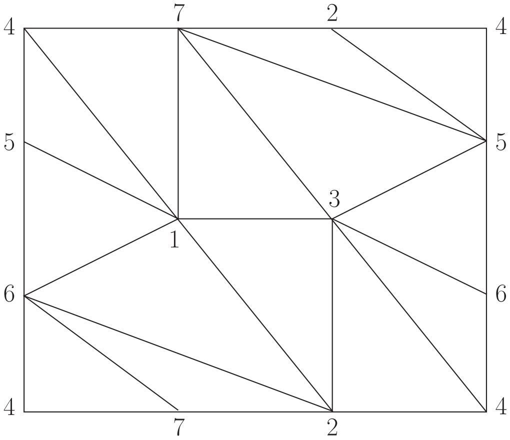
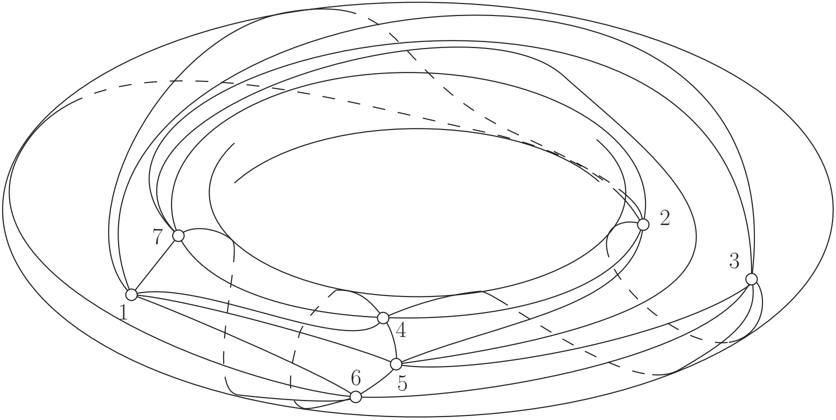
Contents



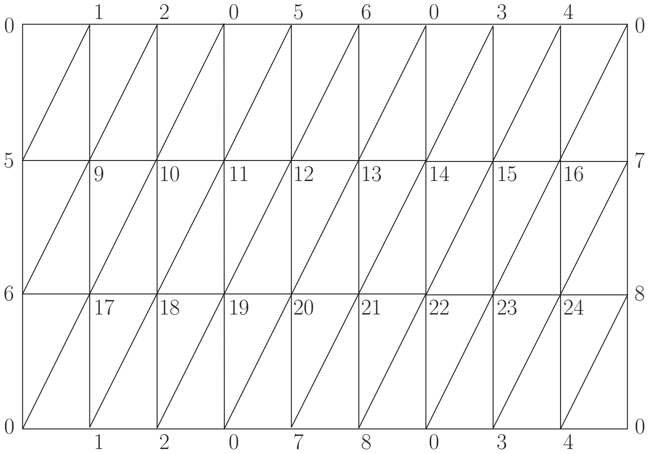
Triangulation of the projective plane



Triangulation of the Csaszar polynomial



Triangulation of the Csaszar torus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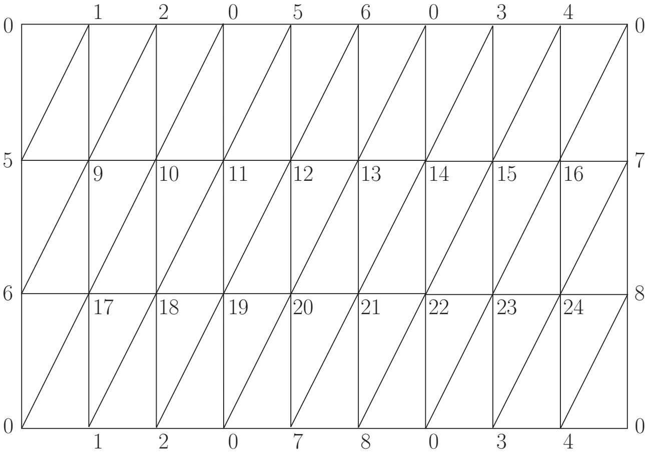
0 0
5 7
9 10 11 12 13 14 15 16
6 8
17 18 19 20 21 22 23 24
0 0
1 2 0 7 8 0 3 4



1 2 0 5 6 0 3 4
0 0
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0 0
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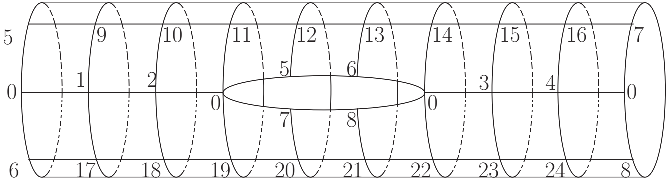

5 9 10 11 12 13 14 15 16 7
1 2 5 6 3 4
0 0 0 0
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6 17 18 19 20 21 22 23 24 8

Contents

Topology — how it emerged

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Early in the twentieth century, topology has split into general or point set theory, mainly invented by Georg Cantor and later developed further by Felix Hausdorff, and what we today call algebraic topology, elaborated as Alexander Grothendieck's cathedral.

Topology as defined by Felix Hausdorff

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iv) For every $U \in \mathcal{U}(p)$ there exists a $V \in \mathcal{U}(p)$ such that $U \in \mathcal{U}(y)$ for all $y \in V$

Contents

Axioms

A heterogeneous relation algebra

- is a category wrt. composition "$;$" and identities $\mathbb{I}$,
- has as morphism sets complete atomic boolean lattices with $\cup, \cap, \overline{}, \mathbb{\bot}, \mathbb{T}, \subseteq$,
- obeys rules for transposition $^\mathsf{T}$ in connection with the latter two that may be stated in either one of the following two ways:

Dedekind rule:
$R;S \cap Q \subseteq (R \cap Q;S^\mathsf{T});(S \cap R^\mathsf{T};Q)$

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Schröder equivalences:
$A;B \subseteq C \iff A^\mathsf{T};\overline{C} \subseteq \overline{B} \iff \overline{C};B^\mathsf{T} \subseteq \overline{A}$

Residuals and the symmetric quotient

- $R \backslash S := \overline{R^{\mathsf{T}} ; \overline{S}}$ left residuum

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  The symmetric quotient sets into relation equal columns.

Illustrating the left residuum

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
US	0	0	0	1	0	1	1	1	0	1	0	0
French	1	0	0	1	0	0	1	0	0	1	0	0
German	1	1	0	0	1	1	0	1	0	0	0	1
British	1	1	0	0	0	0	1	0	1	0	1	1
Spanish	0	0	0	1	0	1	1	1	0	0	0	0

	A	K	Q	J	10	9	8	7	6	5	4	3	2
US	0	0	0	0	0	0	0	0	0	0	0	0	1
French	0	1	0	0	0	0	0	1	0	0	0	0	0
German	0	0	1	0	0	0	1	1	0	1	0	1	0
British	0	1	1	0	0	0	0	1	0	0	0	0	1
Spanish	0	0	0	1	0	0	1	0	0	1	1	0	1

S above R below

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
A	1	1	1	1	1	1	1	1	1	1	1	1
K	1	0	0	0	0	0	1	0	0	0	0	0
Q	1	1	0	0	0	0	0	0	0	0	0	1
J	0	0	0	1	0	1	1	1	0	0	0	0
10	1	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1	1
8	0	0	0	0	0	1	0	1	0	0	0	0
7	1	0	0	0	0	0	0	0	0	0	0	0
6	1	1	1	1	1	1	1	1	1	1	1	1
5	0	0	0	0	0	1	0	1	0	0	0	0
4	0	0	0	1	0	1	1	1	0	0	0	0
3	1	1	0	0	1	1	0	1	0	0	0	1
2	0	0	0	0	0	0	1	0	0	0	0	0

$R \backslash S$

Left residua show how columns of the relation R below the fraction backslash are contained in columns of the relation S above

Illustrating the symmetric quotient

R above

	A	K	Q	J	10	9	8	7	6	5	4	3	2
American	0	0	0	0	0	0	0	0	0	0	0	0	1
French	0	1	0	0	0	0	0	1	0	0	0	0	0
German	0	0	1	0	0	0	1	1	0	1	0	1	0
British	0	1	1	0	0	0	0	1	0	0	0	0	1
Spanish	0	0	0	1	0	0	1	0	0	1	1	0	1

S below

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
American	0	0	0	1	0	1	1	1	0	1	0	0
French	1	0	0	1	0	0	1	0	0	1	0	0
German	1	1	0	0	1	1	0	1	0	0	0	1
British	1	1	0	0	0	0	1	0	1	0	1	1
Spanish	0	0	0	1	0	1	1	1	0	0	0	0

$\mathtt{syq}(R, S)$

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
A	0	0	1	0	0	0	0	0	0	0	0	0
K	0	0	0	0	0	0	0	0	0	0	0	0
Q	0	1	0	0	0	0	0	0	0	0	0	1
J	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	1	0	0	0	0	0	0	0	0	0
9	0	0	1	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0

The symmetric quotient shows which columns of the upper are equal to columns of the lower relation

Set Comprehension

Finding equal columns i, k of relations R, S:

$$\forall n: \quad (n,i) \in R \;\leftrightarrow\; (n,k) \in S$$

$$\forall n: \quad (n,i) \in R \;\rightarrow\; (n,k) \in S \quad \wedge \quad (n,i) \in R \;\leftarrow\; (n,k) \in S$$

$$\forall n: \quad (n,i) \in R \;\rightarrow\; (n,k) \in S \quad \text{and} \quad \forall n: \quad (n,i) \in R \;\leftarrow\; (n,k) \in S$$

$$\overline{\exists n: \quad (n,i) \in R \;\wedge\; (n,k) \notin S} \quad \text{and} \quad \overline{\exists n \quad (n,i) \notin R \;\wedge\; (n,k) \in S}$$

$$(i,k) \in \overline{R^{\mathsf{T}} ; \overline{S}} \cap \overline{\overline{R}^{\mathsf{T}} ; S}$$

Construction of domains

Given a relation algebra, we may extend it in several ways:

- direct product
- direct sum
- direct power
- quotient
- extrusion
- target permutation

Construction of domains

Given any direct products by projections

$\pi : X \times Y \longrightarrow X, \quad \rho : X \times Y \longrightarrow Y,$
$\pi' : U \times V \longrightarrow U, \quad \rho' : U \times V \longrightarrow V,$

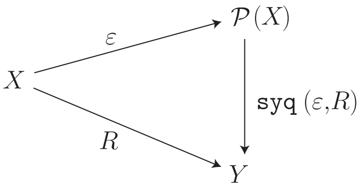
we define the Kronecker product, the fork-, and the join-operator:

i) $(A \otimes B) := \pi{;}A{;}\pi'^{\mathsf{T}} \cap \rho{;}B{;}\rho'^{\mathsf{T}}$

ii) $(C \mathbin{\circledless} D) := C{;}\pi^{\mathsf{T}} \cap D{;}\rho^{\mathsf{T}}$

iii) $(E \mathbin{\circledgtr} F) := \pi{;}E \cap \rho{;}F$

Direct power — up to isomorphism



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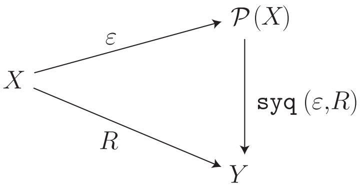

Any relation ε satisfying

- $\mathtt{syq}(\varepsilon, \varepsilon) \subseteq \mathbb{I}$, (i.e., in fact $\mathtt{syq}(\varepsilon, \varepsilon) = \mathbb{I}$)
- $\mathtt{syq}(\varepsilon, R)$ is surjective for every relation R starting in X.

is called a

direct power	interpreted with $\in$-relation
`DirPow x`	$\mathcal{P}(X)$
`Member x`	$\varepsilon : X \longrightarrow \mathcal{P}(X)$

$$
\varepsilon = \begin{array}{c} \\ \spadesuit \\ \heartsuit \\ \diamondsuit \\ \clubsuit \end{array}
\left(\begin{array}{cccccccc|cccccccc}
\overbrace{}^{} & \overbrace{\spadesuit} & \overbrace{\heartsuit} & \overbrace{\spadesuit\heartsuit} & \overbrace{\diamondsuit} & \overbrace{\spadesuit\diamondsuit} & \overbrace{\heartsuit\diamondsuit} & \overbrace{\spadesuit\heartsuit\diamondsuit} & \overbrace{\clubsuit} & \overbrace{\spadesuit\clubsuit} & \overbrace{\heartsuit\clubsuit} & \overbrace{\spadesuit\heartsuit\clubsuit} & \overbrace{\diamondsuit\clubsuit} & \overbrace{\spadesuit\diamondsuit\clubsuit} & \overbrace{\heartsuit\diamondsuit\clubsuit} & \overbrace{\spadesuit\heartsuit\diamondsuit\clubsuit} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1}
\end{array}\right)
$$

$$
\begin{array}{c}
\begin{array}{cccccccc|cccccccc}
\{\} & \{\spadesuit\} & \{\heartsuit\} & \{\spadesuit,\heartsuit\} & \{\diamondsuit\} & \{\spadesuit,\diamondsuit\} & \{\heartsuit,\diamondsuit\} & \{\spadesuit,\heartsuit,\diamondsuit\} & \{\clubsuit\} & \{\spadesuit,\clubsuit\} & \{\heartsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\clubsuit\} & \{\diamondsuit,\clubsuit\} & \{\spadesuit,\diamondsuit,\clubsuit\} & \{\heartsuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\diamondsuit,\clubsuit\}
\end{array}\\
\varepsilon = \begin{array}{c}\spadesuit\\ \heartsuit\\ \diamondsuit\\ \clubsuit\end{array}
\left(\begin{array}{cccccccc|cccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\{\spadesuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\diamondsuit\} & \{\spadesuit,\heartsuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\clubsuit\} & \{\diamondsuit\} & \{\spadesuit,\diamondsuit\} & \{\heartsuit,\diamondsuit\} & \{\spadesuit\} & \{\heartsuit,\clubsuit\} & \{\spadesuit,\clubsuit\} & \{\clubsuit\} & \{\spadesuit,\heartsuit\} & \{\diamondsuit,\clubsuit\} & \{\} & \{\heartsuit,\diamondsuit,\clubsuit\} & \{\heartsuit\}
\end{array}\\
\varepsilon' = \begin{array}{c}\spadesuit\\ \heartsuit\\ \diamondsuit\\ \clubsuit\end{array}
\left(\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1\\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}\right)
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\varepsilon = \begin{array}{c|cccccccc|cccccccc}
 & \{\} & \{\spadesuit\} & \{\heartsuit\} & \{\spadesuit,\heartsuit\} & \{\diamondsuit\} & \{\spadesuit,\diamondsuit\} & \{\heartsuit,\diamondsuit\} & \{\spadesuit,\heartsuit,\diamondsuit\} & \{\clubsuit\} & \{\spadesuit,\clubsuit\} & \{\heartsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\clubsuit\} & \{\diamondsuit,\clubsuit\} & \{\spadesuit,\diamondsuit,\clubsuit\} & \{\heartsuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\diamondsuit,\clubsuit\} \\
\hline
\spadesuit & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\heartsuit & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\diamondsuit & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
\clubsuit & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

$$
\varepsilon' = \begin{array}{c|cccccccccccccccc}
 & \{\spadesuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\diamondsuit\} & \{\spadesuit,\heartsuit,\diamondsuit,\clubsuit\} & \{\spadesuit,\heartsuit,\clubsuit\} & \{\diamondsuit\} & \{\spadesuit,\diamondsuit\} & \{\heartsuit,\diamondsuit\} & \{\spadesuit\} & \{\heartsuit,\clubsuit\} & \{\spadesuit,\clubsuit\} & \{\clubsuit\} & \{\spadesuit,\heartsuit\} & \{\diamondsuit,\clubsuit\} & \{\} & \{\heartsuit,\diamondsuit,\clubsuit\} & \{\heartsuit\} \\
\hline
\spadesuit & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\heartsuit & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\diamondsuit & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\clubsuit & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}
$$

$P := \mathtt{syq}(\varepsilon, \varepsilon')$ satisfies $\varepsilon ; \mathtt{syq}(\varepsilon, \varepsilon') = \varepsilon'$

Membership relations

$$U = \varepsilon ; e \qquad e = \texttt{syq}(\varepsilon, U)$$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}		U
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1		0
b	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1		1
c	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1		0
d	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1		1

$$(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0) = e^{\mathsf{T}}$$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}		e
{}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		0
{a}	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1		0
{b}	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1		0
{a,b}	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1		0
{c}	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1		0
{a,c}	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1		0
{b,c}	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1		0
{a,b,c}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1		0
{d}	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1		0
{a,d}	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1		0
{b,d}	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1		1
{a,b,d}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1		0
{c,d}	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1		0
{a,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1		0
{b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1		0
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		0

Subset U and corresponding point e in the powerset via ε, Ω

Contents

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A neighborhood topology and the basis of its open sets

A relation $\mathcal{U} : X \longrightarrow \mathbf{2}^X$ will be called a **neighborhood topology** if the following properties are satisfied:

i) $\mathcal{U};\mathbb{T} = \mathbb{T} \quad$ and $\quad \mathcal{U} \subseteq \varepsilon$,

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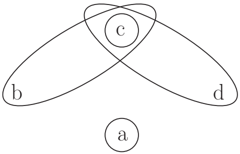
$\mathcal{U} =$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
b	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
c	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
d	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1



Topology given by transition to the open kernel

We call a relation $\mathcal{K} : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ a **mapping-to-open-kernel topology**, if

i) $\mathcal{K}$ is a kernel forming, i.e.,

$$\mathcal{K} \subseteq \Omega^{\mathsf{T}}, \qquad \Omega;\mathcal{K} \subseteq \mathcal{K};\Omega, \qquad \mathcal{K};\mathcal{K} \subseteq \mathcal{K},$$

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$$\begin{array}{ccc} \mathcal{K} \subseteq \Omega^{\mathsf{T}}, & \Omega;\mathcal{K} \subseteq \mathcal{K};\Omega, & \mathcal{K};\mathcal{K} \subseteq \mathcal{K}, \\ \text{contracting} & \text{isotonic} & \text{idempotent} \end{array}$$

ii) $\varepsilon;\mathcal{K}^{\mathsf{T}}$ is total,

iii) $(\mathcal{K} \otimes \mathcal{K});\mathcal{M} \subseteq \mathcal{M};\mathcal{K};\Omega^{\mathsf{T}}$, in fact $(\mathcal{K} \otimes \mathcal{K});\mathcal{M} = \mathcal{M};\mathcal{K}$.

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contracting — isotonic — idempotent

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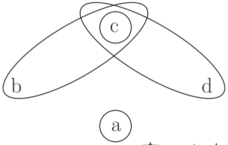
kernel forming commutes with intersection

A topology in different forms



ε

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
b	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
c	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
d	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

$\mathcal{U}$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
b	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
c	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
d	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

$\varepsilon_{\mathcal{O}} := \varepsilon \cap \mathbb{T};\mathcal{O}_V^{\mathsf{T}} = \varepsilon;\mathcal{K} \cap \varepsilon$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
a	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	1
b	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
c	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
d	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

$\mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon)$ indicating $\mathcal{O}_D$ as diagonal, with $\mathcal{O}_V$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}	$\mathcal{O}_V$
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
{a}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
{b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
{a,c}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
{b,c}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
{a,b,c}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
{d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,d}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b,d}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c,d}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
{a,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
{b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1

Non-topological kernel-forming

$\mathcal{K} =$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{c}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,c}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{b,c}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b,c}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,d}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
{b,d}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
{a,b,d}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
{c,d}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,c,d}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
{b,c,d}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Kernel-forming that is not a topology, since not intersection-closed

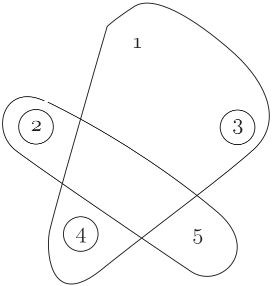
	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
1	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
2	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**
3	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**
4	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**

① 2

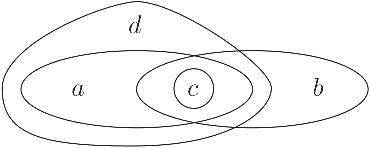
③ 4

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	{5}	{1,5}	{2,5}	{1,2,5}	{3,5}	{1,3,5}	{2,3,5}	{1,2,3,5}	{4,5}	{1,4,5}	{2,4,5}	{1,2,4,5}	{3,4,5}	{1,3,4,5}	{2,3,4,5}	{1,2,3,4,5}
1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1



	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
a	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**
b	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**
c	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**
d	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**



Cryptomorphy of diverse topology concepts

$$\mathcal{U} \quad \mapsto \quad \mathcal{K} := \mathtt{syq}(\mathcal{U}, \varepsilon) : \mathbf{2}^X \longrightarrow \mathbf{2}^X$$

$$\mathcal{K} \quad \mapsto \quad \mathcal{U} := \varepsilon ; \mathcal{K}^{\mathsf{T}} : X \longrightarrow \mathbf{2}^X .$$

$$\mathcal{O}_D \quad \mapsto \quad \mathcal{U} := \varepsilon ; \mathcal{O}_D ; \Omega$$

$$\mathcal{K}, \mathcal{U} \quad \mapsto \quad \mathcal{O}_D := \mathbb{I} \cap \overline{\varepsilon^{\mathsf{T}} ; \overline{\mathcal{U}}} = \mathcal{K}^{\mathsf{T}} ; \mathcal{K}$$

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This means the obligation to prove, e.g.

$$\begin{array}{lcl}
\mathcal{U} ; \mathbb{T} = \mathbb{T}, & & \mathcal{K} \subseteq \Omega^\mathsf{T}, \\
\mathcal{U} \subseteq \varepsilon, & & \Omega ; \mathcal{K} \subseteq \mathcal{K} ; \Omega, \\
\mathcal{U} ; \Omega \subseteq \mathcal{U}, & \Longleftrightarrow & \mathcal{K} ; \mathcal{K} \subseteq \mathcal{K}, \\
(\mathcal{U} \circledless \mathcal{U}) ; \mathcal{M} \subseteq \mathcal{U}, & & \varepsilon ; \mathcal{K}^\mathsf{T} ; \mathbb{T} = \mathbb{T}, \\
\mathcal{U} \subseteq \mathcal{U} ; \overline{\varepsilon^\mathsf{T} ; \overline{\mathcal{U}}}. & & (\mathcal{K} \otimes \mathcal{K}) ; \mathcal{M} = \mathcal{M} ; \mathcal{K}.
\end{array}$$

Separation axioms

Let a topology on X be given via neighborhoods, open sets, kernel mapping as required.

It is T_0-space (sometimes a Kolmogorov space) if for any two points in X an open set exists that contains one of them but not the other.

It is T_1-space when
$\forall x, y : x \neq y \rightarrow \exists U, V \in \mathcal{O} : x \in U \wedge y \notin U \wedge y \in V \wedge x \notin V$.

It is T_2-space, i.e., a topology satisfying the Hausdorff property, when
$\forall x, y : x \neq y \rightarrow \exists U, V \in \mathcal{O} : x \in U \wedge y \in V \wedge \emptyset = U \cap V$.

Separation axioms

Let a topology given in relational form, i.e., by $\mathcal{U}, \mathcal{O}, \mathcal{K}, \varepsilon_{\mathcal{O}}$ as required. It is called a

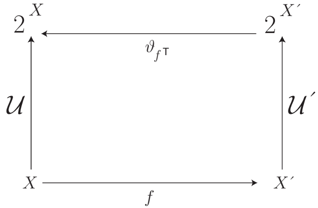
i) T_0-**space** if $\mathtt{syq}(\mathcal{U}^\mathsf{T}, \mathcal{U}^\mathsf{T}) = \mathbb{I}$

ii) T_1-**space** if $\overline{\mathbb{I}} \subseteq \mathcal{U};\overline{\mathcal{U}}^\mathsf{T}$.

iii) T_2-**space** or a **Hausdorff** space if $\overline{\mathbb{I}} \subseteq \mathcal{U};\overline{\varepsilon^\mathsf{T};\varepsilon};\mathcal{U}^\mathsf{T}$.

Contents

Continuity — standard vs. relational definition

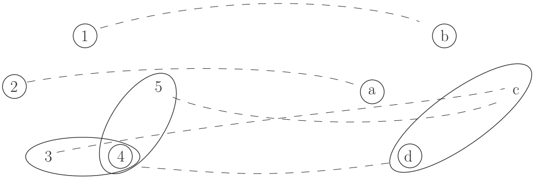


Let any two neighborhood topologies $\mathcal{U}, \mathcal{U}'$ be given on sets X, X', and a mapping $f : X \longrightarrow X'$.

f **continuous** $:\Longleftrightarrow$ For $p \in X$ and every neighborhood $U' \in \mathcal{U}'(f(p))$, there exists a neighborhood $U \in \mathcal{U}(p)$ satisfying $f(U) \subseteq U'$.

$f =$

	a	b	c	d
1	0	1	0	0
2	1	0	0	0
3	0	0	1	0
4	0	0	0	1
5	0	0	1	0



$\vartheta_f =$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{1}	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
{2}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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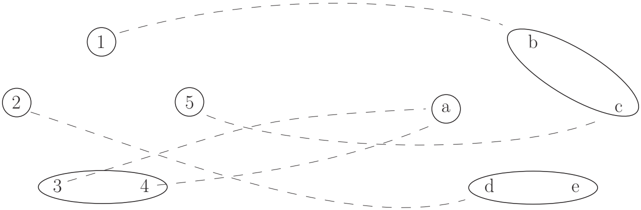
{1,2,3,4,5}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

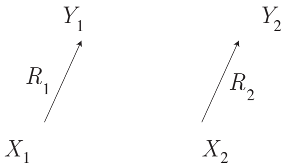
$$f = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}$$



Contents

Structure-preserving mappings

Let be given two "structures" of whatever kind
abstracted to relations $R_1 : X_1 \longrightarrow Y_1$ and $R_2 : X_2 \longrightarrow Y_2$.



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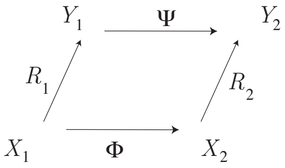

Given mappings $\Phi : X_1 \longrightarrow X_2$ and $\Psi : Y_1 \longrightarrow Y_2$, we may ask whether these mappings transfer the first structure "sufficiently nice" into the second one.

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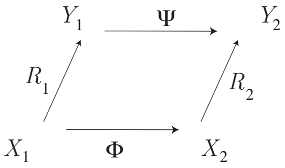

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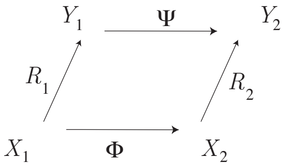

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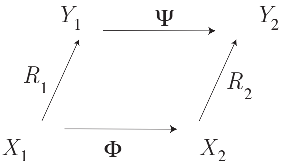

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$$R_1 ; \Psi \subseteq \Phi ; R_2$$

Homomorphism

This concept works for groups, fields and other *algebraic structures*, but also for *relational structures* as, e.g., graphs.

Φ, Ψ is a **homomorphism** from R to R', if
Φ, Ψ are mappings satisfying $R;\Phi \subseteq \Psi;R'$.

Φ, Ψ is an **isomorphism** between R and R', if
Φ, Ψ as well as $\Phi^\mathsf{T}, \Psi^\mathsf{T}$ are homomorphisms.

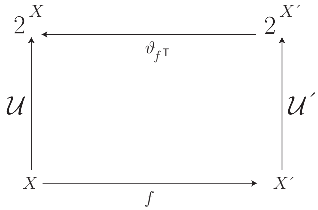
Theorem

If Φ, Ψ *are mappings, then*

$$R;\Psi \subseteq \Phi;R' \iff R \subseteq \Phi;R';\Psi^\mathsf{T} \iff$$
$$\Phi^\mathsf{T};R \subseteq R';\Psi^\mathsf{T} \iff \Phi^\mathsf{T};R;\Psi \subseteq R'$$

If relations Φ, Ψ are not mappings, one cannot fully execute this rolling; there remain different forms of (bi-)simulations.

Continuity compares structures in a different way!



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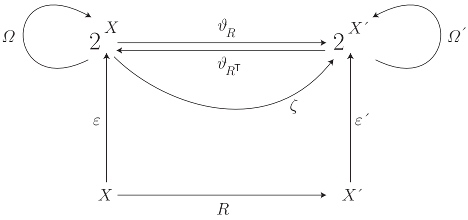
Contents

Existential image of relations



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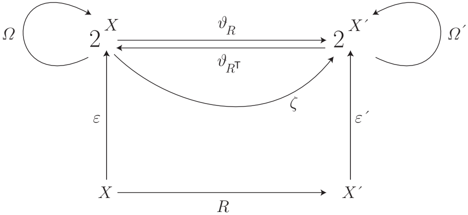
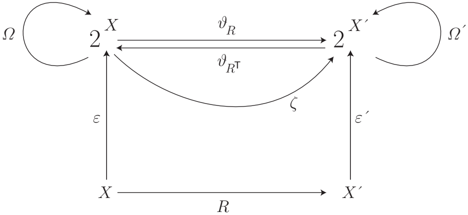


$$\vartheta := \vartheta_R := \mathtt{syq}(R^{\mathsf{T}};\varepsilon, \varepsilon') \quad \textbf{existential image}.$$

Existential image of relations



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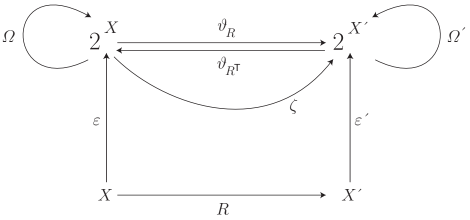

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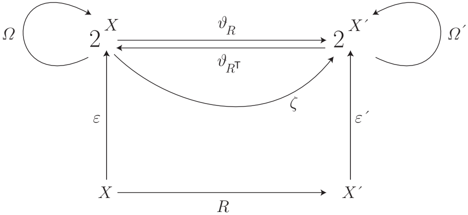
$\vartheta_{\mathbb{I}_X} = \mathbb{I}_{\mathbf{2}^X}$ $\vartheta_{Q;R} = \vartheta_Q;\vartheta_R$ i.e. multiplicative

$\varepsilon^\mathsf{T};R = \vartheta_R;\varepsilon'^\mathsf{T}$ $\varepsilon'^\mathsf{T};R^\mathsf{T} = \vartheta_{R^\mathsf{T}};\varepsilon^\mathsf{T}$ i.e. mutual simulation

R may be re-obtained from ϑ as $R = \overline{\varepsilon;\vartheta;\overline{\varepsilon'}^\mathsf{T}}$

Existential image of relations



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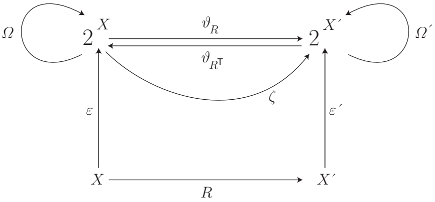
R may be re-obtained from ϑ as $R = \overline{\varepsilon;\vartheta;\overline{\varepsilon'}^\mathsf{T}}$

but there exist many relations W satisfying $R = \overline{\varepsilon;W;\overline{\varepsilon'}^\mathsf{T}}$

Existential image



$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$\vartheta_R =$

	{}	{a}	{b}	{a,b}	{c}	{a,c}	{b,c}	{a,b,c}	{d}	{a,d}	{b,d}	{a,b,d}	{c,d}	{a,c,d}	{b,c,d}	{a,b,c,d}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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{1,2}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
{3}	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
{1,3}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
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{4}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
{1,4}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
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{3,4}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
{1,3,4}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
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{1,3,5}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
{2,3,5}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
{1,2,3,5}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
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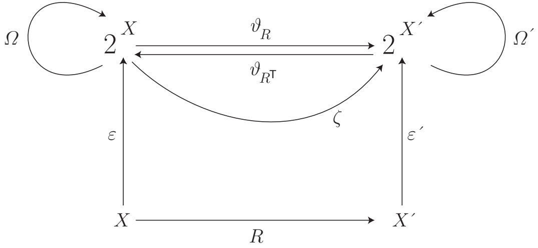
{2,3,4,5}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
{1,2,3,4,5}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Inverse image



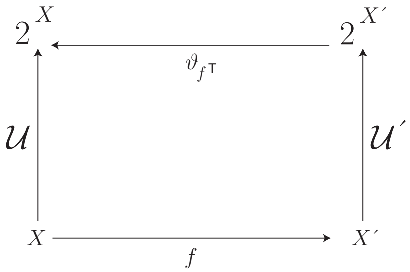
$$R = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 \end{array}$$

$\vartheta_{R^\intercal} =$

	{}	{1}	{2}	{1,2}	{3}	{1,3}	{2,3}	{1,2,3}	{4}	{1,4}	{2,4}	{1,2,4}	{3,4}	{1,3,4}	{2,3,4}	{1,2,3,4}	{5}	{1,5}	{2,5}	{1,2,5}	{3,5}	{1,3,5}	{2,3,5}	{1,2,3,5}	{4,5}	{1,4,5}	{2,4,5}	{1,2,4,5}	{3,4,5}	{1,3,4,5}	{2,3,4,5}	{1,2,3,4,5}
{}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
{b}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
{c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
{a,c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
{b,c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
{a,b,c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
{d}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
{b,d}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
{a,b,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
{c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
{a,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
{b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
{a,b,c,d}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Contents

Continuity compares structures in a different way!



Let any two neighborhood topologies $\mathcal{U}, \mathcal{U}'$ be given on sets X, X', and a mapping $f : X \longrightarrow X'$.

f **continuous** $:\Longleftrightarrow$ For $p \in X$ and every neighborhood $U' \in \mathcal{U}'(f(p))$, there exists a neighborhood $U \in \mathcal{U}(p)$ satisfying $f(U) \subseteq U'$.

Lifting the continuity condition

For all $p \in X$, all $V \in \mathcal{U}'(f(p))$, exists a $U \in \mathcal{U}(p)$ with $f(U) \subseteq V$.

$$\forall p \in X : \forall V \in \mathcal{U}'(f(p)) : \exists U \in \mathcal{U}(p) : f(U) \subseteq V$$

$$\forall p \in X : \forall v \in \mathbf{2}^{X'} : \mathcal{U}'_{f(p),v} \longrightarrow \left(\exists u : \mathcal{U}_{p,u} \wedge \left[\forall y : \varepsilon_{yu} \rightarrow \varepsilon'_{f(y),v}\right]\right)$$

$$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \left(\exists u : \mathcal{U}_{pu} \wedge \left[\forall y : \varepsilon_{yu} \rightarrow (f;\varepsilon')_{yv}\right]\right)$$

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$$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \left(\exists u : \mathcal{U}_{pu} \wedge \overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon'}}_{uv}\right)$$

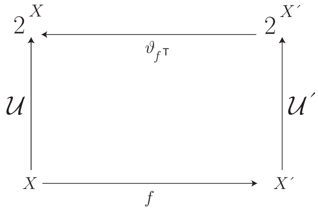
$$\forall p : \forall v : (f;\mathcal{U}')_{pv} \longrightarrow \left(\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon'}}\right)_{pv}$$

$$f;\mathcal{U}' \subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon'}}$$

$f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}$ The last step is proved as follows:

$$\begin{aligned}
\mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon'}} &\subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon';\vartheta_{f^{\mathsf{T}}};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}} && \text{because } \vartheta_{f^{\mathsf{T}}} \text{ is total} \\
&= \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{f;\varepsilon';\mathtt{syq}(f;\varepsilon',\varepsilon);\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}} && \text{by definition of } \vartheta_{f^{\mathsf{T}}} \\
&\subseteq \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}}} && \text{cancellation} \\
&= \mathcal{U};\overline{\varepsilon^{\mathsf{T}};\overline{\varepsilon}};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} && \text{since } \vartheta_{f^{\mathsf{T}}} \text{ is a mapping} \\
&= \mathcal{U};\Omega;\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}} = \mathcal{U};\vartheta^{\mathsf{T}}_{f^{\mathsf{T}}}
\end{aligned}$$

Continuity — standard vs. relational definition



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$$f \text{ **continuous** } :\Longleftrightarrow \quad f;\mathcal{U}';\vartheta_{f^\mathsf{T}} \subseteq \mathcal{U}$$

$$\Longleftrightarrow \quad f;\mathcal{U}' \subseteq \mathcal{U};\vartheta^\mathsf{T}_{f^\mathsf{T}}$$

Cryptomorphy of continuity concepts

Given sets X, X' with topologies, we consider a mapping $f : X \longrightarrow X'$ together with its inverse image $\vartheta_{f^\mathsf{T}} : \mathbf{2}^{X'} \longrightarrow \mathbf{2}^{X}$. Then we say that the pair $(f, \vartheta_{f^\mathsf{T}})$ is

i) $\mathcal{K}$-**continuous** $:\Longleftrightarrow$ $\mathcal{K}_2^\mathsf{T} ; \vartheta_{f^\mathsf{T}} \subseteq \overline{\varepsilon_2{}^\mathsf{T} ; f^\mathsf{T} ; \overline{\varepsilon_1}} ; \mathcal{K}_1^\mathsf{T}$

ii) $\mathcal{O}_D$-**continuous** $:\Longleftrightarrow$ $\mathcal{O}_{D2} ; \vartheta_{f^\mathsf{T}} \subseteq \vartheta_{f^\mathsf{T}} ; \mathcal{O}_{D1}$

iii) $\mathcal{O}_V$-**continuous** $:\Longleftrightarrow$ $\vartheta_{f^\mathsf{T}}^\mathsf{T} ; \mathcal{O}'_V \subseteq \mathcal{O}_V$

iv) $\varepsilon_{\mathcal{O}}$-**continuous** $:\Longleftrightarrow$ $f ; \varepsilon_{\mathcal{O}_2} ; \vartheta_{f^\mathsf{T}} \subseteq \varepsilon_{\mathcal{O}_1}$

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iv) $\varepsilon_\mathcal{O}$-**continuous** $:\Longleftrightarrow$ $f;\varepsilon_{\mathcal{O}_2};\vartheta_{f^\mathsf{T}} \subseteq \varepsilon_{\mathcal{O}_1}$

Again, there is an obligation to prove

f is $\mathcal{K}$-**continuous** $\Longleftrightarrow$ f is $\mathcal{O}_D$-**continuous** $\Longleftrightarrow$
f is $\mathcal{O}_V$-**continuous** $\Longleftrightarrow$ f is $\varepsilon_\mathcal{O}$-**continuous**

Thank you!

Language and system

Systems to support work with relations

- RelView: RBDD-Implementierung; auch für große Relationen
- **TituRel** eine relationale Sprache, transformierbar, interpretierbar
- Ralf: weiland ein guter Formel-Manipulator und Beweis-Assistent
- RATH: Exploring (finite) relation algebras with tools written in Haskell

Aims in designing TituRel

- Formulate all problems so far tackled with relational methods
- Transform relational terms and formulae in order to optimize them
- Interpret the relational constructs as boolean matrices, in RelView, in the TituRel substrate, or in Rath
- Prove relational formulae with system support in the style of Ralf or Rasiowa-Sikorski
- Translate relational formulae into TeX-representation, or to first-order predicate logic, e.g.

Recalling syntax vs. semantics for PL/I:

tokens	K const. φ fct. p pred.	interpretation I in supporting set	an element K_I for K a function table φ_I for φ s subset p_I for p

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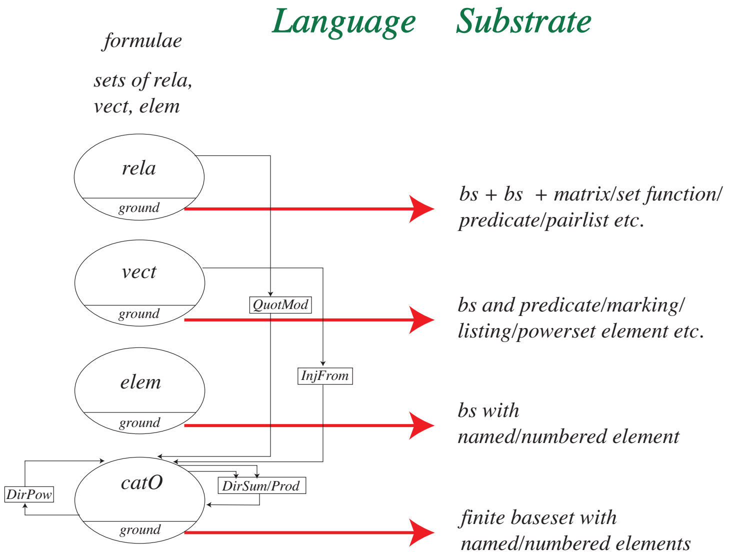
and formulae interpreted

$$\models_{I,v} p(t) \;:\Longleftrightarrow\; v^*(t) \subseteq p_I \qquad \models_{I,v} \neg F \;:\Longleftrightarrow\; \not\models_{I,v} F$$

$$\models_{I,v} \forall x : F \quad :\Longleftrightarrow \quad \text{For all } s \text{ holds} \models_{I,v_{x \leftarrow s}} F$$

Relational language



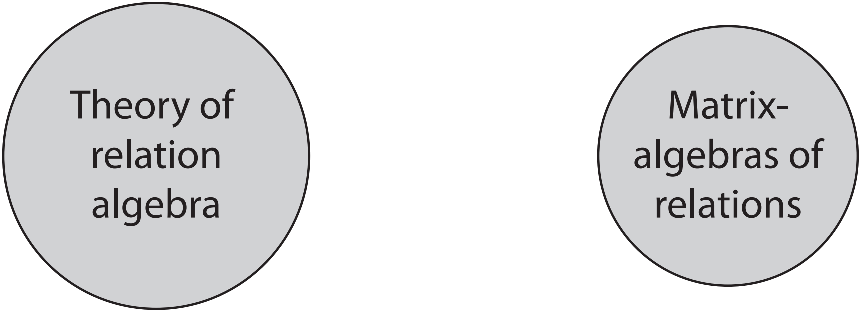
The system TITUREL runs under one of the following acronym interpretations

- This is the ultimate relation system
- Towards improved techniques using relations
- Teaching informaticians to use relations
- Try it, to use relations
- Toolkit intended to use relations
- Testing innovative tools using relations
- Think innovative - try using relations

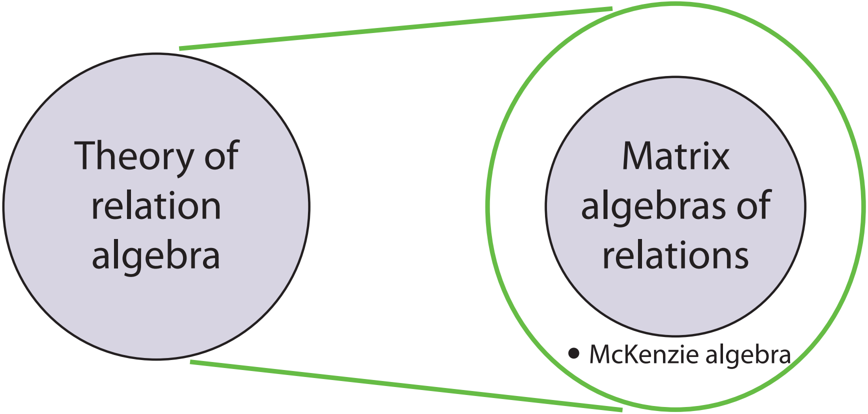
TITUREL ontvangt de Heilige Graal en de Heilige Speer uit handen van een Engelenschaar die neder daalt uit de hemel. Hij bouwt een Tempel voor deze heilige relikwien, de Graalburcht Montsalvat. Ridders die tot de Graal worden geroepen vormen de ridderschap van de Heilige Graal, hun Koning is Titurel. Op hoge leeftijd draagt hij zijn ambt over op zijn zoon Amfortas.

Model questions

Model problem



Model problem



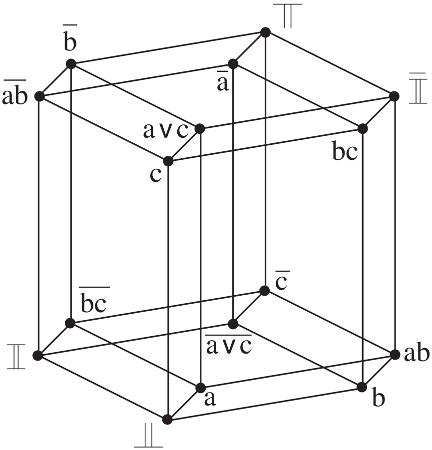
Predicate logic vs. relational logic

RRA (representable relation algebras, i.e. the Boolean matrix algebras) are not finitely axiomatizable. (Don Monk)

RA can express any (and up to logical equivalence, exactly the) first-order logic formulas containing no more than three variables.

RRA is axiomatizable by a universal Horn theory.

Model problem



$$\mathbb{I}^{\mathsf{T}} = \mathbb{I}$$

$$a^{\mathsf{T}} = c \qquad b^2 = \overline{b} = \mathbb{I} \cup a \cup c$$

$$b^{\mathsf{T}} = b \qquad a;c = c;a = \mathbb{T}$$

$$c^{\mathsf{T}} = a \qquad a;b = b;a = a \cup b$$

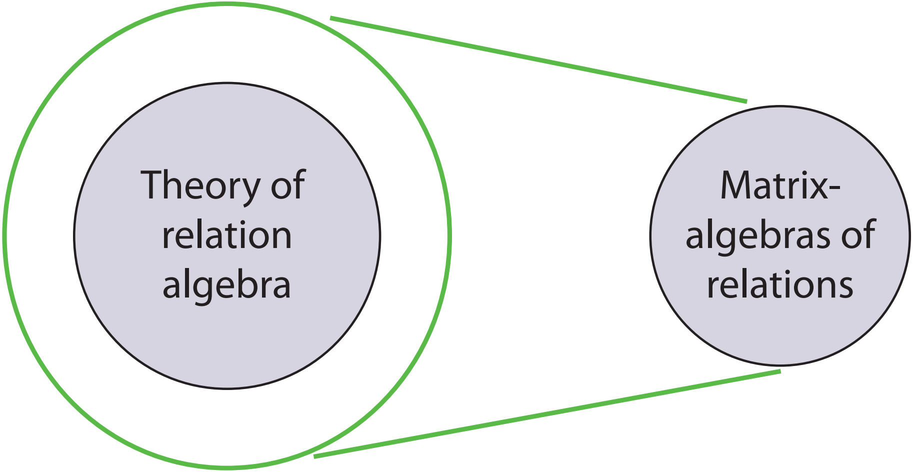
$$a^2 = a \qquad c;b = b;c = c \cup b$$

$$c^2 = c$$

Ralph McKenzie's homogeneous non-representable RA

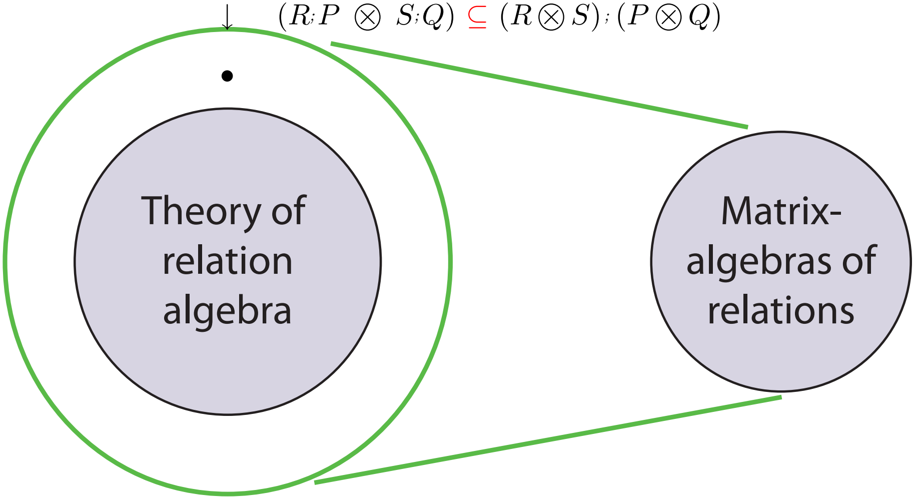
The element a cannot be conceived as a Boolean matrix.

Model problem



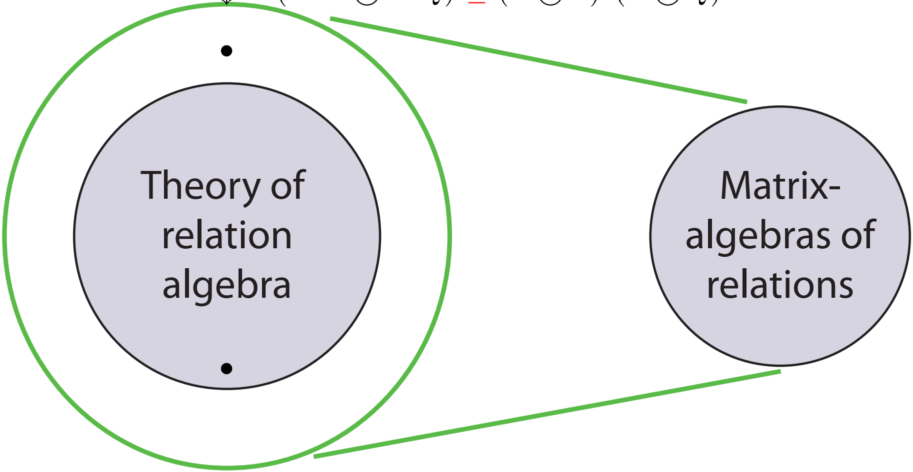
Model problem

$$(R;P \otimes S;Q) \subseteq (R \otimes S);(P \otimes Q)$$



Model problem

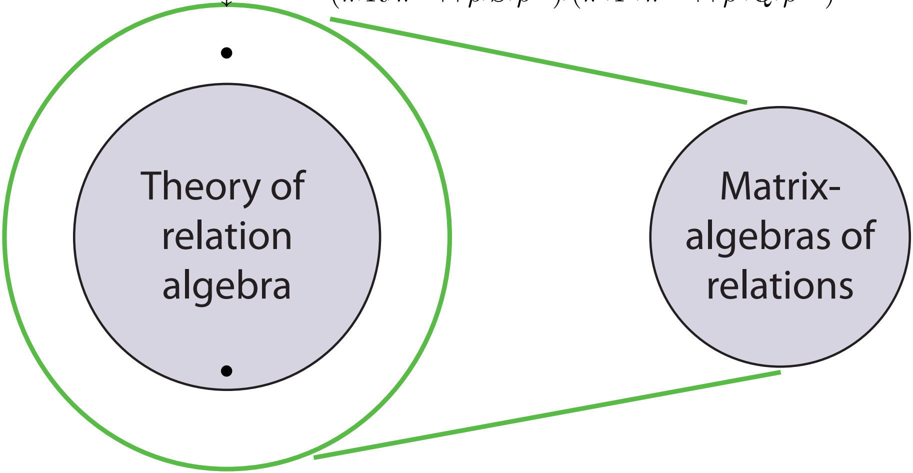
$$\downarrow \quad (R;P \otimes S;Q) \subseteq (R \otimes S);(P \otimes Q)$$



$$\uparrow \quad (R;P \otimes S;Q) \supseteq (R \otimes S);(P \otimes Q)$$

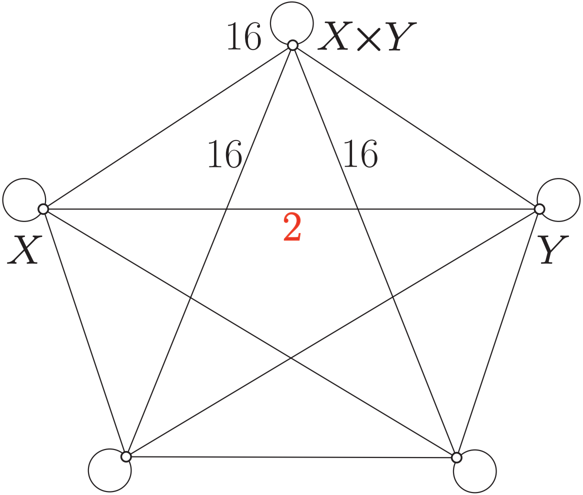
Model problem

$$\downarrow \quad \pi;R;P;\pi''^{\mathsf{T}} \cap \rho;S;Q;\rho''^{\mathsf{T}} \subseteq (\pi;R;\pi'^{\mathsf{T}} \cap \rho;S;\rho'^{\mathsf{T}});(\pi';P;\pi''^{\mathsf{T}} \cap \rho';Q;\rho''^{\mathsf{T}})$$



$$\uparrow \quad \pi;R;P;\pi''^{\mathsf{T}} \cap \rho;S;Q;\rho''^{\mathsf{T}} \supseteq (\pi;R;\pi'^{\mathsf{T}} \cap \rho;S;\rho'^{\mathsf{T}});(\pi';P;\pi''^{\mathsf{T}} \cap \rho';Q;\rho''^{\mathsf{T}})$$

Model problem



4 morphisms in any other case

Model problem

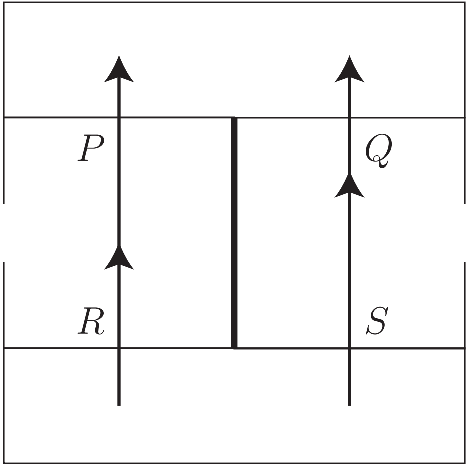
It is, however, possible to prove that

$$(Q \otimes \mathbb{I}_X) ; (\mathbb{I}_B \otimes R) \;=\; (Q \otimes R) \;=\; (\mathbb{I}_A \otimes R) ; (Q \otimes \mathbb{I}_Y)$$

This does express correctly that Q and R may *with one execution thread* be executed *in either order*; i.e., with meandering "coroutines".

But *no two execution threads* are provided to execute in parallel.

Model problem



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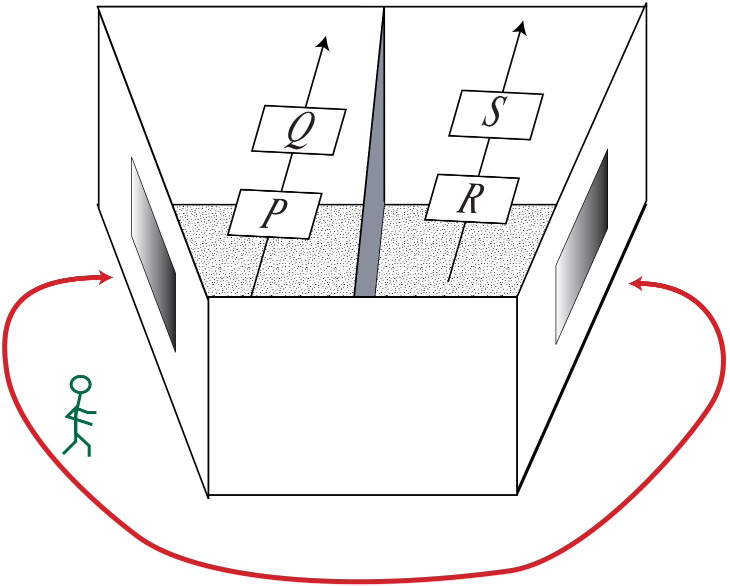
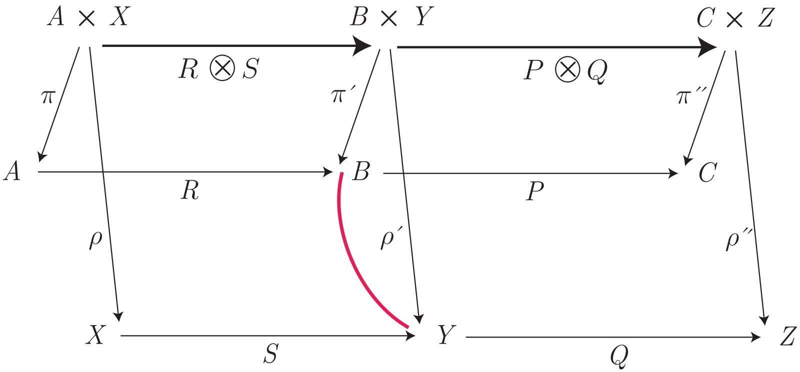


Model problem



History of relations

History of relations

Relations were being developed at a time when

- formal semantics was not yet known
  - language and interpretation
  - typing and unification
- the idea that several models of a theory may exist, was close to being completely unknown
  (non-Euclidian geometry: Bolyai, Lobatschevskij $\approx$ 1840)
- one was still bound to handle the following in the respective natural language, namely in English, German, Latin, Greek, Japanese, Russian, Arabic ...!

quantification	$\forall, \exists$
conversion	R^T
composition	$A ; B$

  but also „brother“, „father“, „uncle“

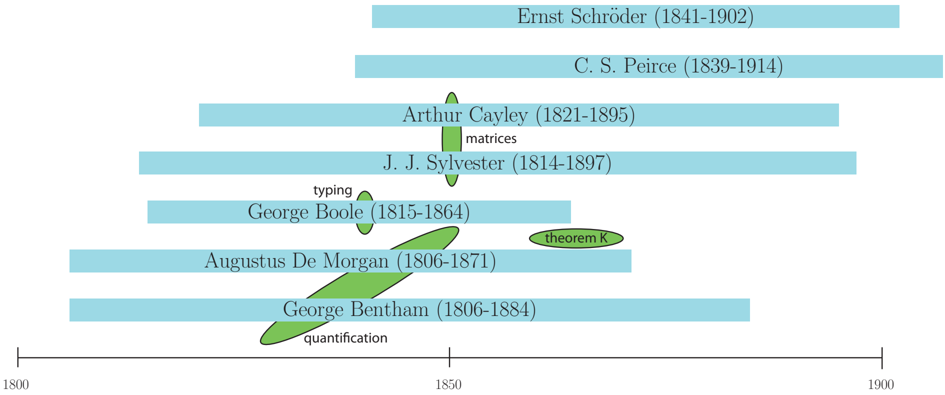
  and only gradually developed a more standardized language
- the concept of a matrix had not yet been coined
  (Cayley, Sylvester 1850's)

History of relations

George Boole's investigations on the laws of thought of 1854:

In every discourse, whether of the mind conversing with its own thoughts, or of the individual in his intercourse with others, there is an assumed or expressed limit within which the subjects of its operation are confined. The most unfettered discourse is that in which the words we use are understood in the widest possible application, and for them the limits of discourse are co-extensive with those of the universe itself. But more usually we confine ourselves to a less spacious field. . . . Furthermore, this universe of discourse is in the strictest sense the ultimate subject of the discourse. The office of any name or descriptive term employed under the limitations supposed is not to raise in the mind the conception of all the beings or objects to which that name or description is applicable, but only of those which exist within the supposed universe of discourse.

History of relations



Closure and contact

Closure and contact

Definition

Let some ordered set $(V, \leq)$ be given. A mapping $\rho : V \longrightarrow V$ is called a closure operation, if it is

i) expanding $x \leq \rho(x)$,

ii) isotonic $x \leq y \longrightarrow \rho(x) \leq \rho(y)$,

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$$\forall x, y : \; x \leq y \; \longrightarrow \; \rho(x) \leq \rho(y)$$

which makes 18 symbols in standard mathematics notation.
This will now shrink down to just 7.

Theorem

Assume an ordering $E : X \longrightarrow X$ *and a mapping* $\rho : X \longrightarrow X$*. Then* ρ *is a closure operator if and only if*

$$\rho \subseteq E \qquad E;\rho \subseteq \rho;E \qquad \rho;\rho \subseteq \rho$$

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We convince ourselves, that the intentions of the preceding definition are met when lifting in this way, starting from $\rho(\rho(x)) \leq \rho(x)$:

$$\forall x, y, z : \rho_{xy} \wedge \rho_{yz} \rightarrow [\exists w : \rho_{xw} \wedge E_{zw}]$$
$$\iff \quad \forall x, y, z : \rho_{xy} \wedge \rho_{yz} \rightarrow (\rho;E^{\mathsf{T}})_{xz}$$
$$\iff \quad \neg\big(\exists x, z : (\exists y : \rho_{xy} \wedge \rho_{yz}) \wedge \overline{[\rho;E^{\mathsf{T}}]_{xz}}\big)$$
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$$\begin{aligned}
&\forall x, y, z : \rho_{xy} \ \wedge \rho_{yz} \rightarrow [\exists w : \rho_{xw} \ \wedge E_{zw}] \\
\iff \quad & \forall x, y, z : \rho_{xy} \ \wedge \rho_{yz} \rightarrow (\rho;E^{\mathsf{T}})_{xz} \\
\iff \quad & \neg\big(\exists x, z : (\exists y : \rho_{xy} \ \wedge \rho_{yz}) \ \wedge \overline{[\rho;E^{\mathsf{T}}]_{xz}}\big) \\
\iff \quad & \neg\big(\exists x, z : (\rho;\rho)_{xz} \ \wedge \overline{[\rho;E^{\mathsf{T}}]_{xz}}\big) \\
\iff \quad & \forall x, z : (\rho;\rho)_{xz} \rightarrow [\rho;E^{\mathsf{T}}]_{xz} \\
\iff \quad & \rho;\rho \subseteq \rho;E^{\mathsf{T}}
\end{aligned}$$

Together with the others, we get $\quad \iff \quad \rho;\rho \subseteq \rho$

Closure and contact

Definition

We consider a set related to its powerset, with a membership relation $\varepsilon : X \longrightarrow \mathcal{P}(X)$ and a powerset ordering $\Omega : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$. A relation $C : X \longrightarrow \mathcal{P}(X)$ is called an Aumann contact relation, provided

i) it contains the membership relation, i.e., $\varepsilon \subseteq C$,

ii) an element x in contact with a set Y *all of whose elements* are in contact with a set Z, will be in contact with Z, the so-called *infectivity* of contact, i.e., $C;\overline{\varepsilon^{\mathsf{T}};\overline{C}} \subseteq C$, or equivalently, $C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C}$.

One will easily show that C forms an upper cone, i.e., $C;\Omega \subseteq C$:
$C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{\varepsilon} = \overline{\Omega}$

Closure and contact

Theorem

Given a closure operator $\rho : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ on some powerset defined via a membership relation $\varepsilon : X \longrightarrow \mathcal{P}(X)$, the construct $C := \varepsilon;\rho^\mathsf{T}$ turns out to be an Aumann contact relation.

Beweis.

i) $\varepsilon \subseteq \varepsilon;\rho^\mathsf{T} \qquad \iff \qquad \varepsilon;\rho \subseteq \varepsilon \Longleftarrow \qquad \varepsilon;\Omega \subseteq \varepsilon.$

ii) $C;\overline{\varepsilon^\mathsf{T};\overline{C}} = \varepsilon;\rho^\mathsf{T};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon;\rho^\mathsf{T}}} = \varepsilon;\rho^\mathsf{T};\overline{\varepsilon^\mathsf{T};\overline{\varepsilon}};\rho^\mathsf{T}$ since ρ is a mapping

$= \varepsilon;\rho^\mathsf{T};\Omega;\rho^\mathsf{T}$

$\subseteq \varepsilon;\Omega;\rho^\mathsf{T};\rho^\mathsf{T}$ with the second closure property

$\subseteq \varepsilon;\Omega;\rho^\mathsf{T}$ with the third closure property

$= \varepsilon;\rho^\mathsf{T} = C$ since $\varepsilon;\Omega = \varepsilon$

□

Closure and contact

Theorem

Given any Aumann contact relation $C : X \longrightarrow \mathcal{P}(X)$, forming the construct $\rho := \mathtt{syq}(C, \varepsilon)$ results in a closure operator.

Proof: i) $\rho = \mathtt{syq}(C, \varepsilon) \subseteq \overline{C^\mathsf{T};\overline{\varepsilon}} \subseteq \overline{\varepsilon^\mathsf{T};\overline{\varepsilon}} = \Omega$

ii) We recall $\varepsilon;\mathtt{syq}(\varepsilon, Y) = Y$ and $\overline{\varepsilon};\mathtt{syq}(\varepsilon, Y) = \overline{Y}$ for

$$\rho;\overline{\Omega};\rho^\mathsf{T} = \mathtt{syq}(C, \varepsilon);\varepsilon^\mathsf{T};\overline{\varepsilon};\mathtt{syq}(\varepsilon, C) = C^\mathsf{T};\overline{C} \subseteq \varepsilon^\mathsf{T};\overline{\varepsilon} = \overline{\Omega}.$$

Since ρ is a mapping, we may proceed with

$$\overline{\rho;\Omega;\rho^\mathsf{T}} \subseteq \overline{\Omega} \qquad \Omega \subseteq \rho;\Omega;\rho^\mathsf{T} \qquad \Omega;\rho \subseteq \rho;\Omega$$

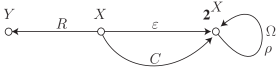
iii) We prove $\rho;\rho \subseteq \rho$, i.e., $\mathtt{syq}(C, \varepsilon);\mathtt{syq}(C, \varepsilon) \subseteq \mathtt{syq}(C, \varepsilon)$ or

$$(\overline{C}^\mathsf{T};\varepsilon \cup C^\mathsf{T};\overline{\varepsilon});\mathtt{syq}(\varepsilon, C) \subseteq \overline{C}^\mathsf{T};\varepsilon \cup C^\mathsf{T};\overline{\varepsilon}$$

Now, the two terms on the left are treated separately.

Example

Let an arbitrary relation $R : X \longrightarrow Y$ be given.



Then $C := \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}}$ is always an Aumann contact relation. To show this, we have to prove

$\varepsilon \subseteq \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}} = C$, which is trivial using Schröder equivalences.

$$C^{\mathsf{T}};\overline{C} \subseteq \varepsilon^{\mathsf{T}};\overline{C} \iff \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}}^{\mathsf{T}};\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon} \subseteq \varepsilon^{\mathsf{T}};\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}$$

$$\Longleftarrow \quad \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}}^{\mathsf{T}};\overline{R} \subseteq \varepsilon^{\mathsf{T}};\overline{R}$$

$$\iff \quad \overline{\varepsilon^{\mathsf{T}};\overline{R}};\overline{R}^{\mathsf{T}} \subseteq (\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon})^{\mathsf{T}}$$

The construct $C := \overline{\overline{R};\overline{\overline{R}^{\mathsf{T}};\varepsilon}}$ may be read as follows: It declares those combinations $x \in X$ and $S \subseteq X$ to be in contact C, for which every relationship $(x, y) \notin R$ implies that there exists also an $x' \in S$ in relation $(x', y) \notin R$.

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