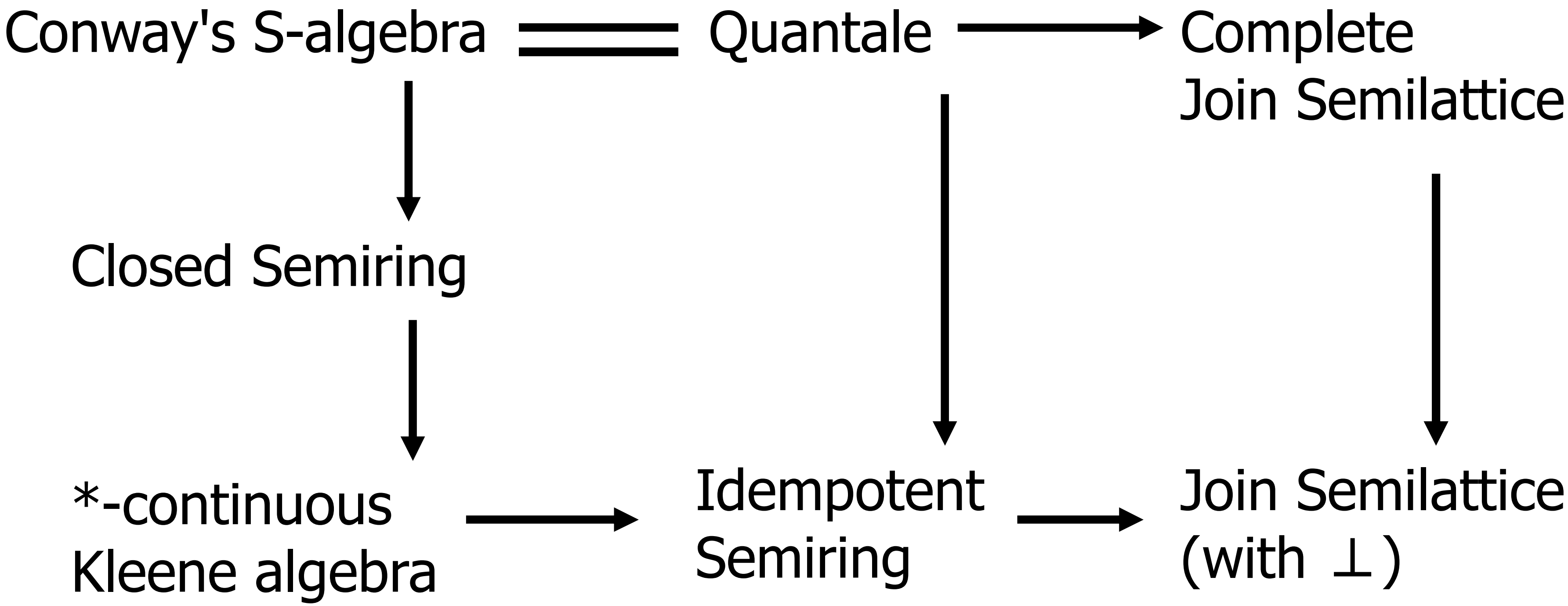
A Sufficient Condition for Liftable Adjunctions between Eilenberg-Moore Categories

Koki Nishizawa (Kanagawa Univ.)
with Hitoshi Furusawa（Kagoshima Univ.）

RAMiCS 2014/4/30

Background

- Some left adjoints to forgetful functors are given by construction of ideals.



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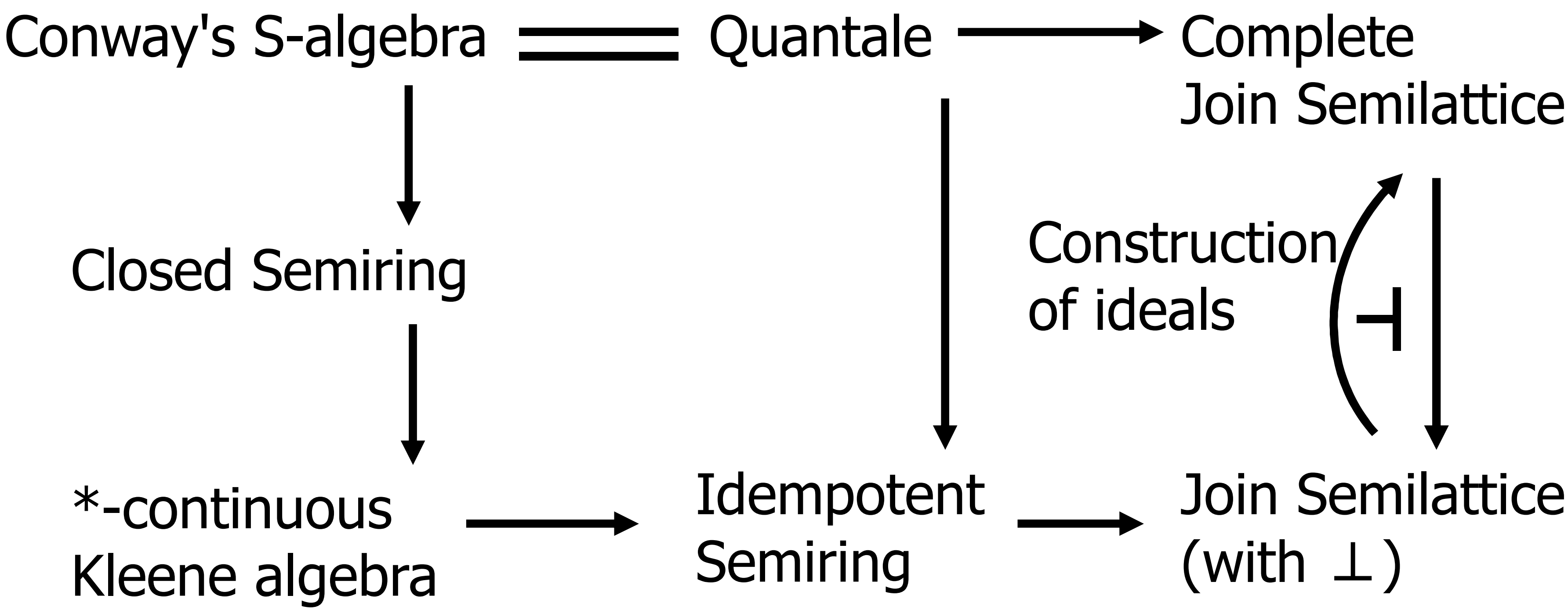

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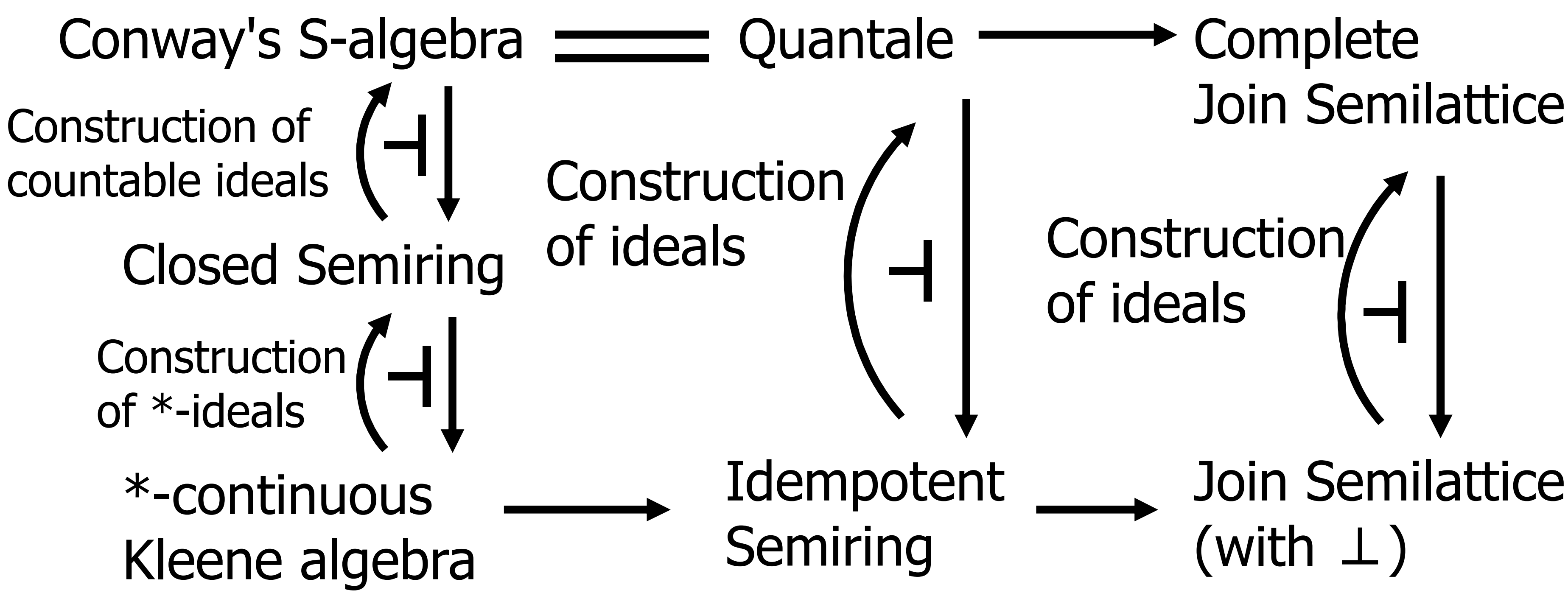
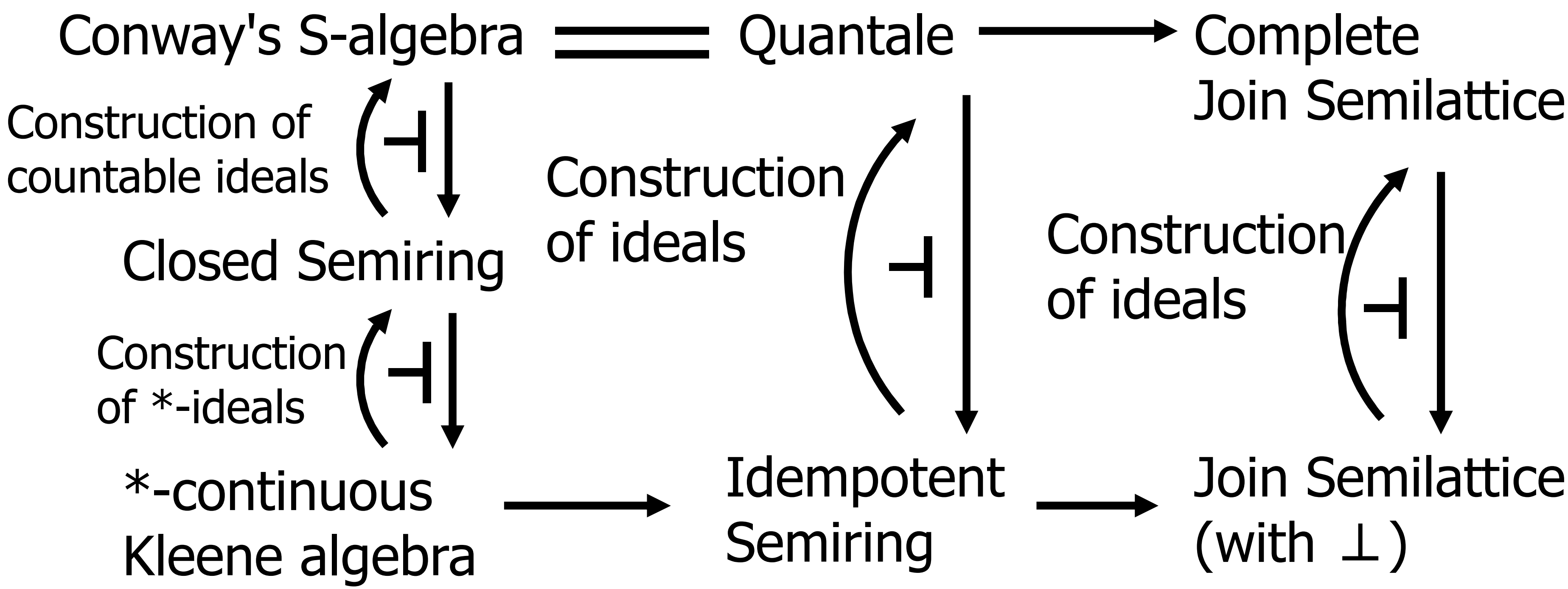


Background

- Some left adjoints to forgetful functors are given by construction of ideals. ➡ Why ? When ?



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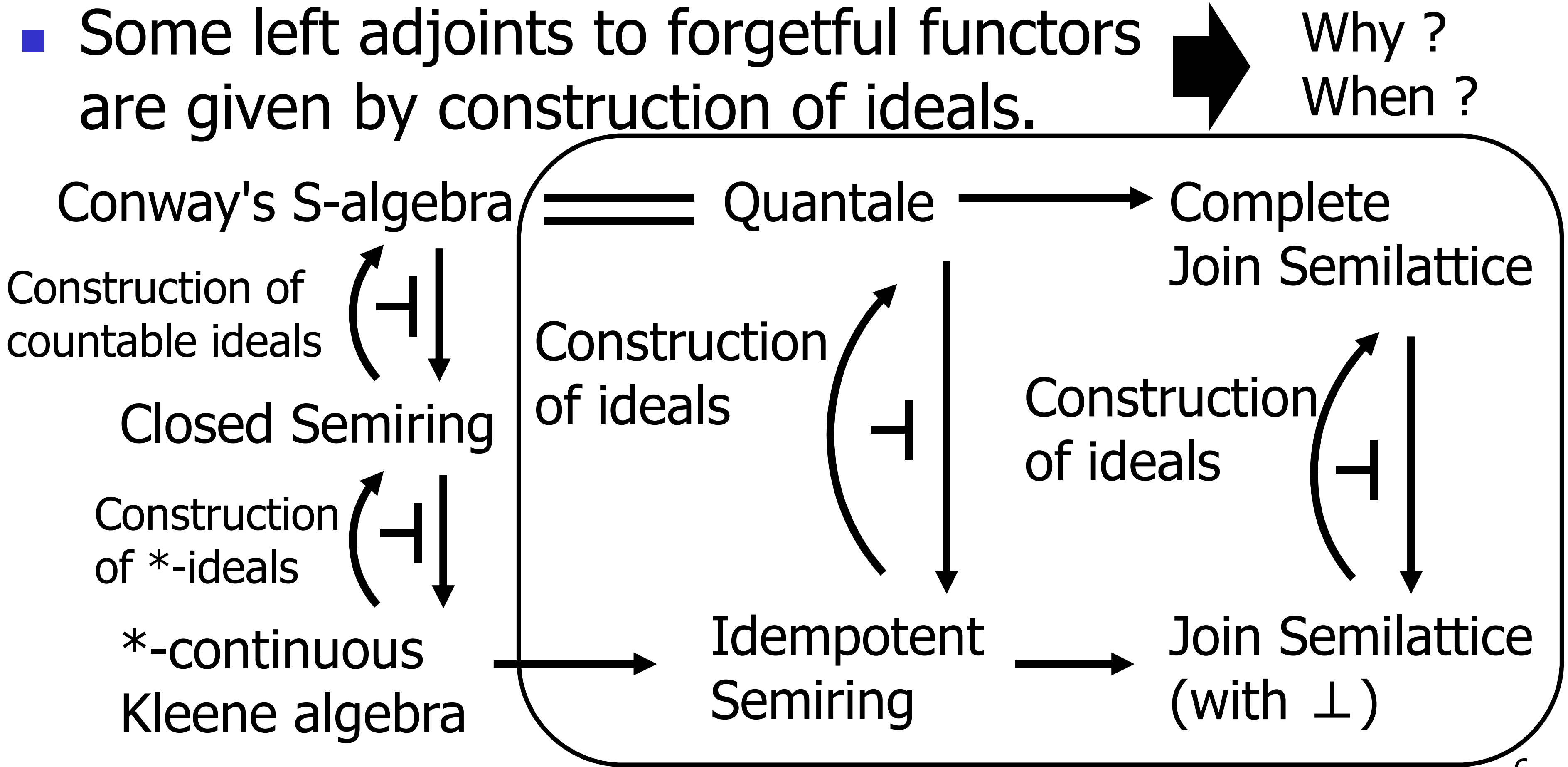
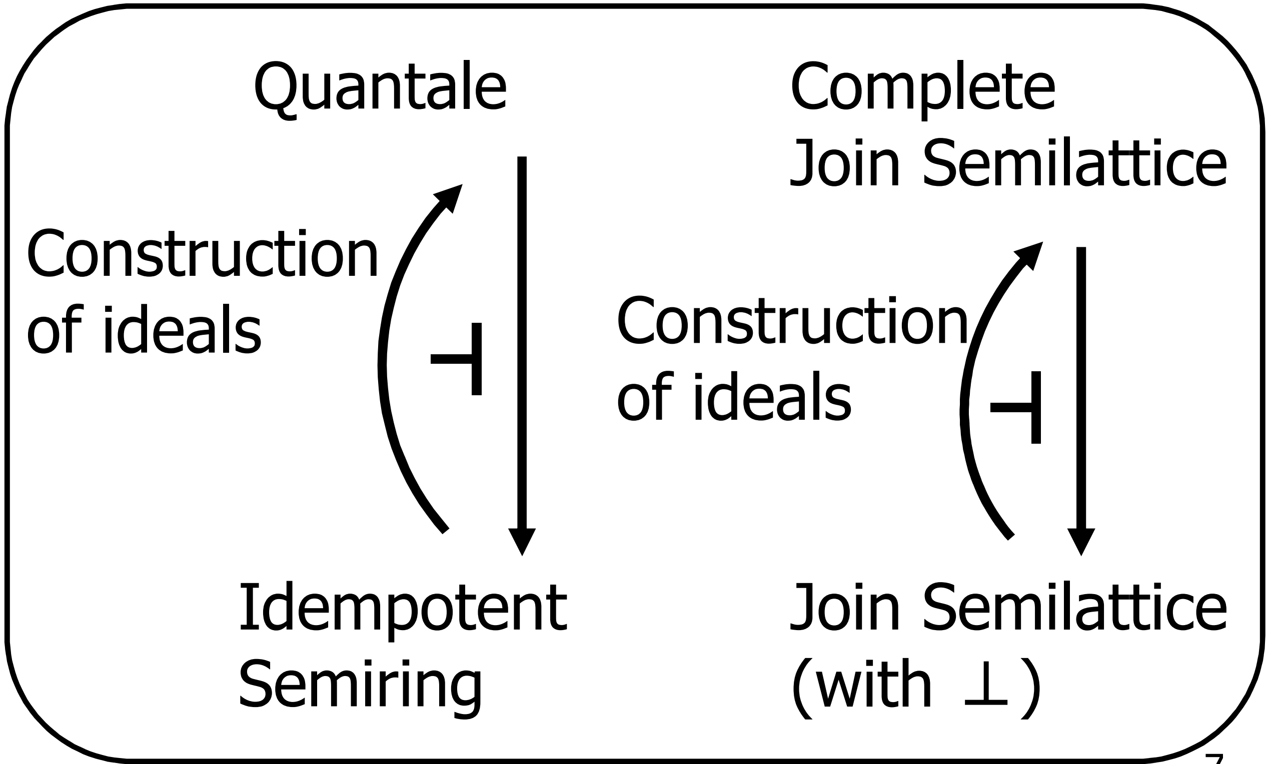
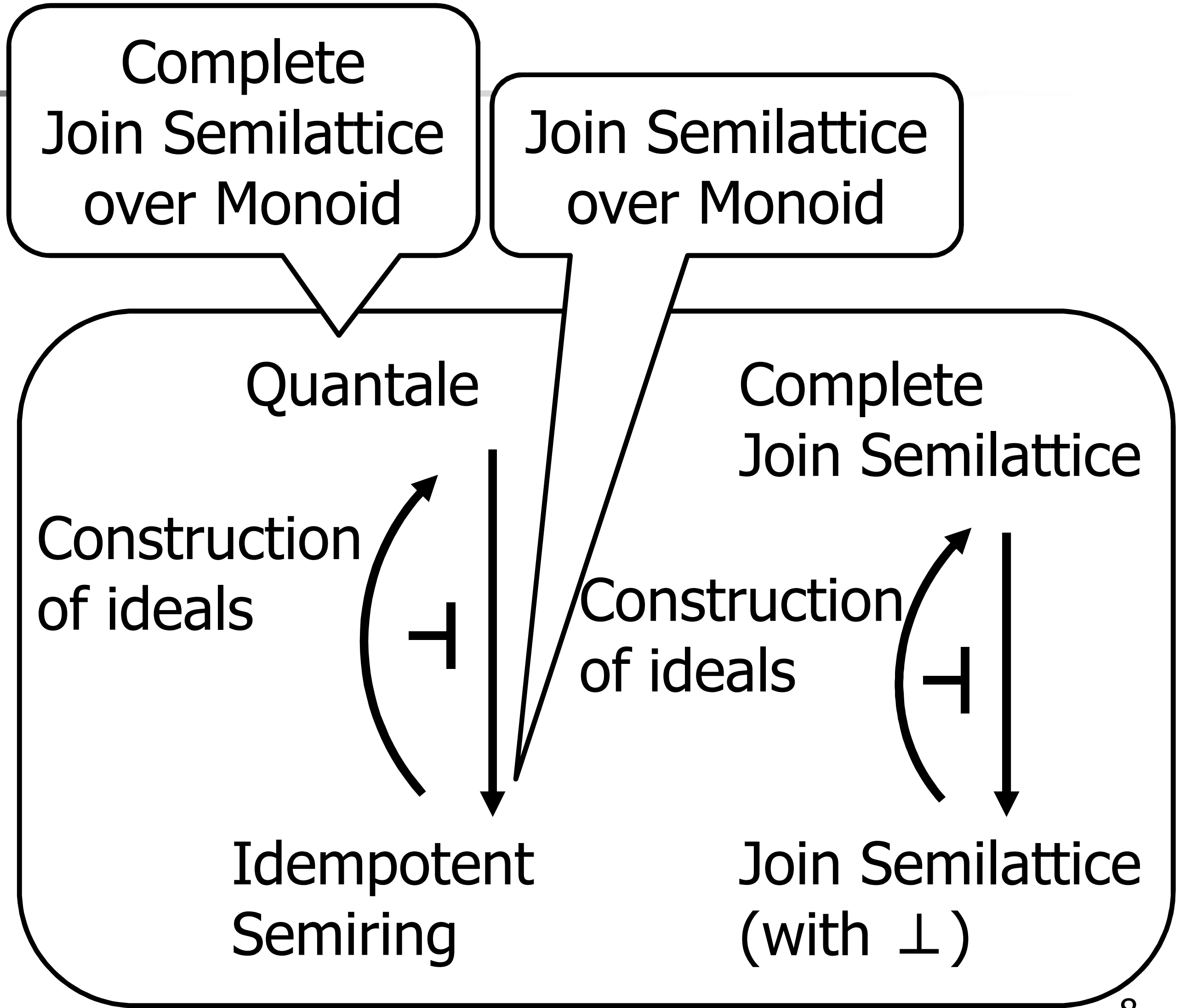


Result:


Quantale
Complete
Join Semilattice
Construction
of ideals
⊣
Construction
of ideals
⊣
Idempotent
Semiring
Join Semilattice
(with ⊥)

Result:


Complete
Join Semilattice
over Monoid
Join Semilattice
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Join Semilattice
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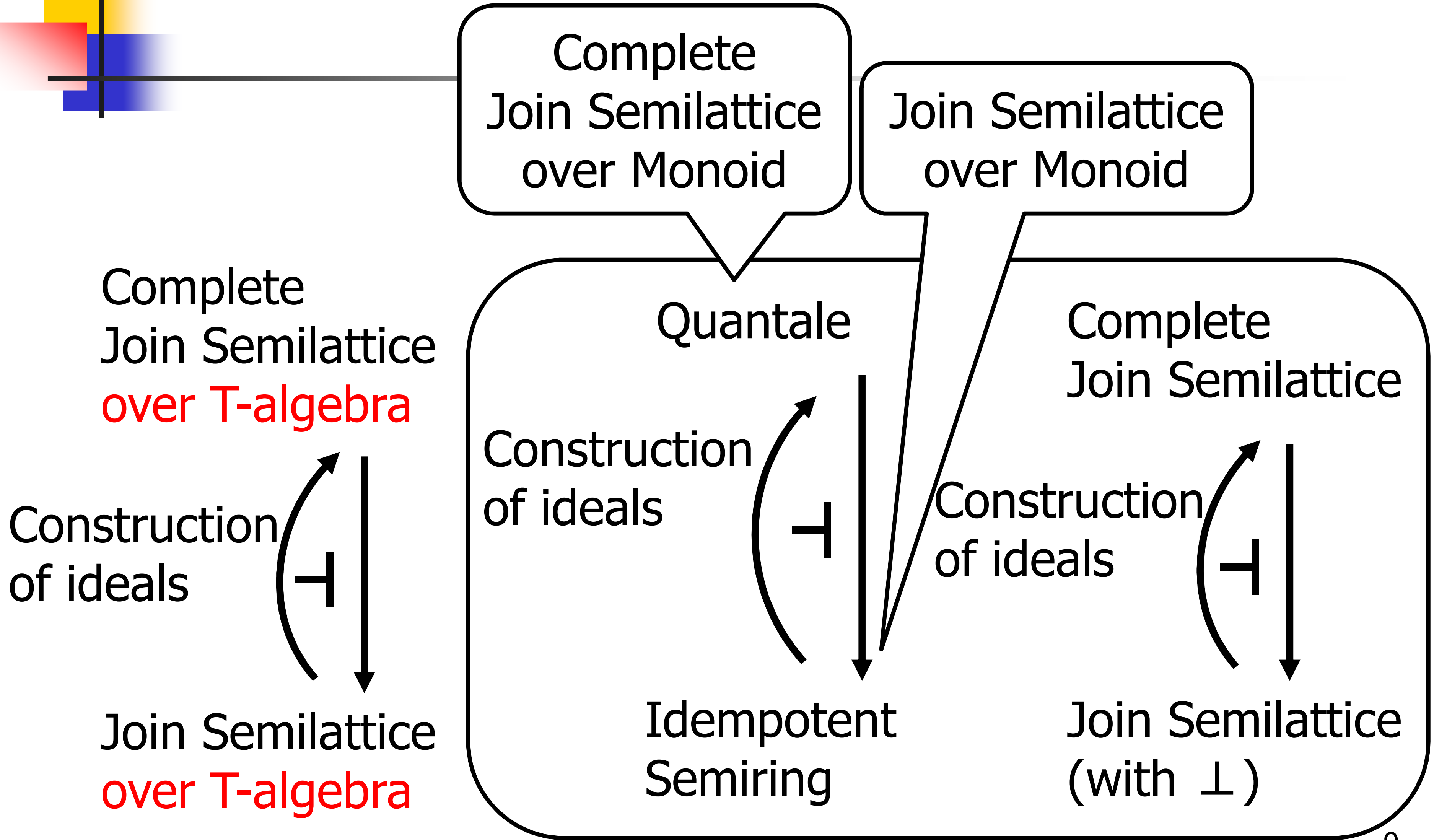
Result: from Monoid to T-algebra



Contents

1. Construction of left adjoint
   from join semilattice
   to complete join semilattice
2. Construction of left adjoint
   from join semilattice over T-algebra
   to complete join semilattice over T-algebra
3. Conclusion

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Ideal in Join Semilattice

Def. S is a join semilattice if

- S is a partially ordered set
- S has a join $\bigvee X$ for each finite subset X of S.

Def. S is a complete join semilattice if

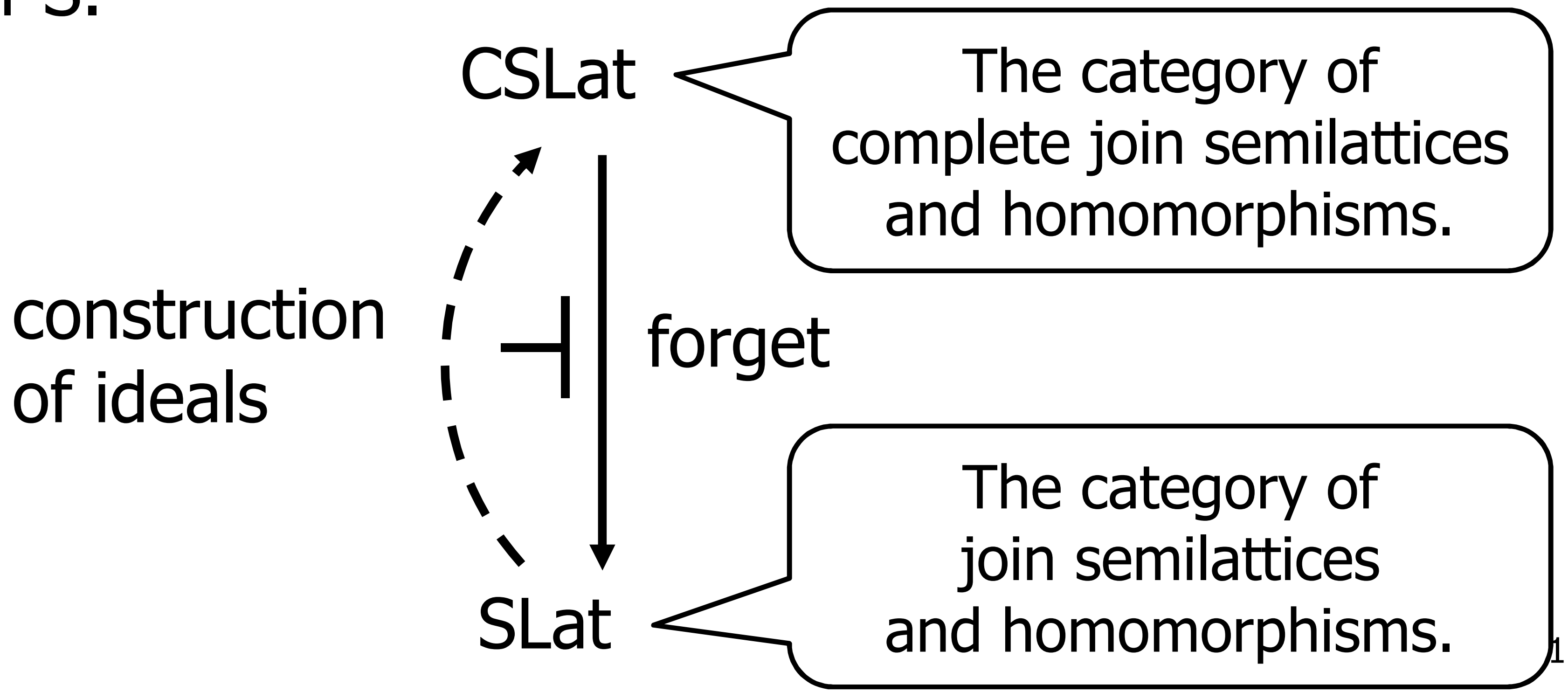
- S is a partially ordered set
- S has a join $\bigvee X$ for each subset X of S.

Def. A subset X of a join semilattice S is an ideal if

- X is closed under finite joins
- X is downward-closed

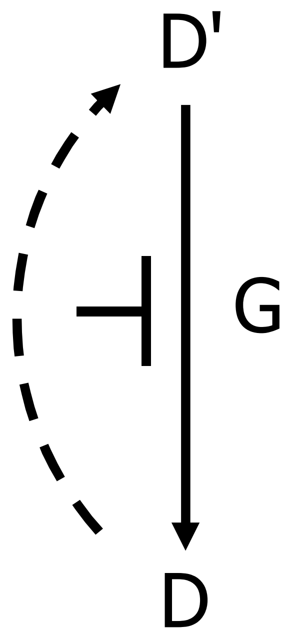
Goal

Prop. The forgetful functor from CSLat to SLat has a left adjoint, which sends S to the set of all ideals of S.



A sufficient condition of adjunction [Toposes, Triples and Theories]

- G:D'→D has a left adjoint if



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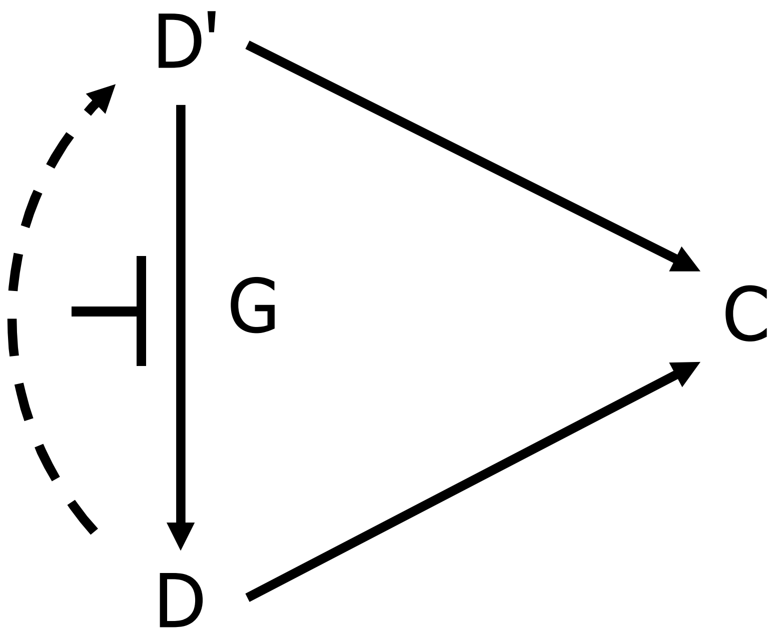

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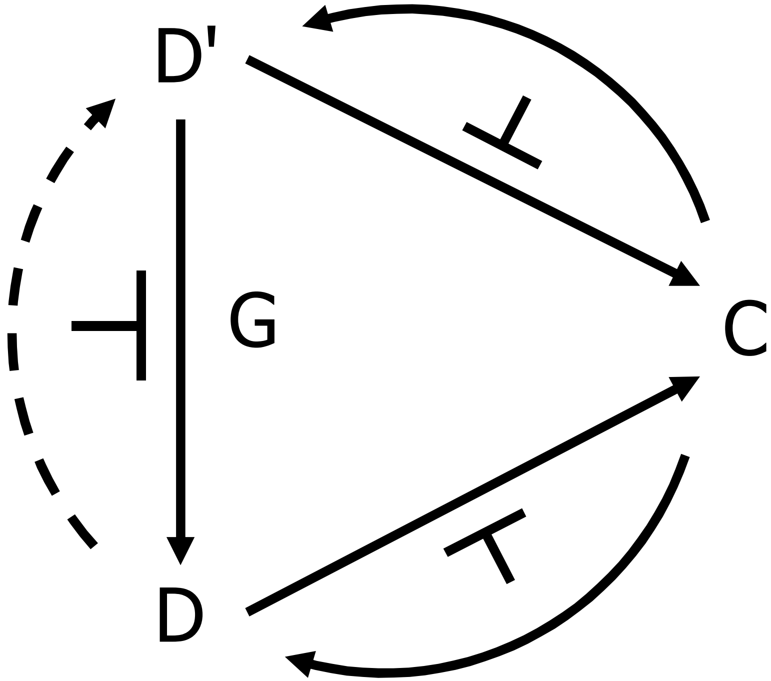

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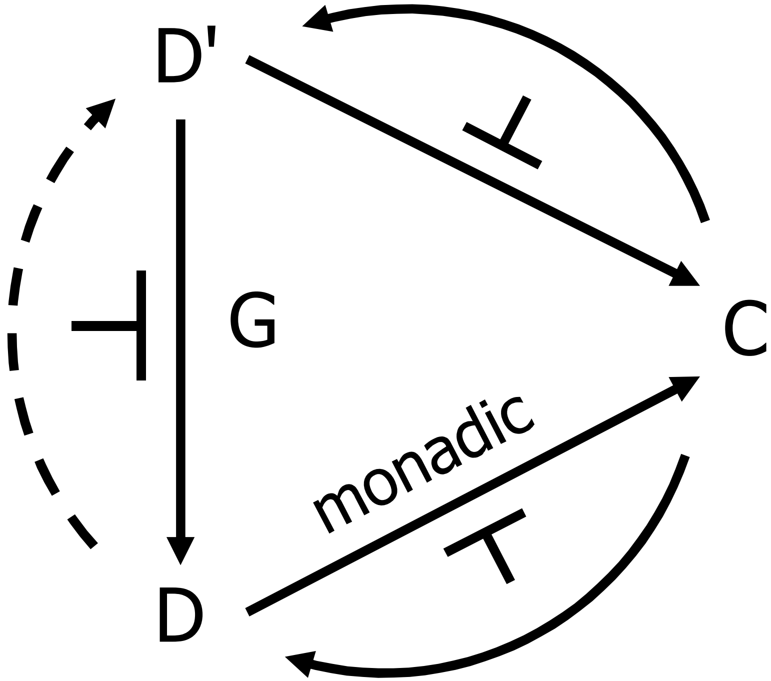

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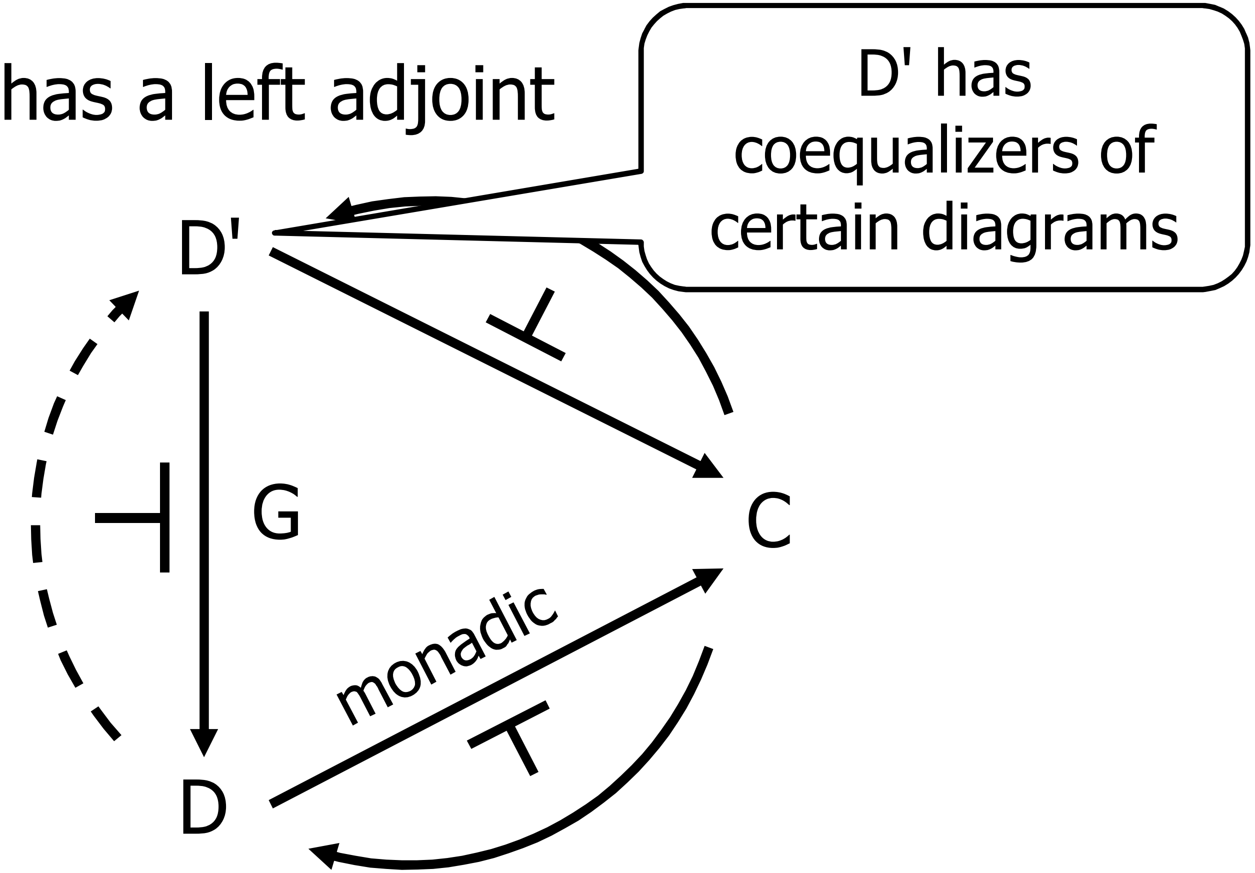
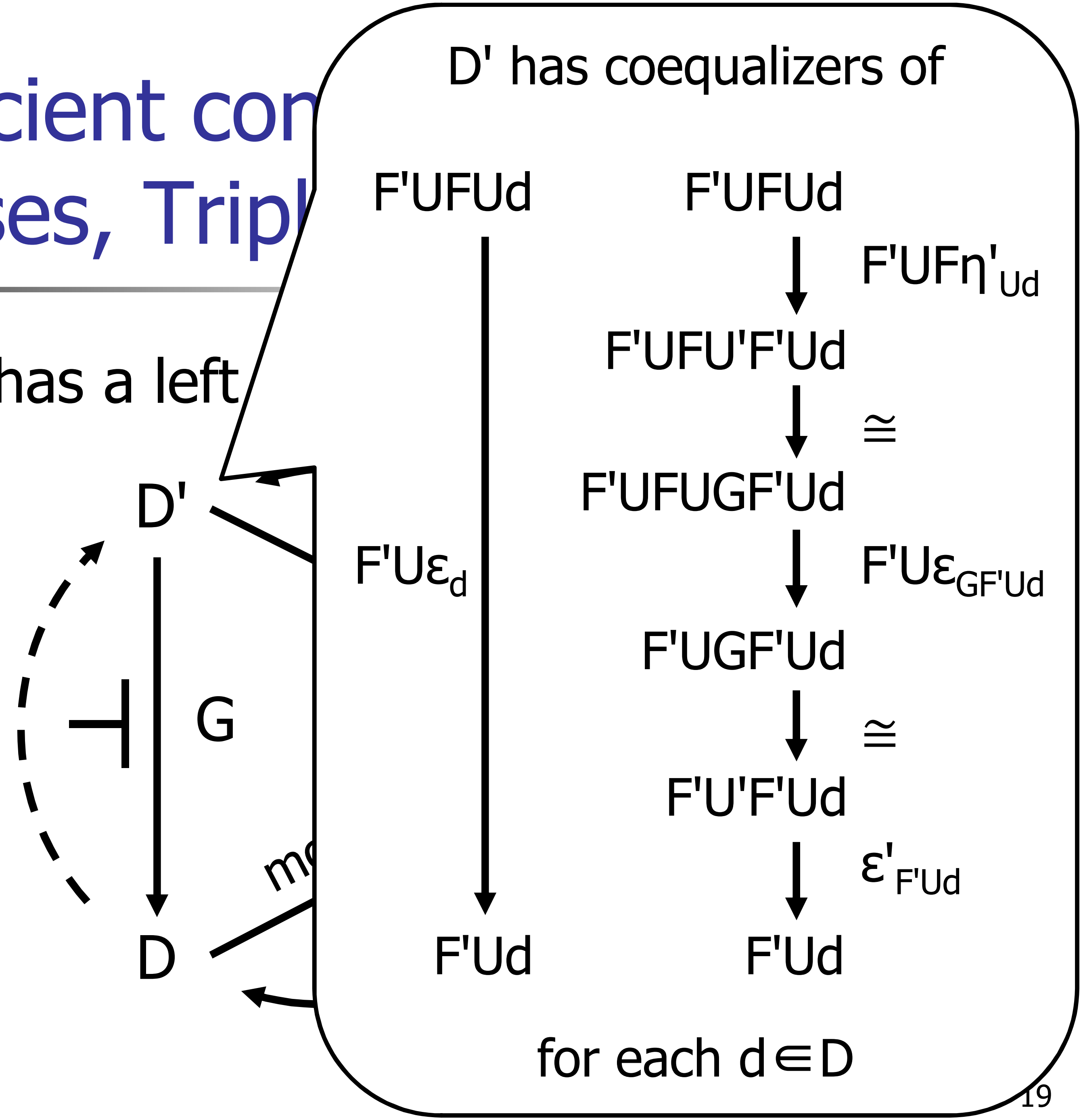


A sufficient con[…]
[Toposes, Tripl[…]

- G:D'→D has a left […]
  if



+ Beck's Theorem

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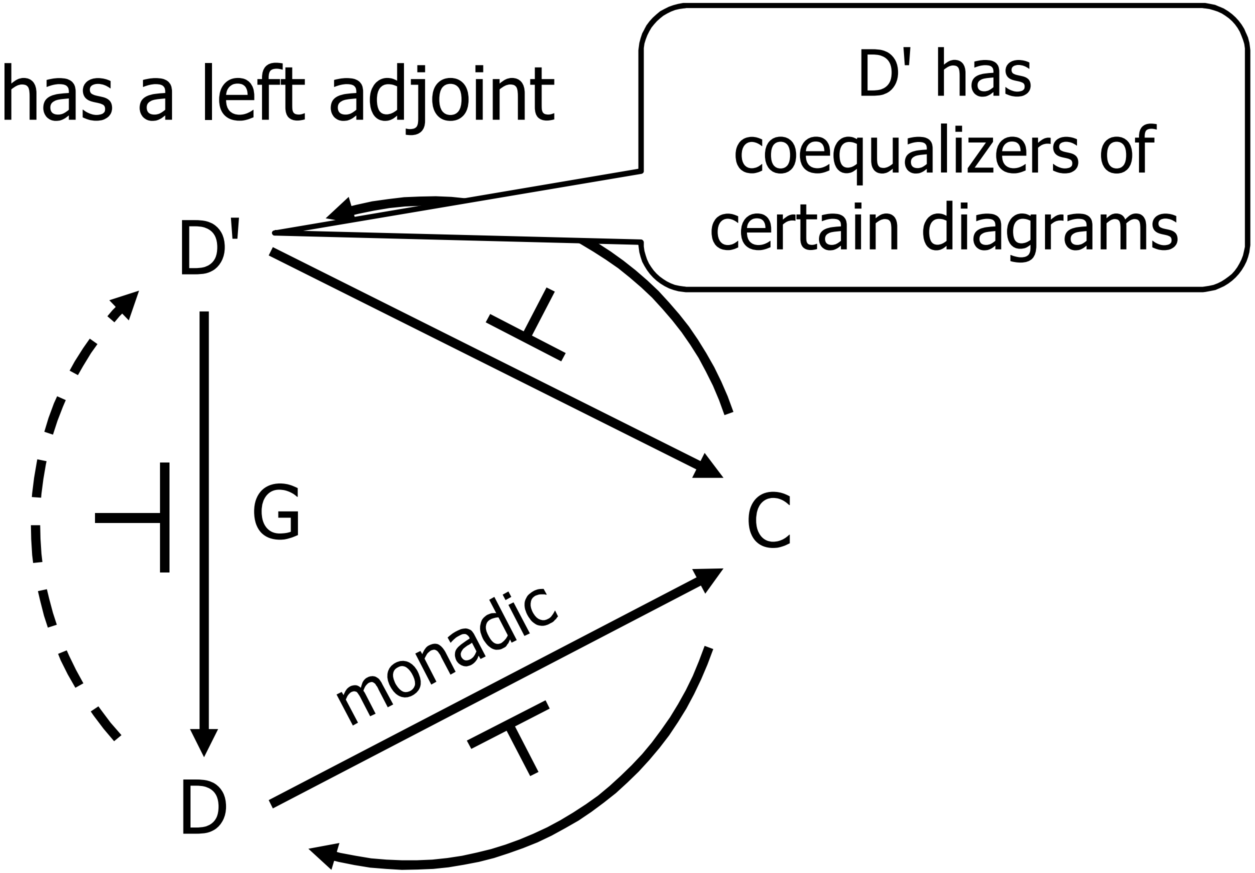

Beck's Theorem:
A monadic functor U creates a coequalizer of parallel arrows f,g if Uf and Ug has an absolute coequalizer.

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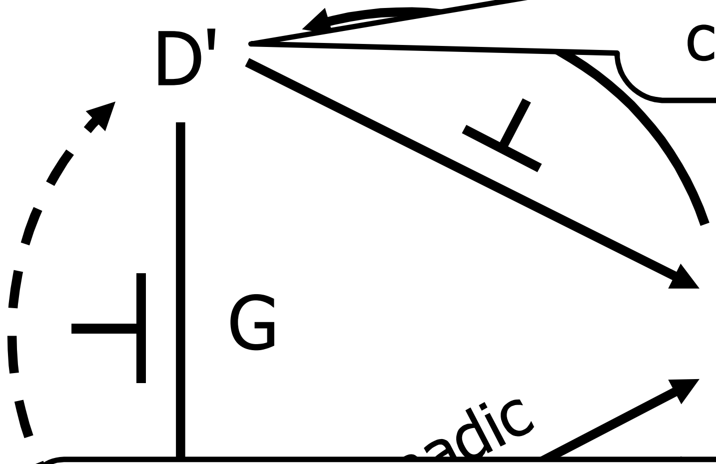

D' has coequalizers of certain diagrams

absolute coequalizer
= coequalizer preserved by all functors

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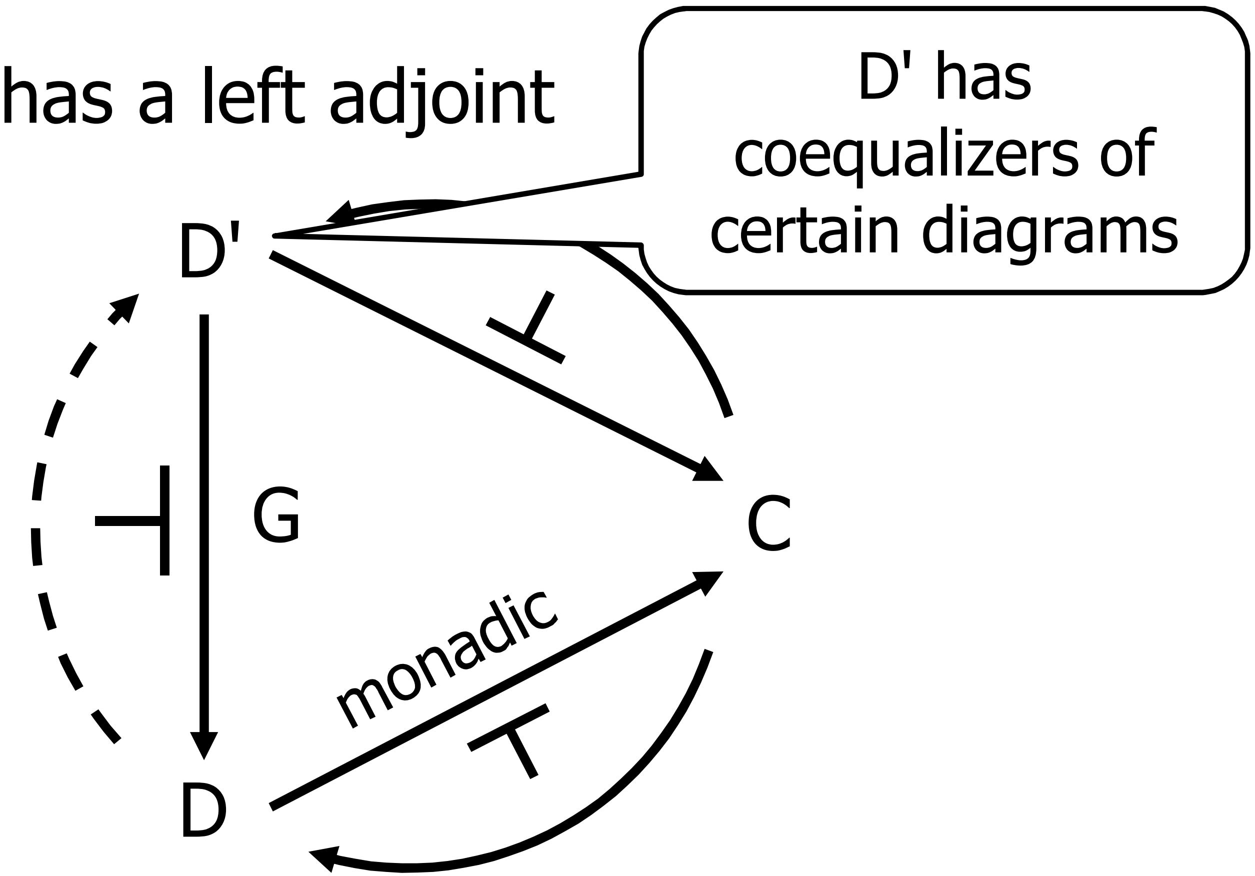

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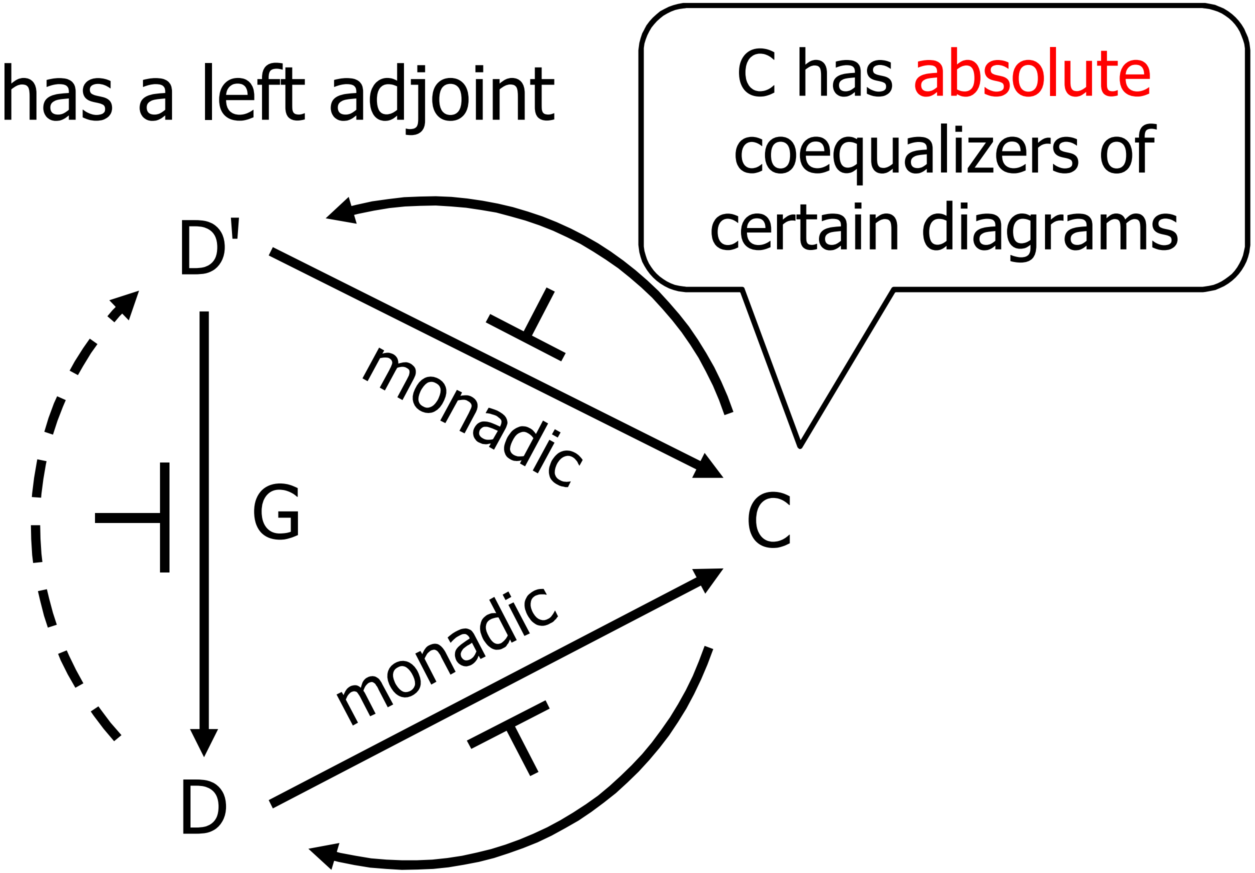

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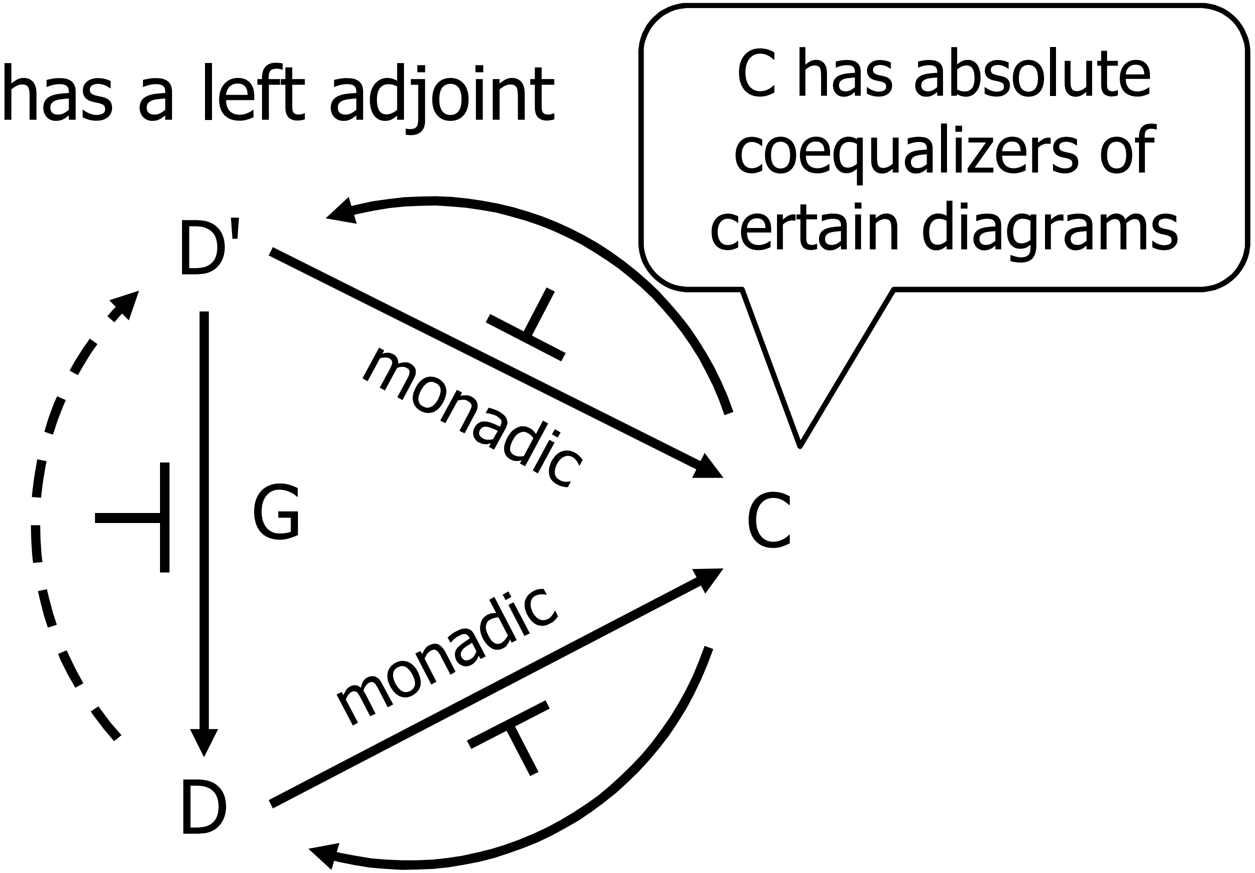

When G is the functor which pre-composes a monad map

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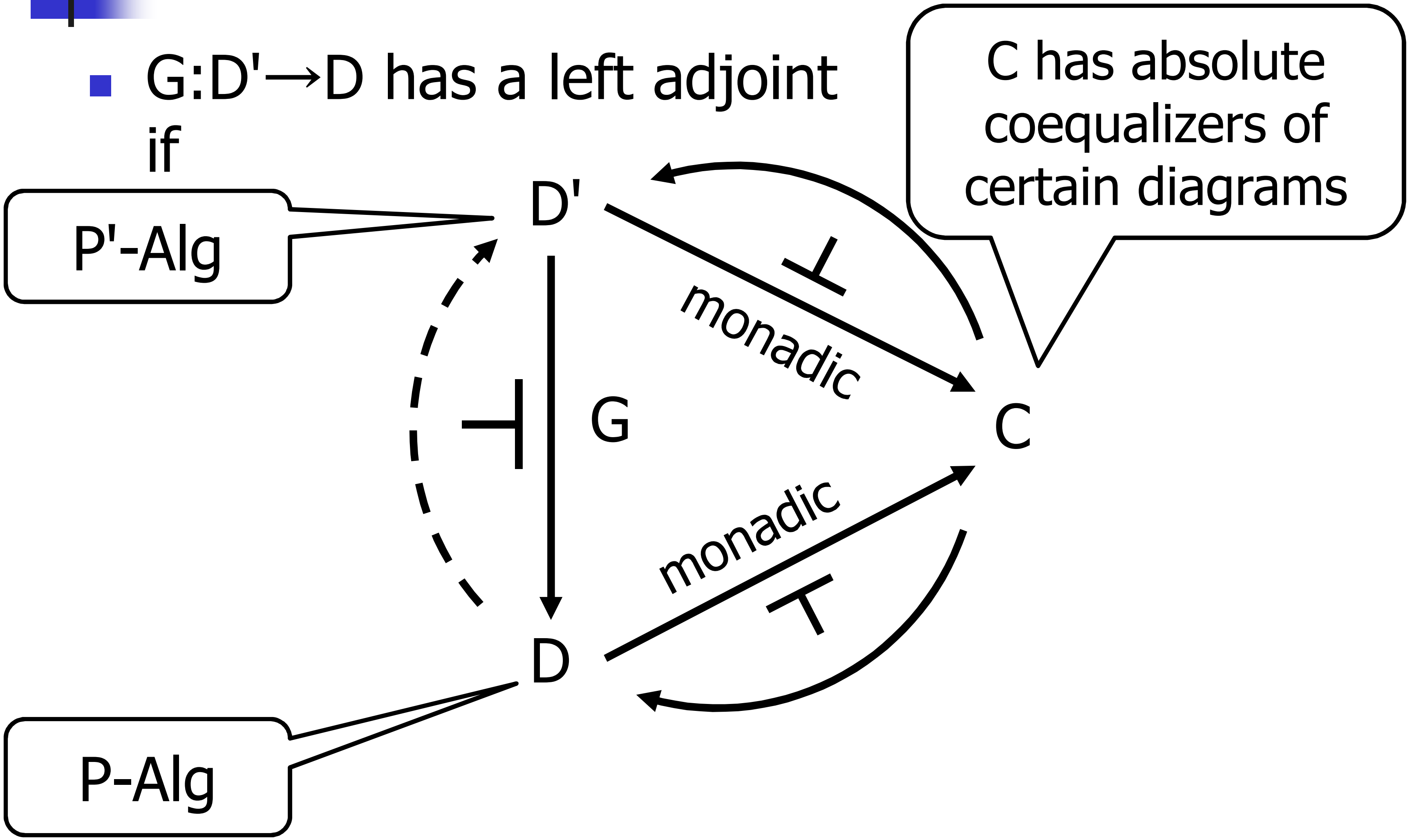

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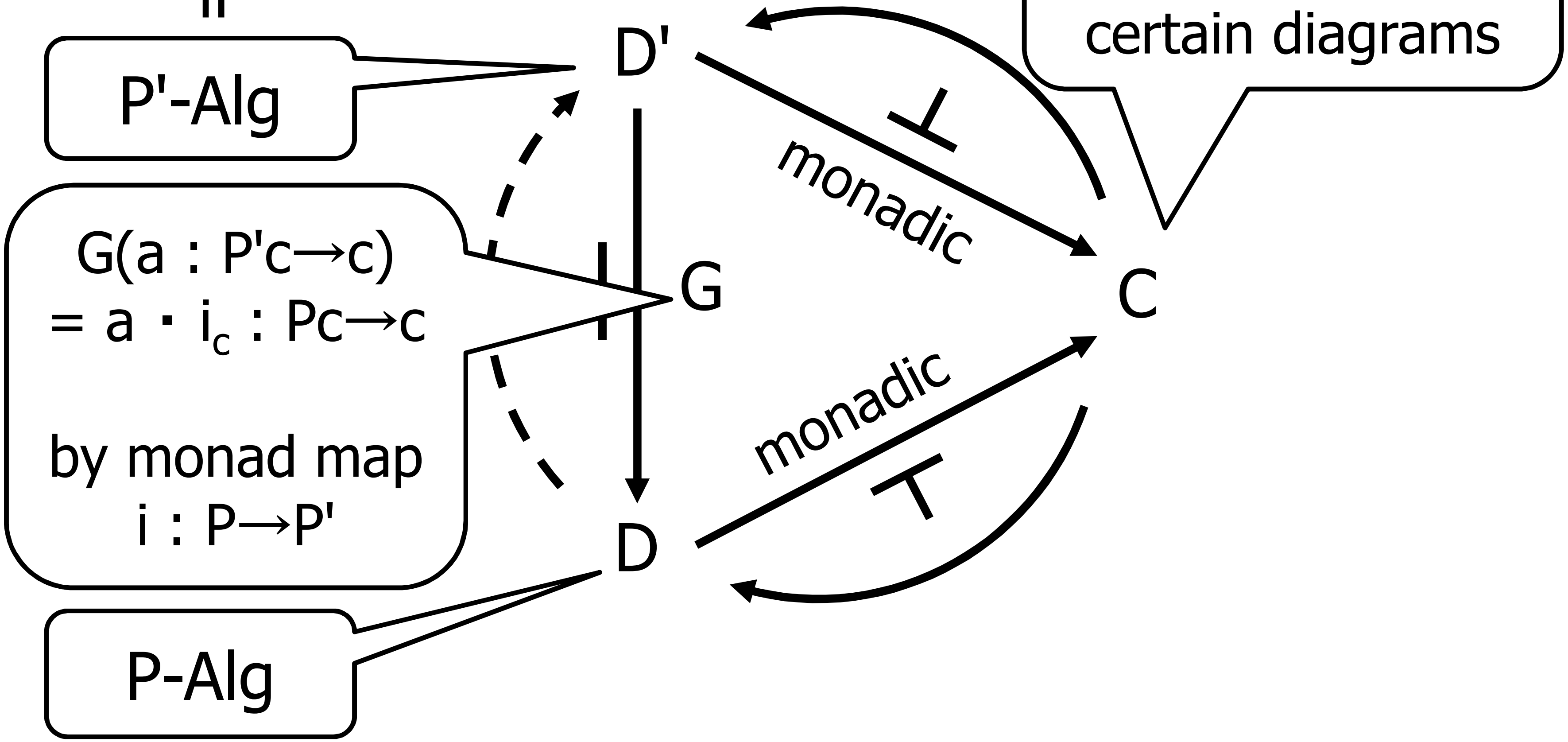
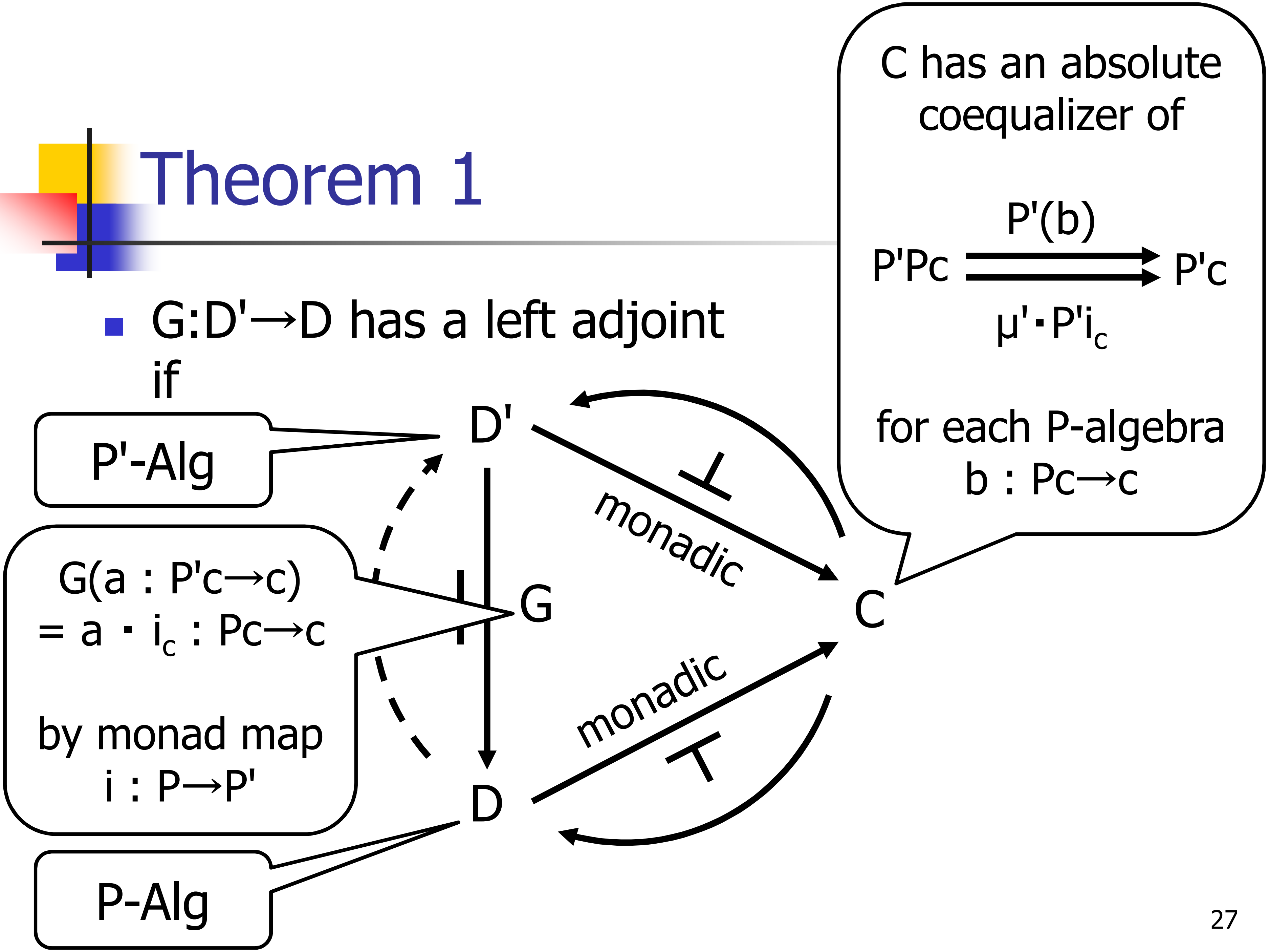


Theorem 1

- G:D'→D has a left adjoint if

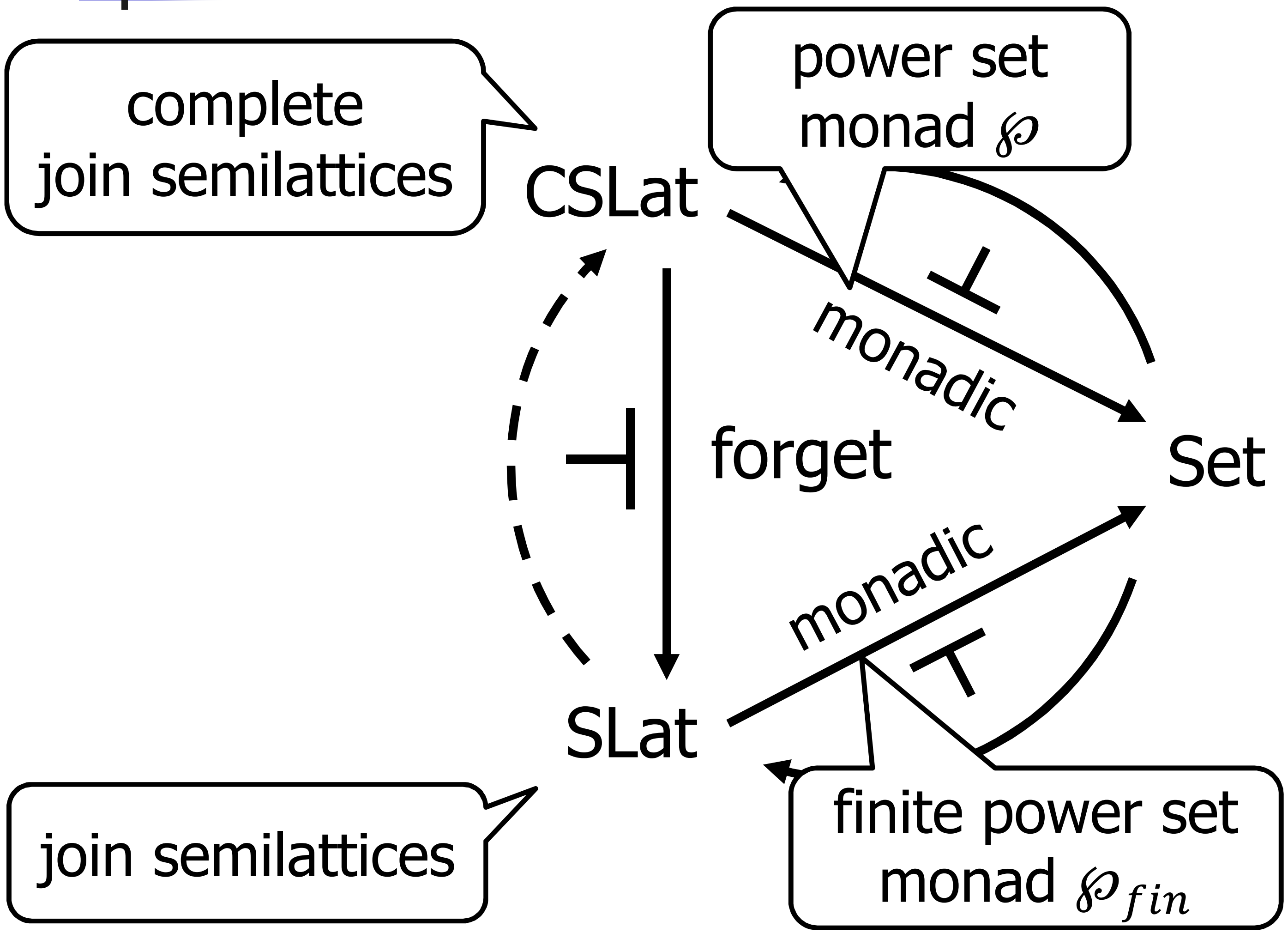


Theorem 1

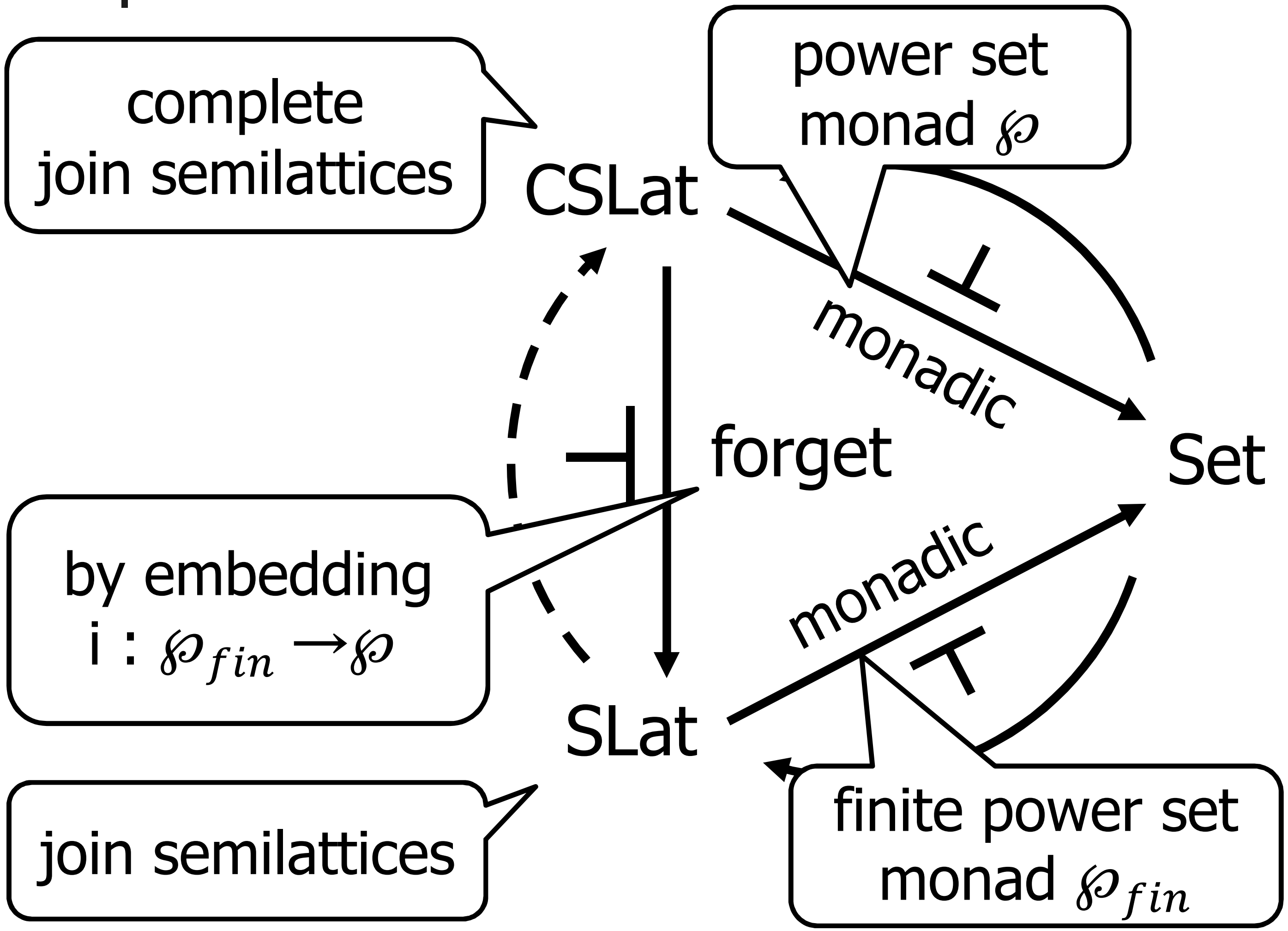
- $G : P'\text{-Alg} \rightarrow P\text{-Alg}$ has a left adjoint if
  - P, P' are monads on C
  - $i : P \rightarrow P'$ is a monad map
  - C has an absolute coequalizer of for each P-algebra $(b : Pc \rightarrow c)$
  - $G(a : P'c \rightarrow c) = (a \cdot i_c : Pc \rightarrow c)$

$$P'Pc \underset{\mu' \cdot P'i_c}{\overset{P'(b)}{\rightrightarrows}} P'c$$

Corollary 1


complete
join semilattices
power set
monad ℘
CSLat
monadic
forget
Set
monadic
SLat
join semilattices
finite power set
monad $\wp_{fin}$

Corollary 1



complete
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power set
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forget
Set
by embedding
i : $\wp_{fin} \to \wp$
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SLat
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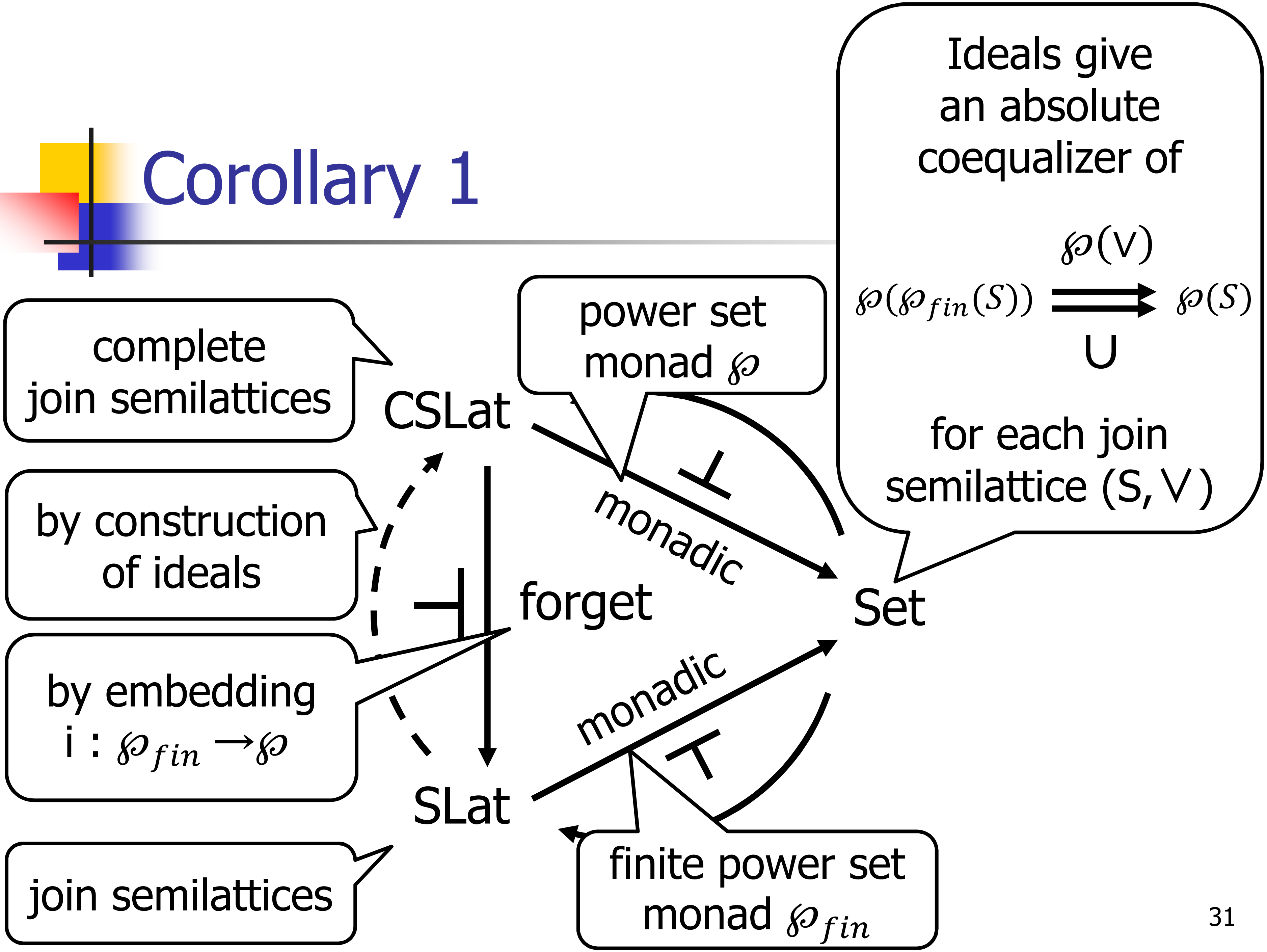
Corollary 1


Ideals give
an absolute
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$\wp(\vee)$
$\wp(\wp_{fin}(S))$
$\wp(S)$
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for each join
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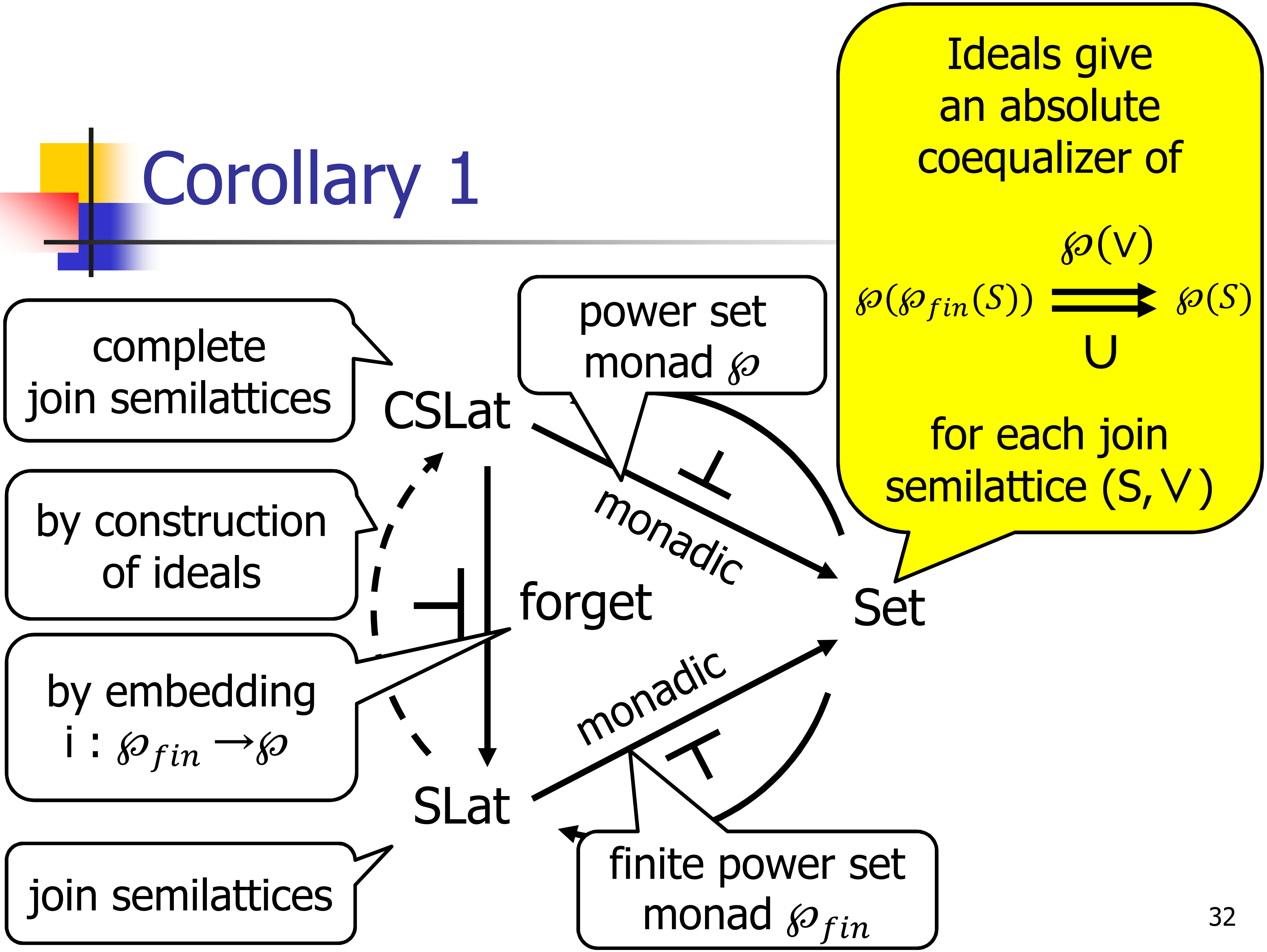

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Lemma

For each join semilattice S, ideal completion of S gives an absolute coequalizer of $\wp(\vee)$ and $\cup$ in Set.

$$\wp(\wp_{fin}(S)) \overset{\wp(\vee)}{\underset{\cup}{\rightrightarrows}} \wp(S) \xrightarrow[\text{Ideal Completion}]{} \text{Ideal(S)}$$

Lemma

For each join semilattice S, ideal completion of S gives an absolute coequalizer of $\wp(\vee)$ and $\cup$ in Set.

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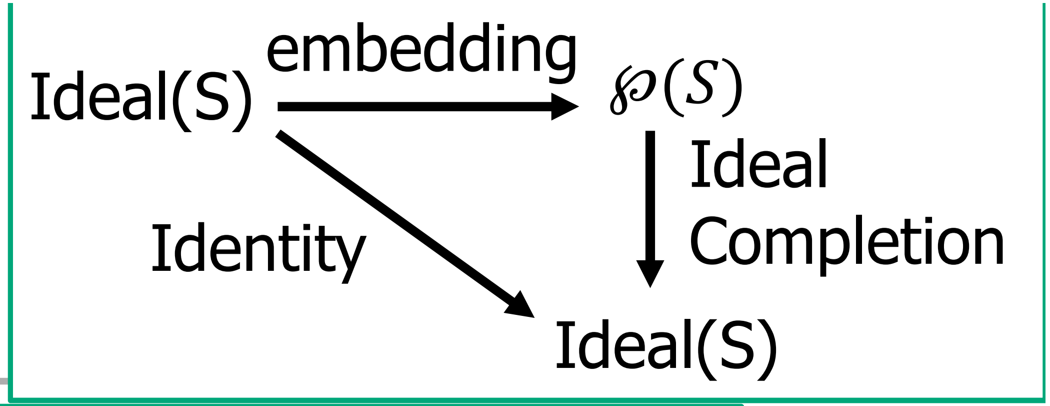
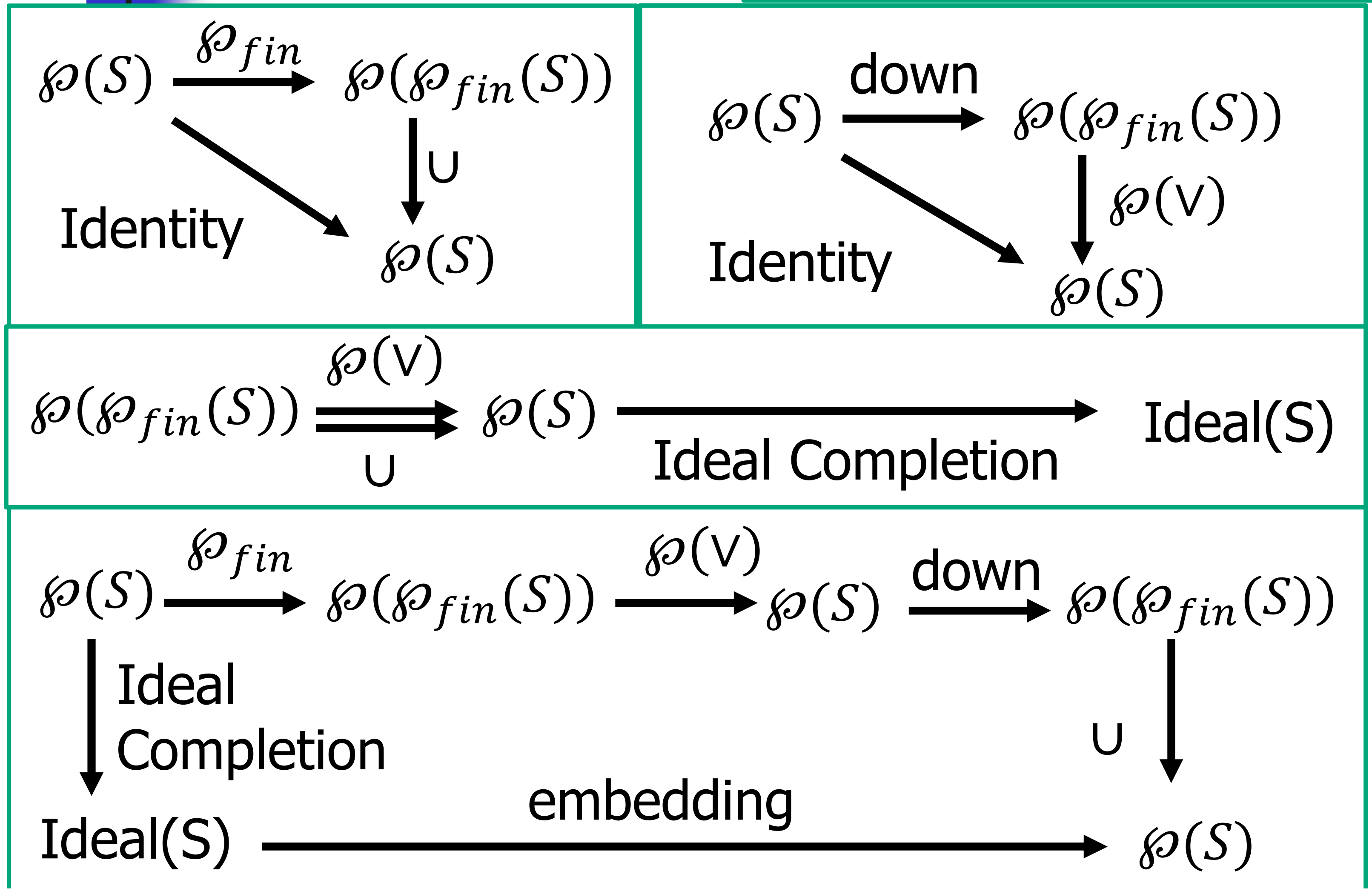
Proof of the Lemma





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Definitions

Def. $(S,\vee,\cdot,e)$ is an idempotent semiring if

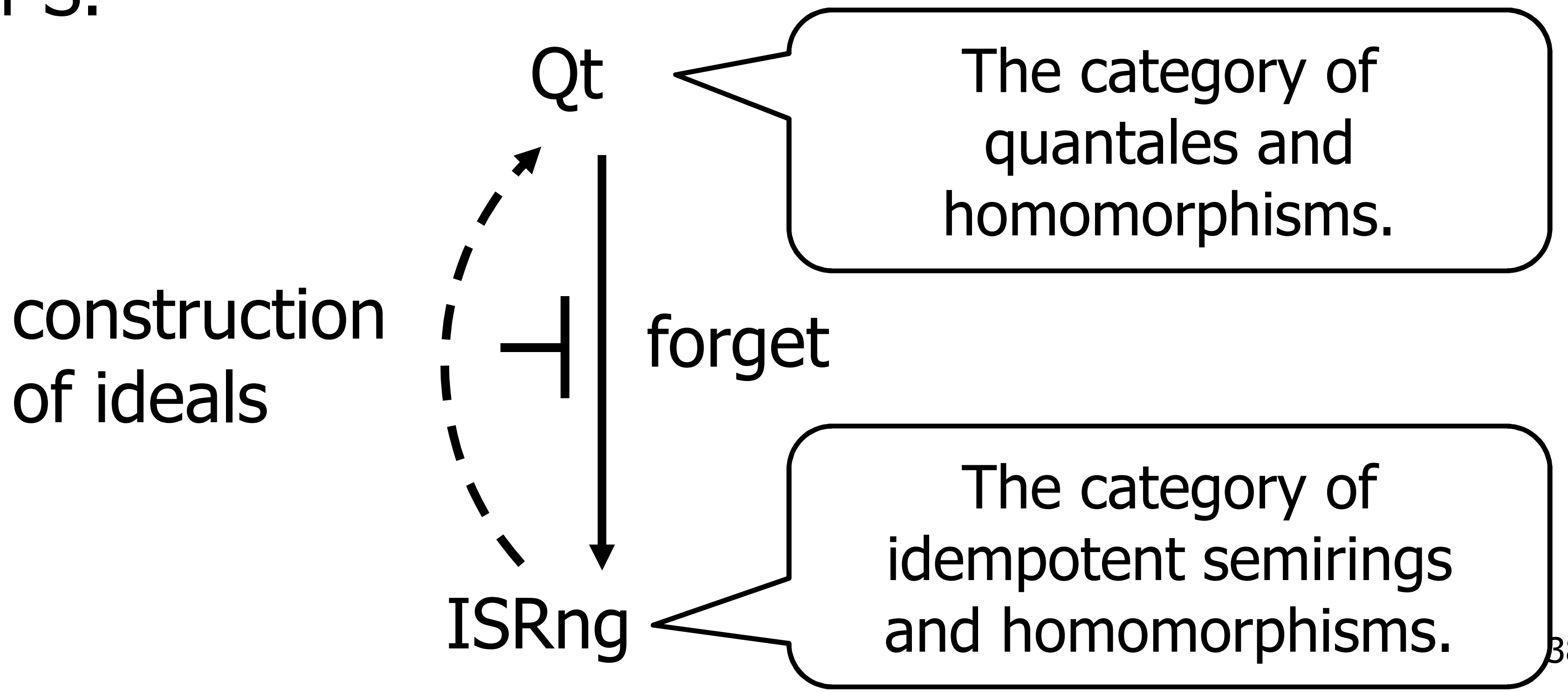
- $(S,\vee)$ is a join semilattice
- $(S,\cdot,e)$ is a monoid
- $a\cdot(\vee X)=\vee\{a\cdot x \mid x\in X\}$ and $(\vee X)\cdot a=\vee\{x\cdot a \mid x\in X\}$ for each $a\in S$ and each finite subset X of S

Def. $(S,\vee,\cdot,e)$ is a quantale
(i.e., complete idempotent semiring) if

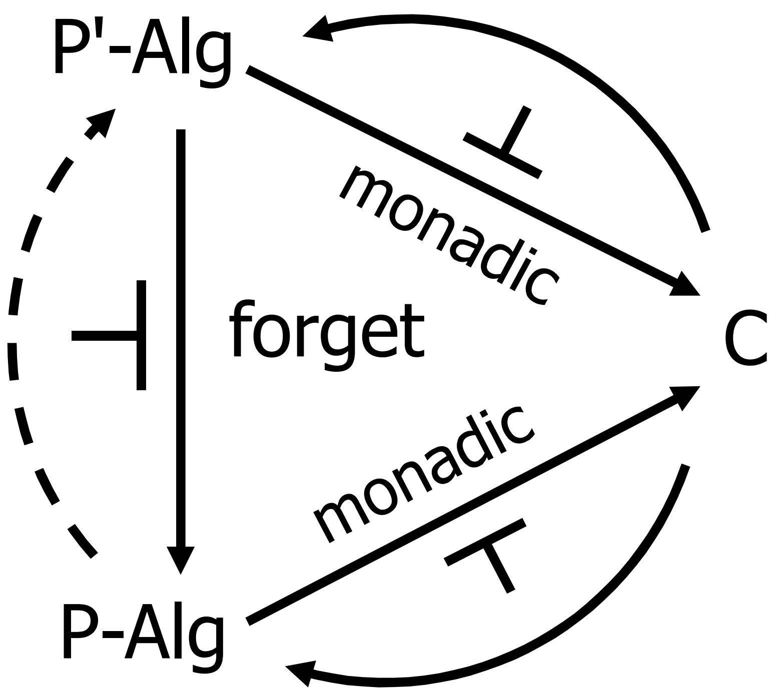
- $(S,\vee)$ is a complete join semilattice
- $(S,\cdot,e)$ is a monoid
- $a\cdot(\vee X)=\vee\{a\cdot x \mid x\in X\}$ and $(\vee X)\cdot a=\vee\{x\cdot a \mid x\in X\}$ for each $a\in S$ and each subset X of S

Goal

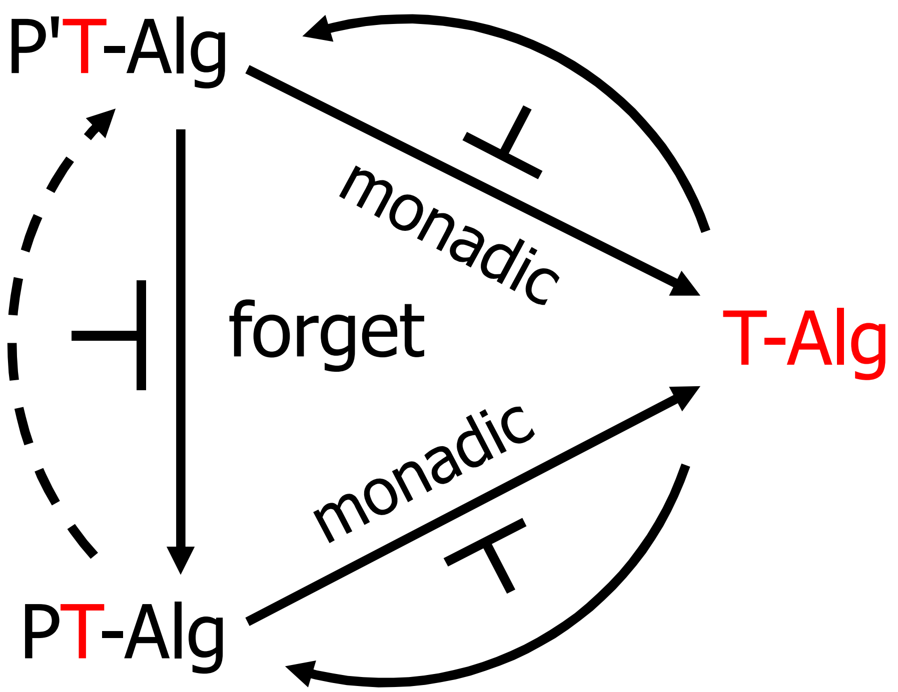
Prop. The forgetful functor from Qt to ISRng has a left adjoint, which sends S to the set of all ideals of S.



Theorem 1



Theorem 2



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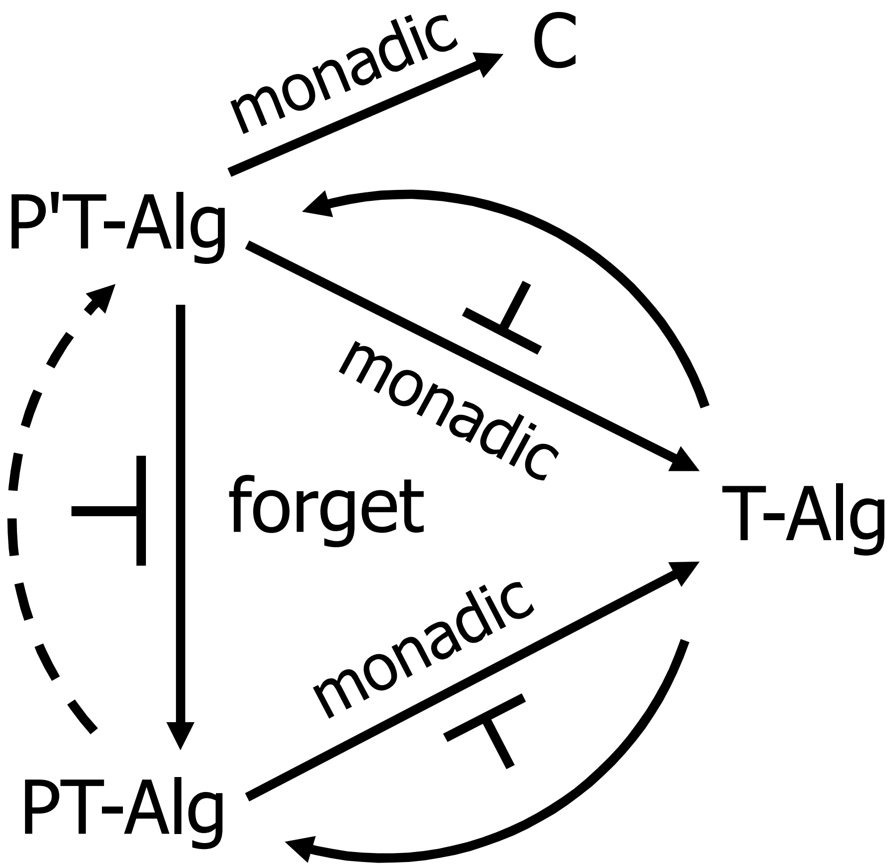
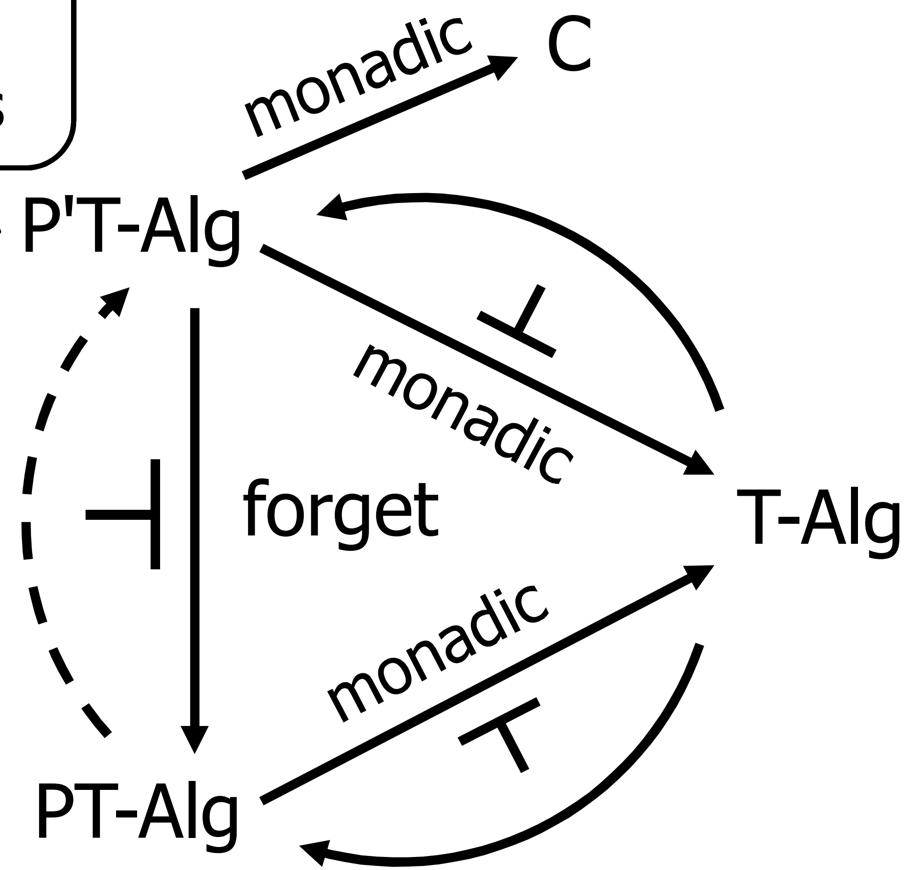


Theorem 2

P'T-Alg has coequalizers of certain diagrams



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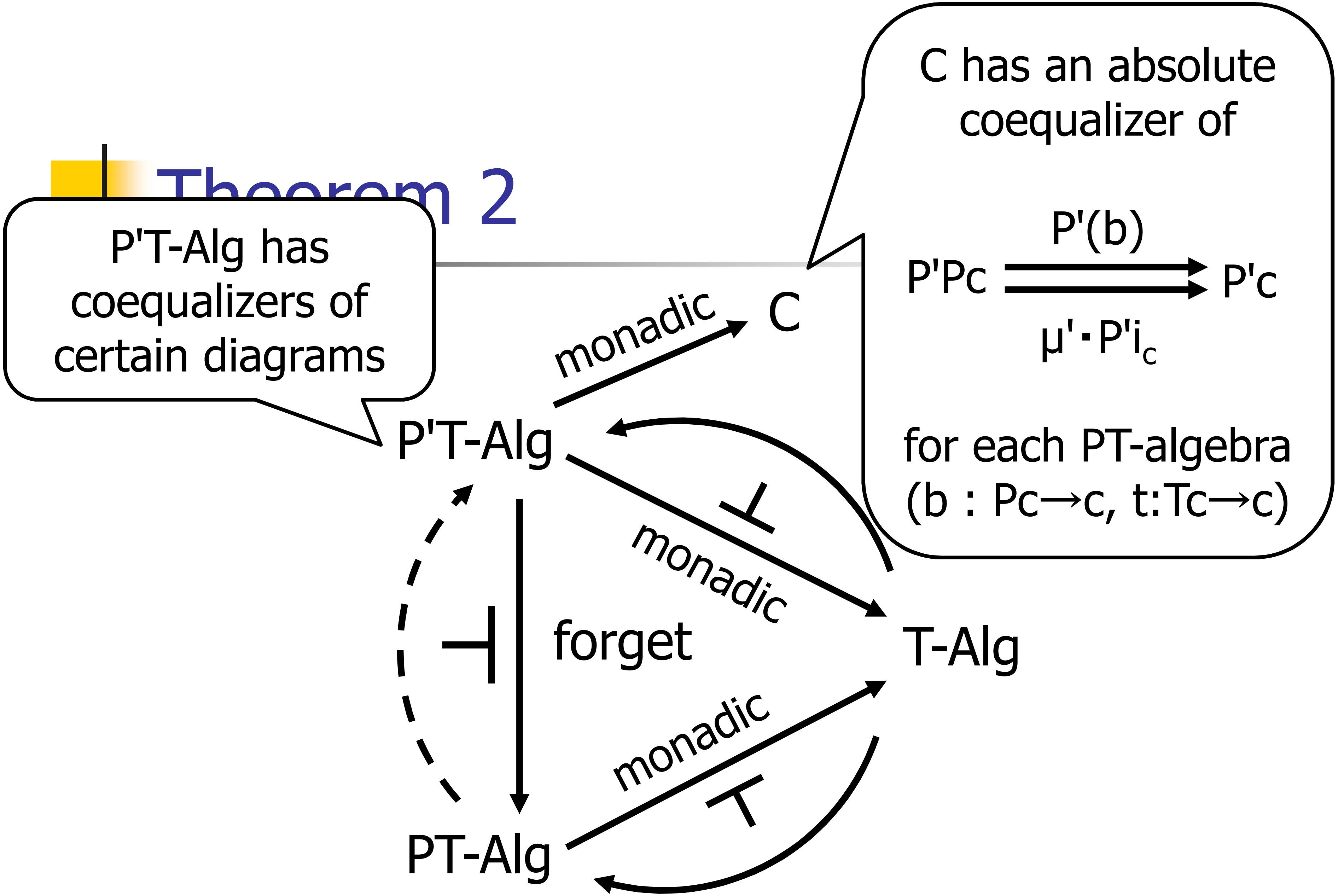
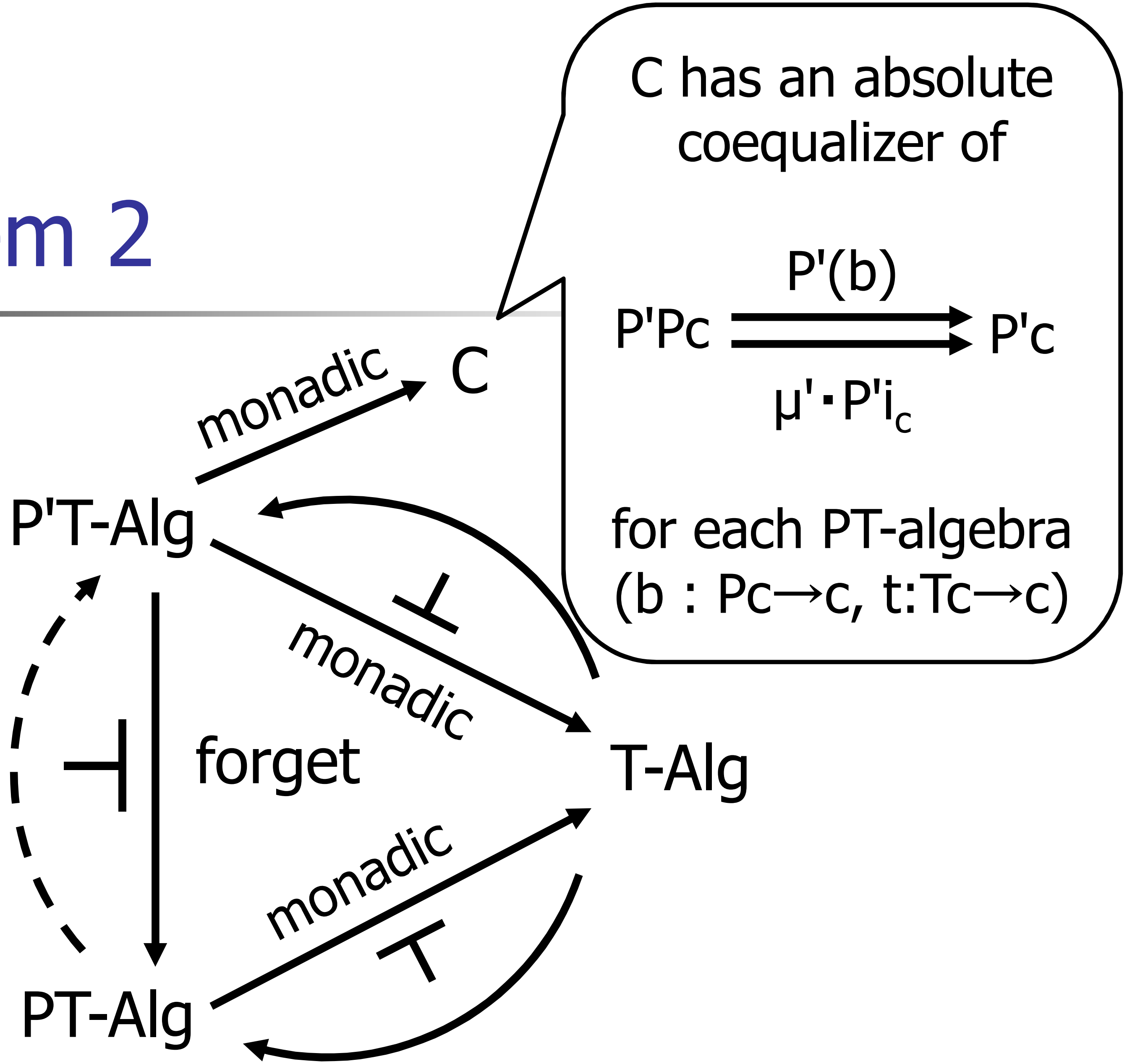


P'T-Alg has coequalizers of certain diagrams
C has an absolute coequalizer of
P'(b)
P'Pc
P'c
μ'·P'i_c
for each PT-algebra (b : Pc→c, t:Tc→c)
monadic
C
P'T-Alg
monadic
T-Alg
forget
monadic
PT-Alg

Theorem 2



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T-Alg
monadic
PT-Alg

Theorem 2

- G : P'T-Alg→PT-Alg has a left adjoint if

Assumption of Theorem 1:

- P,P' are monads on C
- i : P→P' is a monad map
- C has an absolute coequalizer of $P'Pc \underset{\mu' \cdot P' i_c}{\overset{P'(b)}{\rightrightarrows}} P'c$

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Theorem 2

- $G : P'T\text{-Alg} \to PT\text{-Alg}$ has a left adjoint if

Assumption of Theorem 1:

- P, P' are monads on C
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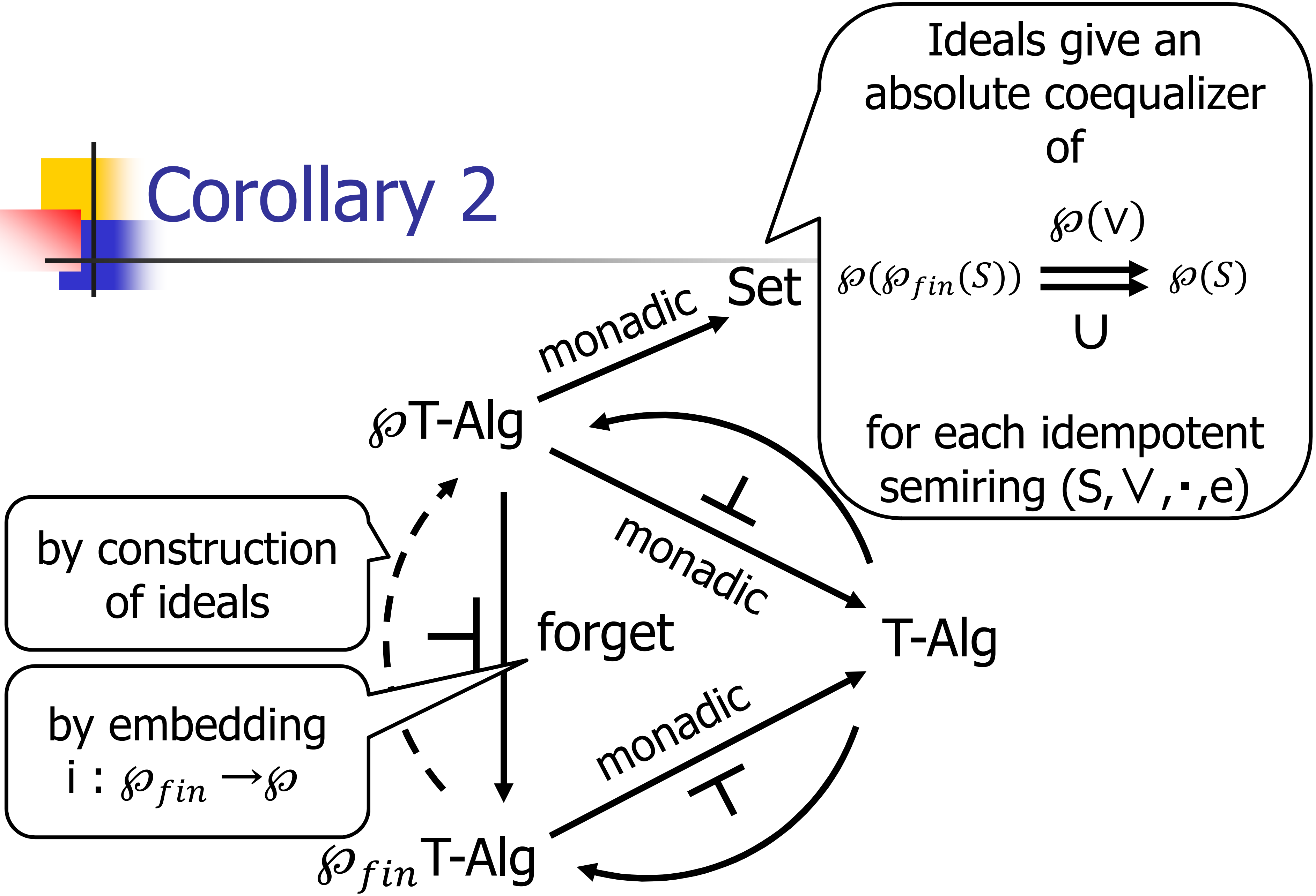
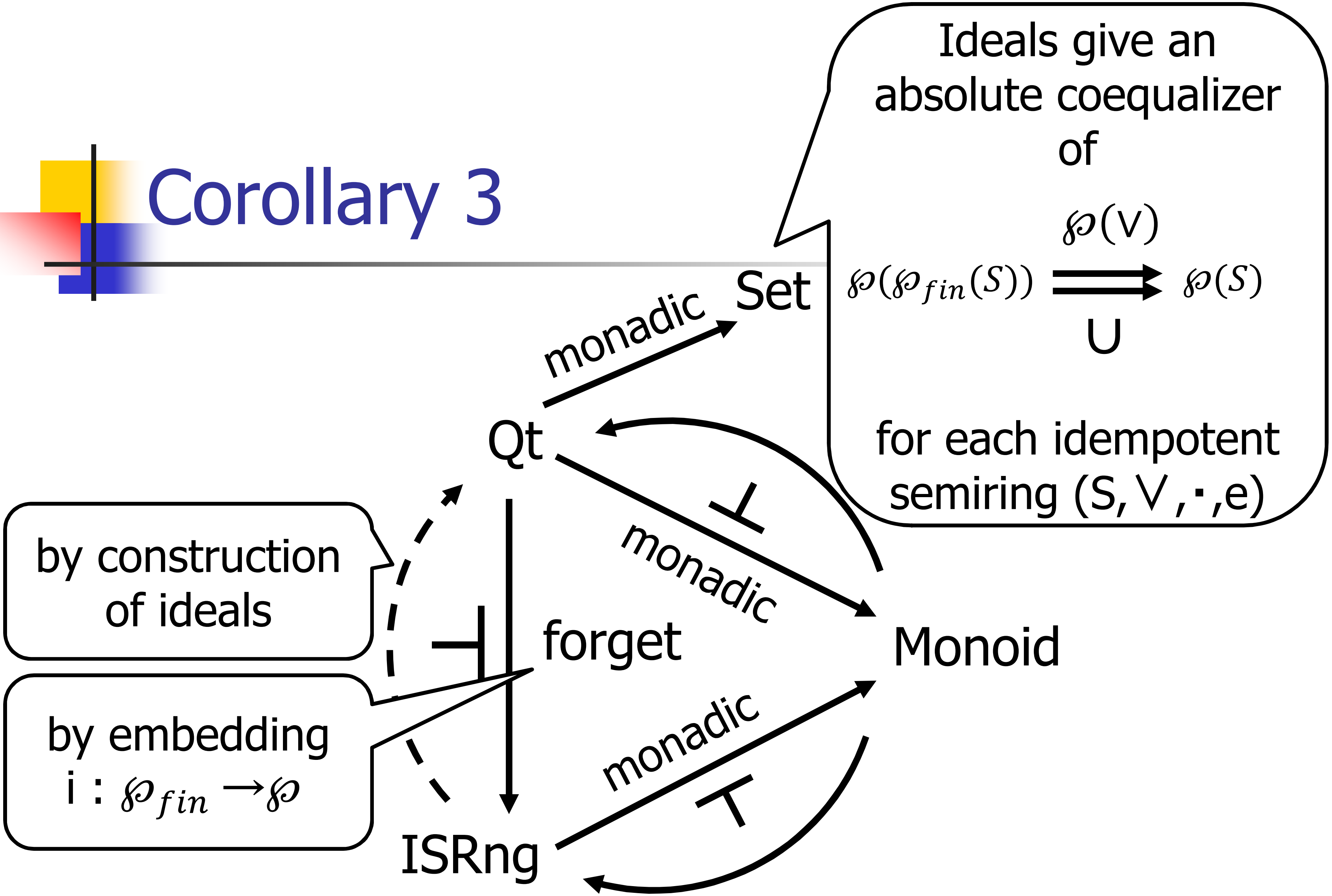
  for each PT-algebra $(b : Pc \to c,\ t : Tc \to c)$

- T is a monad on C
- $\theta : TP \to PT$ is a distributive law
- $\theta' : TP' \to P'T$ is a distributive law
- $(id, i) : \theta \to \theta'$ is a map between distributive laws
- $G(a : P'c \to c,\ \ t : Tc \to c) = (a \cdot i_c : Pc \to c,\ t : Tc \to c)$

Corollary 2


Ideals give an absolute coequalizer of
℘(∨)
℘(℘fin(S)) ⇉ ℘(S)
∪
for each idempotent semiring (S,∨,·,e)
Set
monadic
℘T-Alg
monadic
T-Alg
forget
monadic
℘finT-Alg
by construction of ideals
by embedding i : ℘fin → ℘

Corollary 3


Ideals give an absolute coequalizer of
$\wp(\vee)$
$\wp(\wp_{fin}(S)) \rightrightarrows \wp(S)$
$\cup$
for each idempotent semiring $(S, \vee, \cdot, e)$
Set
monadic
Qt
monadic
Monoid
forget
monadic
ISRng
by construction of ideals
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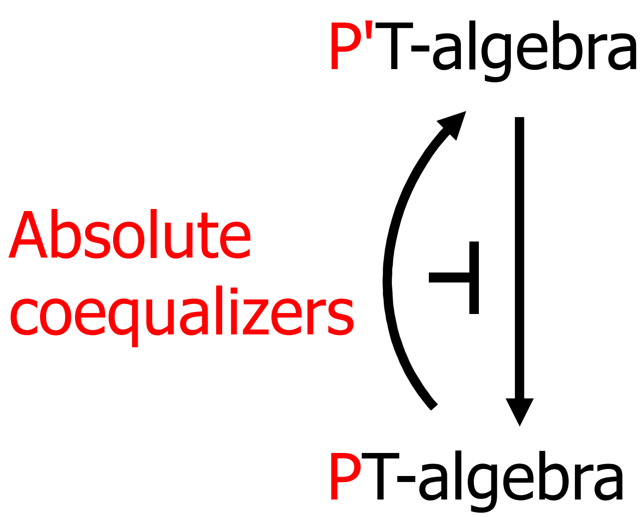
Main Theorem and Corollary

- To give a sufficient condition for monad P,P',T such that



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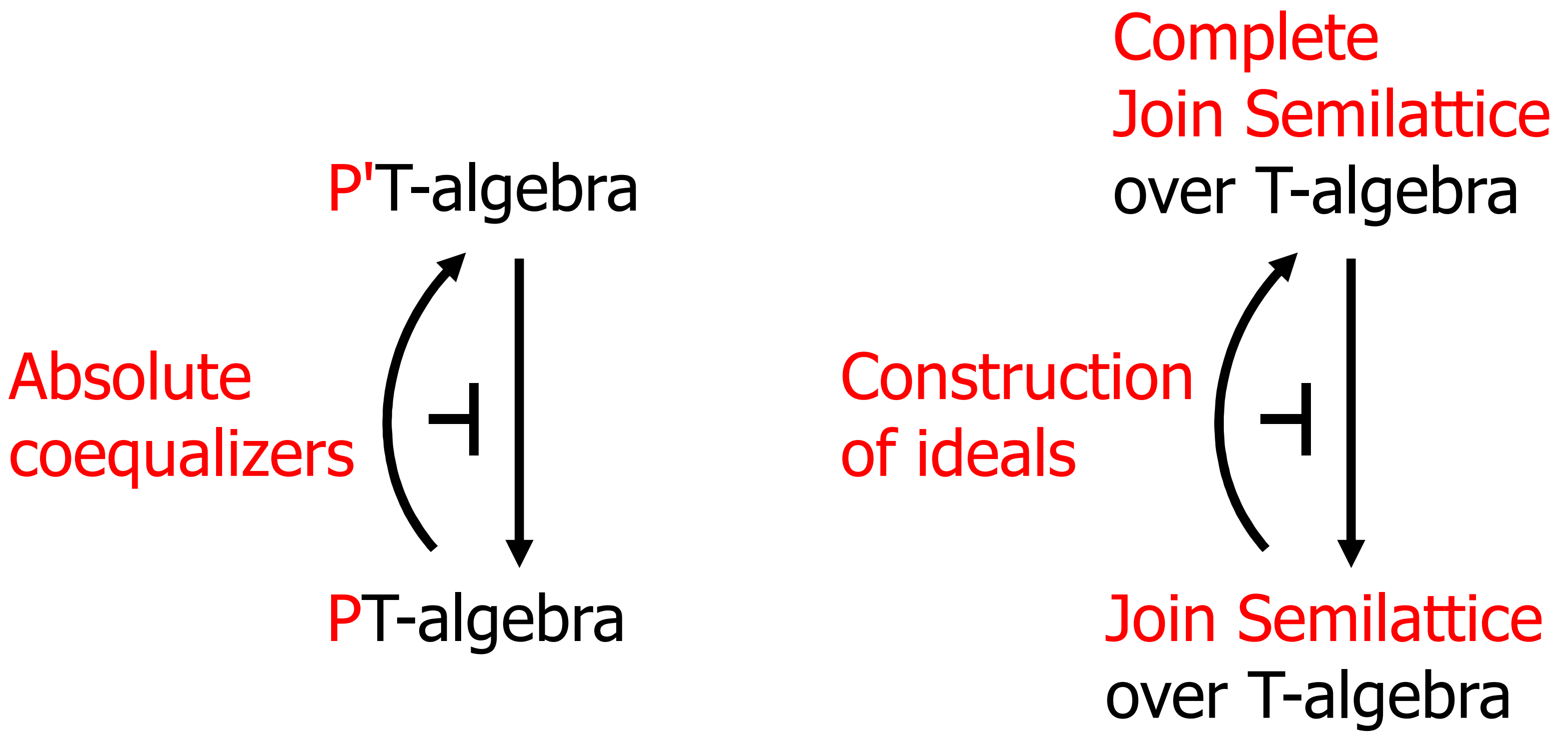
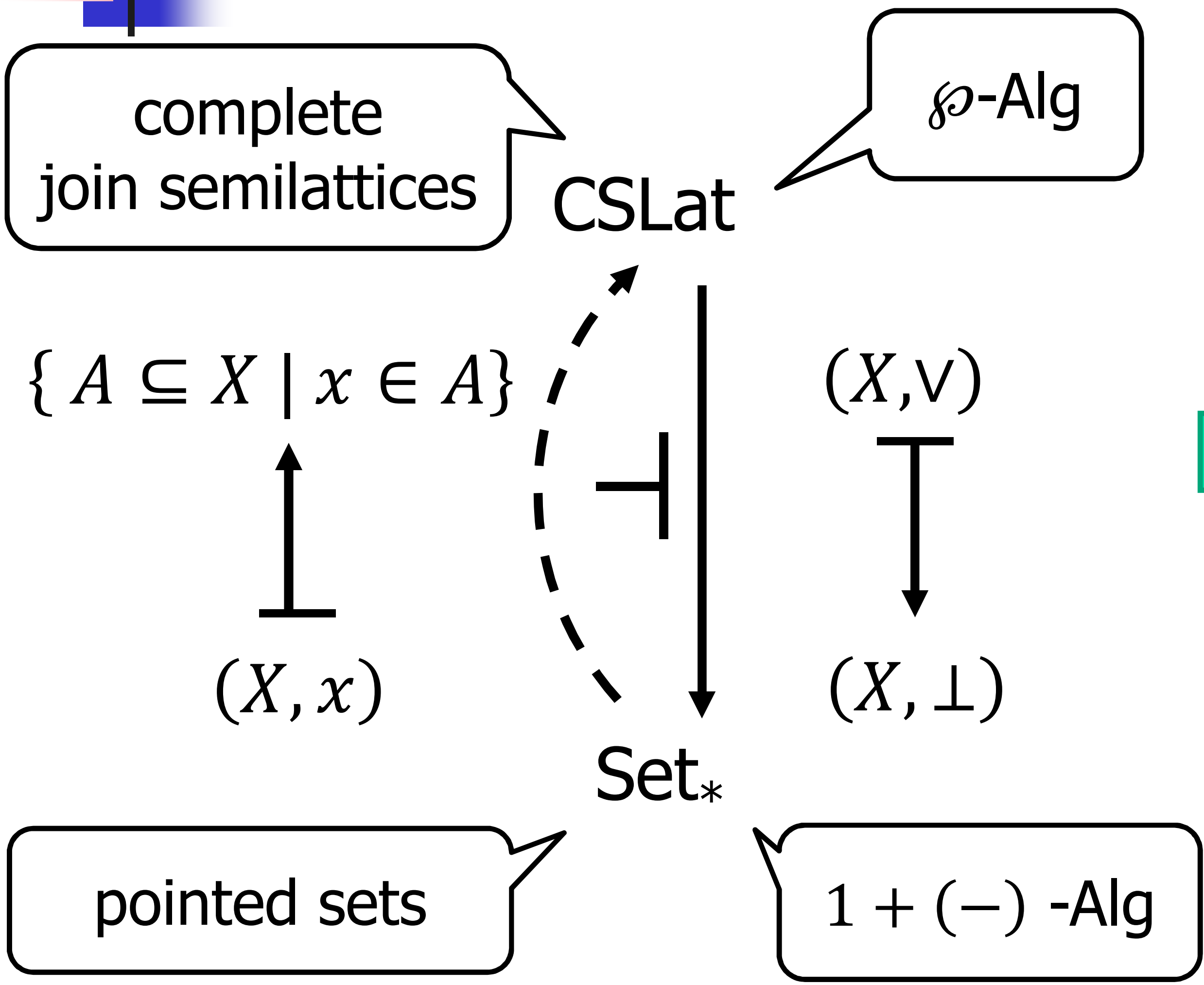
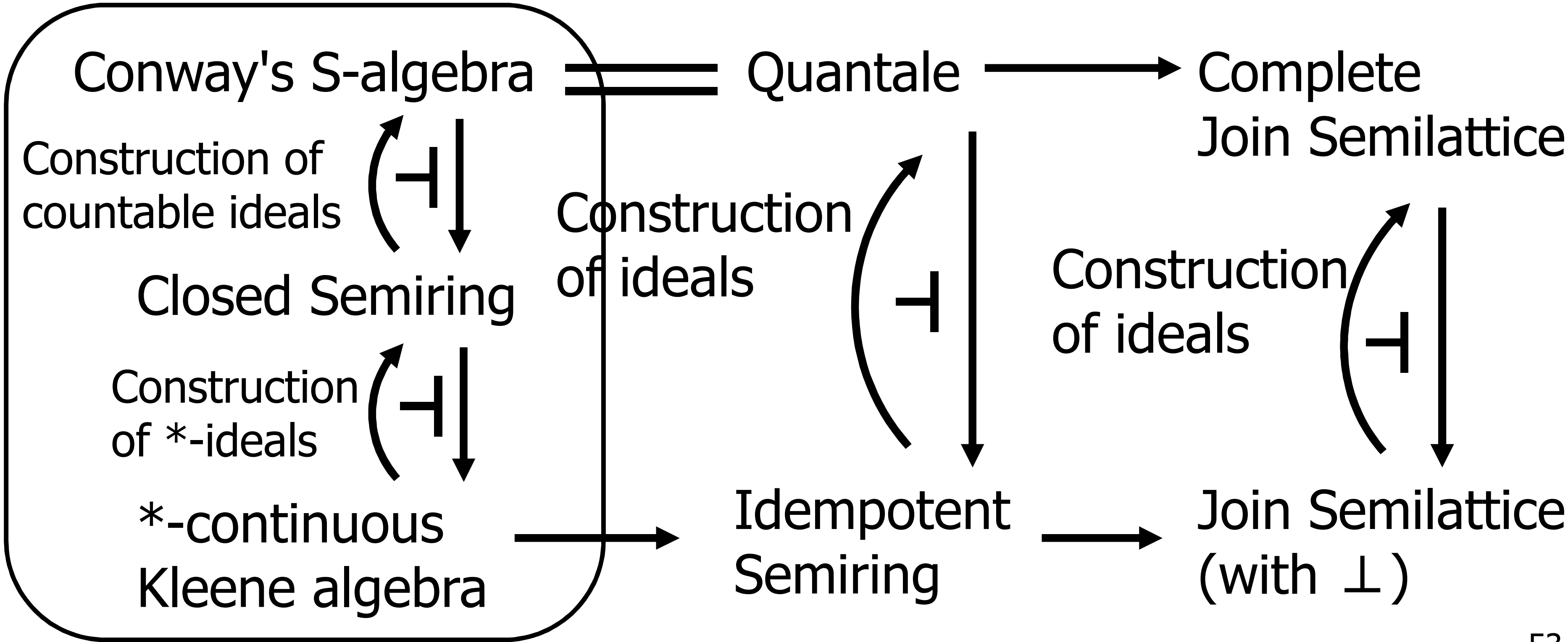


Non-ideal Example



over
T-algebra

Future Work


Conway's S-algebra
Construction of
countable ideals
Closed Semiring
Construction
of *-ideals
*-continuous
Kleene algebra
Quantale
Complete
Join Semilattice
Construction
of ideals
Construction
of ideals
Idempotent
Semiring
Join Semilattice
(with ⊥)

Thank you