Nominal sets over algebraic atoms

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Limited access to data

Infinite alphabets that can only be accessed in limited ways.

- equality only [**Gabbay and Pitts 90s**]
- relational structure, e.g. ordered atoms [**Bojańczyk et al. 2011**]

What about atoms with algebraic structure?

origins in set theory [**Fraenkel 20s, Mostowski 30s**]

Atoms

A - countably infinite set of *atoms*

Atoms - algebraic structure

$$(\mathbb{N}, =) \quad (\mathbb{Q}, \leq) \quad (\mathbb{Q}, \leq, +1, -1)$$

together with its group of automorphisms

Atoms	Automorphisms group
$(\mathbb{N}, =)$	all permutations
$(\mathbb{Q}, \leq)$	monotone permutations
$(\mathbb{Q}, \leq, +1, -1)$	monotone permutations that preserve $x \mapsto x + 1$

The choice of atoms is a parameter of the notion of a nominal set.

Finitely generated supports

X - a set equipped with an action of $\mathrm{Aut}(A)$

A $\quad$ $\{\{a,b\} \mid a \neq b\}$ $\quad$ A^* $\quad$ $A^{(n)}$ (no repetitions)

A *substructure* of A is closed under applying functions.

A substructure S of A is a *support* of $x \in X$ iff

$$\pi|_S = \mathrm{id}|_S \quad \Rightarrow \quad x \cdot \pi = x \qquad \text{for all } \pi \in \mathrm{Aut}(A).$$

Equality atoms $(\mathbb{N}, =)$
$a \in A$ $\quad$ supported by $\{a\}$

Timed atoms $(\mathbb{Q}, \leq, +1, -1)$
$a \in A$ $\quad$ supported by $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

Nominal sets

X is *nominal* iff every $x \in X$ is supported by a finitely generated substructure of A.

$f : X \to Y$ is *equivariant* iff

$$f(\pi(x)) = \pi(f(x)) \qquad \text{for every } x \in X \text{ and every } \pi \in \mathrm{Aut}(A).$$

Examples

Equality atoms $(\mathbb{N}, =)$

$X = A^{(2)} \rightrightarrows (a, b) \in X$ supported by $\{a, b\}$

$X = A^* \rightrightarrows abbcaa \in A^*$ supported by $\{a, b, c\}$

Total order atoms $(\mathbb{Q}, \leq)$

$X =$ the set of all open intervals $\rightrightarrows (0;1) \in X$ supported by $\{0, 1\}$

Timed atoms $(\mathbb{Q}, \leq, +1, -1)$

$X =$ the set of infinite words $a_1 a_2 \dots$ such that $a_{i+1} = a_i + 1 \rightrightarrows$
$a_1 a_2 \dots \in X$ supported by a substructure generated by $\{a_1\}$

Integer atoms $(\mathbb{Z}, \leq)$ (automorphisms $\rightarrow$ translations $x \mapsto x + k$)

$\pi|_{\{0\}} = \mathrm{id}|_{\{0\}} \Rightarrow \pi = \mathrm{id} \rightrightarrows$ everything supported by $\{0\}$

In integer atoms every set is nominal!

Least finitely generated supports

Equality atoms $(\mathbb{N}, =)$

$(1, 2, 3) \in \mathbb{N}^3$ supported by:

- $\{1, 2, 3\}$,
- $\{1, 2, 3, 4, 100, 123\}$.

supports $\rightrightarrows$ closed under adding atoms

Atoms are *supportable* iff every element of every nominal set has a least finitely generated support.

Least finitely generated supports

Equality atoms $(\mathbb{N}, =)$

$(1, 2, 3) \in \mathbb{N}^3$ supported by:

- $\{1, 2, 3\}$, $\Leftarrow$ least finitely generated support!
- $\{1, 2, 3, 4, 100, 123\}$.

supports $\rightrightarrows$ closed under adding atoms

Atoms are *supportable* iff every element of every nominal set has a least finitely generated support.

Orbit-finite sets

An *orbit of $x \in X$* is the set

$$\{x \cdot \pi \mid \pi \text{ is an automorphisms of atoms}\} \subseteq X.$$

An *orbit-finite* set is a finite union of orbits.

The orbit-finite sets play the role of finite sets.

Examples

Equality atoms $(\mathbb{N}, =)$ $\quad \mathbb{N}^2$ has two orbits

$$(1,1)\cdot \mathrm{Aut}(\mathbb{N}) \qquad (0,2)\cdot \mathrm{Aut}(\mathbb{N})$$

Total order atoms $(\mathbb{Q}, \leq)$ $\quad \mathbb{Q}^2$ has three orbits

$$(1,1)\cdot \mathrm{Aut}(\mathbb{Q}) \qquad (2,4)\cdot \mathrm{Aut}(\mathbb{Q}) \qquad (8,0)\cdot \mathrm{Aut}(\mathbb{Q})$$

Integer atoms $(\mathbb{Z}, \leq)$ $\quad \mathbb{Z}^2$ has infinitely many orbits

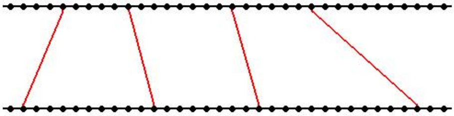
$$\ldots \;\; (1,-1)\cdot \mathrm{Aut}(\mathbb{Z}) \;\; (1,0)\cdot \mathrm{Aut}(\mathbb{Z}) \;\; (1,1)\cdot \mathrm{Aut}(\mathbb{Z}) \;\; (1,2)\cdot \mathrm{Aut}(\mathbb{Z}) \;\; \ldots$$

A product of two orbit-finite sets might not be orbit-finite.

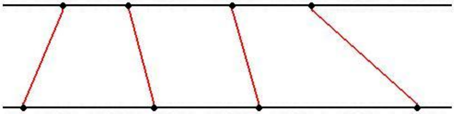
Can we guarantee that orbit-finite sets are well behaved?

Homogeneous atoms

An algebraic structure is *homogeneous* if every isomorphism between its finitely generated substructures extends to a full automorphism.



Integer atoms $(\mathbb{Z}, \leq)$ - not homogeneous



Total order atoms $(\mathbb{Q}, \leq)$ - homogeneous

Structure representation

$S \subseteq A$ - finitely generated substructure

$X =$ the set of embeddings $u\colon S \to A$, where: $u \cdot \pi = \pi \circ u$

Equality atoms $(\mathbb{N}, =)$

finite substructure of atoms: $S = \{1, 3\}$

set of embeddings $\rightrightarrows$ isomorphic to $\mathbb{N}^{(2)}$

set of embeddings is a nominal set $\to$ an embedding is supported by its image

set of embeddings has one orbit $\to$ because the atoms are homogeneous

Structure representation

$S \subseteq A$ - finitely generated substructure

H - some group of automorphisms of S (not necessarily all)

A *structure representation* $[S, H]$ is the set of embeddings $u \colon S \to A$, quotinted by the equivalence relation:

$$u \equiv_H v \quad \Leftrightarrow \quad u \circ \sigma = v \text{ for some } \sigma \in H,$$

with an action of the automorphisms group defined by:

$$[u]_H \cdot \pi = [\pi \circ u]_H.$$

This is also a single-orbit nominal set.

Examples

Equality atoms $(\mathbb{N}, =)$

finite substructure of atoms: $S = \{1, 3\}$

$H = \mathrm{Aut}(S)$ (the identity and transposition)

$[S, H] \rightrightarrows$ isomorphic to all size 2 subsets of atoms

Timed atoms $(\mathbb{Q}, \leq, +1, -1)$

finitely generated substructure of atoms:
$S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$H_1 = \{\mathrm{id}\} \leq \mathrm{Aut}(S)$

$[S, H_1] \rightrightarrows$ isomorphic to the set of atoms

$H_2 = \mathrm{Aut}(S) = \mathbb{Z}$ (translations)

$[S, H_2] \rightrightarrows$ isomorphic to the interval $[0, 1)$

Representation theorem

for homogeneous, supportable atoms

Representation theorem

Every single-orbit nominal set X is isomorphic to $[S, H]$, where S is a finitely generated substructure of atoms and H is some group of automorphisms of structure S.

For relational atoms $\rightrightarrows$ proved by Bojańczyk, Klin and Lasota.

Consequences

An algebraic structure is *locally finite* iff its finitely generated substructures are finite.

For atoms that are locally finite:

- There are countably many single-orbit nominal sets.
- We can represent them in a finite way.

Timed atoms $(\mathbb{Q}, \leq, +1, -1)$
we also obtain finite representation $\rightrightarrows$ automorphisms groups of finitely generated substructures of atoms are isomorphic to $\mathbb{Z}$

Future work

- The theorem uses automorphism groups of finitely generated substructures of atoms $\rightrightarrows$ can we represent them in a finite way?
- Characterization of atoms that are ”well-behaved" $\rightrightarrows$ more natural criteria that would be easier to verify.