Varieties of Boolean Semilattices

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Motivating Construction

Let $\mathbf{G} = \langle G, \cdot \rangle$ be a groupoid (i.e., 1 binary operation)
Form the *complex algebra*

$$\mathbf{G}^+ = \langle \mathrm{Sb}(G), \cap, \cup, \sim, \cdot, \emptyset, G \rangle$$

$$X \cdot Y = \{\, x \cdot y : x \in X,\ y \in Y \,\} \text{ “complex operation”}$$

- $\mathbf{G}^+$ is an expansion of a Boolean algebra
- $X \cdot \emptyset = \emptyset \cdot X = \emptyset$ *(normality)*
- $X \cdot (Y \cup Z) = (X \cdot Y) \cup (X \cdot Z)$ and
  $(Y \cup Z) \cdot X = (Y \cdot X) \cup (Z \cdot X)$ *(additivity)*

In fact, complete and atomic, and completely additive

Boolean Groupoids

Definition

A *Boolean groupoid* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, ', \cdot, 0, 1 \rangle$ such that

$$\langle B, \wedge, \vee, ', 0, 1 \rangle \text{ is a Boolean algebra}$$
$$x \cdot 0 \approx 0 \cdot x \approx 0$$
$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z)$$
$$(y \vee z) \cdot x \approx (y \cdot x) \vee (z \cdot x).$$

BG $=$ the variety of Boolean groupoids

Suppose $\mathbf{G} = \langle G, \cdot \rangle$ is a *semilattice,* i.e. associative, commutative, and idempotent. Then $\mathbf{G}^+$ will be a *Boolean semilattice*.

Definition

A *Boolean semilattice* is a Boolean groupoid satisfying

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$$
$$x \cdot y \approx y \cdot x$$
$$x \leq x \cdot x$$

The variety is denoted BSl.

$\mathrm{Sl}^+ = \{\, \mathbf{S}^+ : \mathbf{S} \text{ a semilattice} \,\}$

Note that Sl^+ is not closed under any of **H**, **S**, **P**

Clearly $\mathbf{V}(\mathsf{SI}^+) \subseteq \mathsf{BSI}$.
One wishes that these are equal. They are not.

Example

Let H be a boolean algebra with atoms a, b. Define multiplication by

$\cdot$	a	b
a	a	b
b	b	$a \vee b$

$\mathbf{H} \in \mathsf{BSI}$
but $\mathsf{SI}^+ \models x \wedge (y \cdot 1) \leq x \cdot y$ while $\mathbf{H}$ fails this identity

Open Questions

1. Is $\mathbf{V}(\mathsf{SI}^+)$ finitely based? Is the equational theory decidable?
2. Is either BSI or $\mathbf{V}(\mathsf{SI}^+)$ generated by its finite members?

Theorem

Let $\mathcal{K}$ be one of following classes.

- *groupoids*
- *commutative groupoids*
- *idempotent groupoids*
- *commutative idempotent groupoids*
- *left-zero semigroups*
- *rectangular bands*

Then $\mathbf{V}(\mathcal{K}^+)$ *is finitely based.*

Let Sg denote the variety of semigroups.

Theorem (Jipsen)

$\mathbf{V}(\mathrm{Sg}^+)$ *is not finitely axiomatizable*

Algebraic Theory of BSI

Let $\mathbf{B} \in$ BSI, $x \in B$.

$\downarrow x = x \cdot 1$

Theorem

'$\downarrow$' yields a closure operator on B:
$x \leq \downarrow x = \downarrow\downarrow x$ *and* $x \leq y \implies \downarrow x \leq \downarrow y$

x is *closed* if $x = \downarrow x$

For a semilattice $\mathbf{S}$ and $X \in \mathbf{S}^+$

$$\downarrow X = X \cdot S = \{\, y \in S : (\exists x \in X)\ y \leq x \,\}$$

the downset generated by X.

Congruence Ideals

Let **B** be a Boolean semilattice

θ is a Boolean semilattice congruence $\implies$

θ is a Boolean congruence $\iff$ $I = 0/\theta$ is a boolean ideal

What condition on a boolean ideal I ensures that it comes from a BSI congruence?

Answer: $x \in I \implies {\downarrow}x \in I$

so instead of working with congruences, we can work with congruence ideals

Consequences

Let **B** be a Boolean semilattice

1. Let $a \in \mathbf{B}$. The smallest congruence ideal containing a is $(\downarrow a]$.
2. BSl has equationally definable principal congruences (EDPC)
3. **B** is subdirectly irreducible iff it has a smallest nonzero closed element
4. **B** is simple iff $x > 0 \implies \downarrow x = 1$
   Thus, for $\mathbf{S} \in$ SI, $\mathbf{S}^+$ is SI iff **S** has a least element
   $\mathbf{S}^+$ is simple iff **S** is trivial
5. Every simple algebra is a discriminator algebra

Subvarieties of BSI

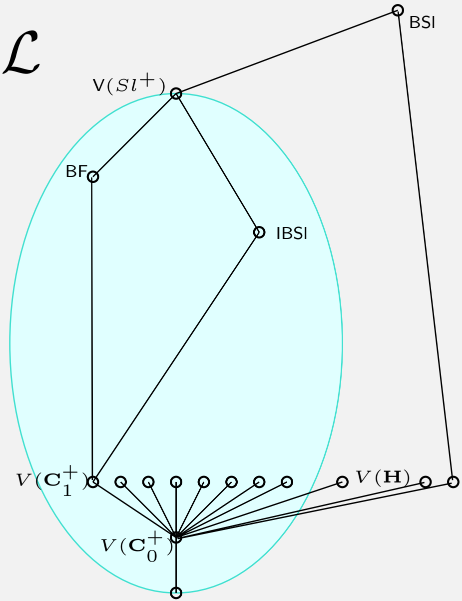
Every nontrivial BSI contains $\{0, 1\}$ as a subalgebra

$\cdot$	0	1
0	0	0
1	0	1

$\cong \mathbf{C}_0^+$

Thus, the lattice of subvarieties has a single atom (Defined by $x \cdot x \approx x \wedge x$)

Unfortunately there seem to be (infinitely?) many covers of the atom


$\mathcal{L}$
BSI
V(Sl^+)
BF
IBSI
$V(\mathbf{C}_1^+)$
$V(\mathbf{H})$
$V(\mathbf{C}_0^+)$

Idempotent Boolean Semilattices

IBSl $= \text{Mod}(x^2 \approx x)$

Let S be a semilattice. $S^+ \models x^2 \approx x$ iff S is linear

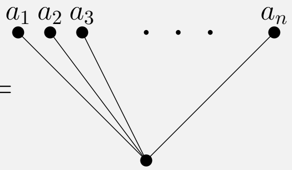
Theorem: IBSl $= \mathbf{V}\,\{\, \mathbf{L}^+ : \mathbf{L}$ linearly ordered $\}$

Problems:

1. Is IBSl $= \mathbf{V}\,\{\, \mathbf{C}_n^+ : n \in \omega \,\}$?
2. Is IBSl $= \mathbf{V}(\mathbb{N}^+)$?
3. is $\mathbb{N}^+ \in \mathbf{V}\,\{\, \mathbf{C}_n^+ : n \in \omega \,\}$?
4. Does *IBSl* have uncountably many subvarieties?

Boolean Fans

A *fan* is a (meet) semilattice of height 1. Let $\mathbf{Y}_n$ denote the fan with an n-element antichain at the top.

$\mathbf{Y}_n =$ 

Theorem:
$\mathrm{BF} = \mathbf{V}\,\{\,\mathbf{Y}^+ : \mathbf{Y} \text{ a fan}\,\}$ "Boolean Fans"
$= \mathrm{Mod}\big((\downarrow x - x)\cdot(\downarrow(x') - x') \approx 0\big)$

Theorem: BF has uncountably many subvarieties

Splitting Algebras (Blok-Pigozzi)

EDPC $\implies$ every finite subdirectly irreducible algebra is splitting.

B is *splitting* iff there is an identity ϵ such that

$$(\forall \mathbf{A} \in \mathrm{BSI}) \;\; \mathbf{B} \in \mathbf{V}(\mathbf{A}) \text{ or } \mathbf{A} \models \epsilon$$

Equivalently, $\mathcal{L} = \left[\mathbf{V}(\mathbf{B})\right) \dot{\cup} \left(\mathrm{Mod}(\epsilon)\right]$

Write $\mathrm{BSI}/\mathbf{B} = \mathrm{Mod}(\epsilon) \cap \mathrm{BSI}$ "Conjugate variety"

Theorem: $\mathrm{BSI}/\mathbf{B} = \{\, \mathbf{A} : \mathbf{B} \notin \mathbf{SH}_\omega(\mathbf{A}) \,\}$

Can we find the conjugate identity for some small S.I. Boolean semilattices?

Theorem: $\mathrm{BF}/\mathbf{Y}_1^+ = \mathrm{Mod}\big((x \cdot x')' \cdot (x \cdot x')' \approx 1\big)$

Problem: Find the defining identity for $\mathrm{BSI}/\mathbf{Y}_1^+$

Papers, notes, etc., available on my web site: `http://www.math.iastate.edu/cbergman/manuscripts/pubs.html`