Algebraic Logic Applied to Relevance Logic: From De Morgan Monoids to Generalized Galois Logics

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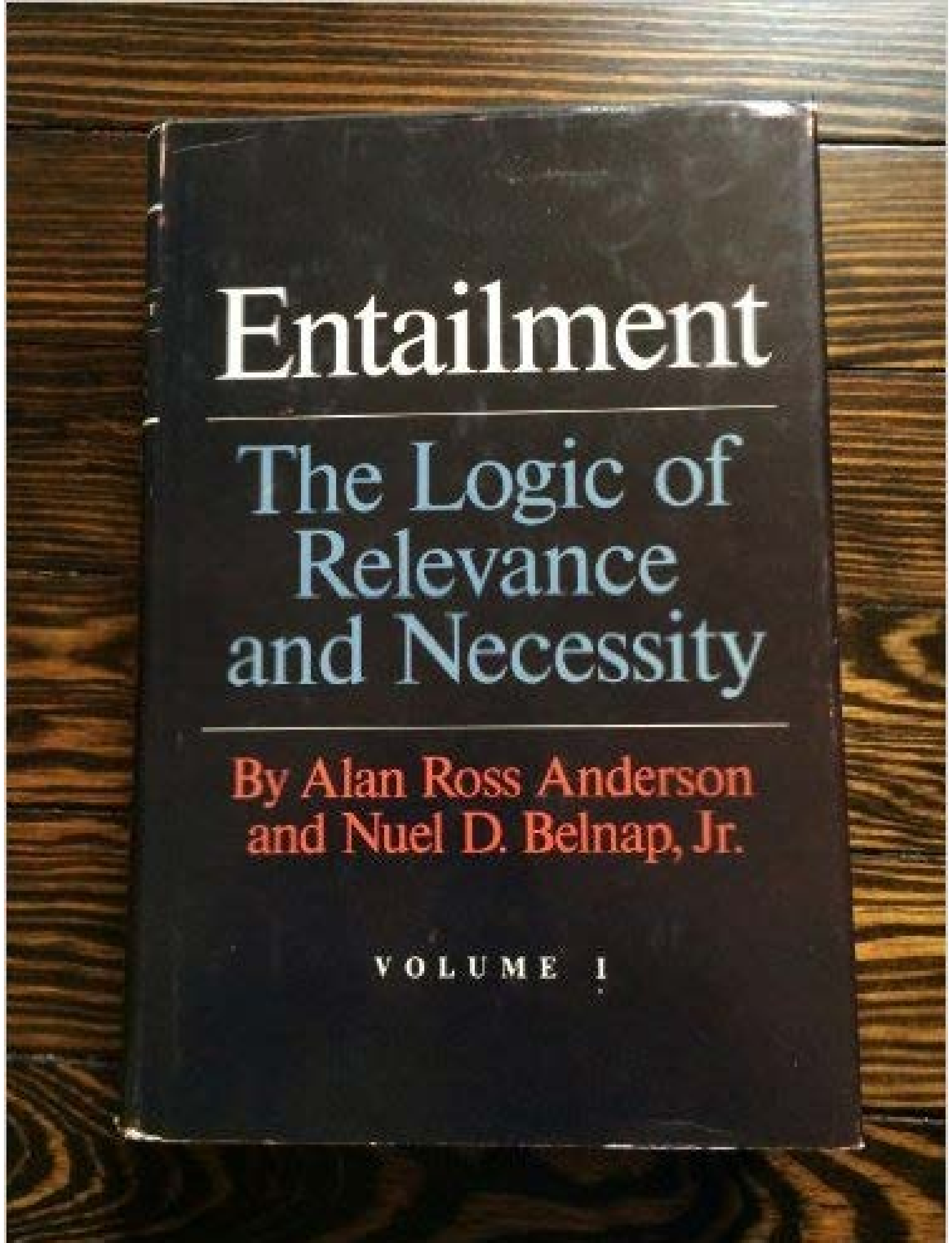
Special Section on Algebraic Logic
Fall Western Sectional Meeting of the
American Mathematical Society
University of Denver
October 8-9, 2016



1975, Princeton University Press

Relevance Logic

In the late 1950's, Alan Ross Anderson and Nuel D. Belnap started to develop their systems **E** of Entailment and **R** of Relevant Implication. Their work was inspired by Wilhelm Ackermann's "Begrundung einer strengen Implikation," *The Journal of Symbolic Logic*, 2:113-128. 1956. They translated "strenge Implikation" as "rigorous implication" to distinguish it from C. I. Lewis's "strict implication" in modal logic. The motivating idea was that in an implication there had to be some relevance between the antecedent and consequent,

and an essential condition was the Variable Sharing Property: (VSP) $A \rightarrow B$ is a theorem of **E** or **R** only if A and B share some propositional variable p.

Important to avoid: $(p \wedge {\sim}p) \rightarrow q$, $p \rightarrow (q \vee {\sim}q)$

Axioms and Rules of **R+**
(Positive Relevant Implication)

Axioms

- $A \to A$ Self-Implication
- $(A \to B) \to [(C \to A) \to (C \to B)]$ Prefixing
- $(A \to B) \to [(B \to C) \to (A \to C)]$ Suffixing (redundant)
- $[A \to (A \to B)] \to (A \to B)$ Contraction
- $[A \to (B \to C)] \to [B \to (A \to C)]$ Permutation
- $A \wedge B \to A$, $A \wedge B \to B$ Conjunction Elimination
- $[(A \to B) \wedge (A \to C)] \to (A \to B \wedge C)$ Conjunction Intro.
- $A \to A \vee B$, $B \to A \vee B$ Disjunction Intro.
- $[(A \to C) \wedge (B \to C)] \to (A \vee B \to C)$ Disjunction Elim.
- $[A \wedge (B \vee C)] \to [(A \wedge B) \vee C]$ Distribution

Rules

- *modus ponens:* $A, A \to B \vdash B$
- adjunction: $A, B \vdash A \wedge B$

Axioms and Rules of **E+** (Logic of Entailment)

E+ is obtained by restricting Permutation axiom

$$A \to (B \to C)] \to [B \to (A \to C)]$$

so that B must be an implication:

$$[A \to ((B \to B') \to C)] \to [(B \to B') \to (A \to C)]$$

$$\text{Restricted Permutation}$$

Axioms and Rules of **B+** (Basic or Minimal Relevance Logic)

Modify **R+** as indicated

Axioms

- $A \to A$ Self-Implication
- ~~$(A \to B) \to [(C \to A) \to (C \to B)]$ Prefixing~~
- ~~$(A \to B) \to [(B \to C) \to (A \to C)]$ Suffixing (redundant)~~
- ~~$[A \to (A \to B)] \to (A \to B)$ Contraction~~
- ~~$[A \to (B \to C)] \to [B \to (A \to C)]$ Permutation~~
- $(A \wedge B) \to A$, $(A \wedge B) \to B$ Conjunction Elimination
- $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$ Conjunction Intro.
- $A \to (A \vee B)$, $B \to (A \vee B)$ Disjunction Intro.
- $[A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$ Disjunction Elim.
- $[A \wedge (B \vee C)] \to [(A \wedge B) \vee C]$ Distribution

Rules

- *modus ponens:* $A, A \to B \vdash B$
- adjunction: $A, B \vdash A \wedge B$
- Prefixing: $A \to B \vdash (C \to A) \to (C \to B)$
- Suffixing: $A \to B \vdash (B \to C) \to (A \to C)$

Add negation (~) with these axioms to get full **R** or **E**

1. $(A \to {\sim}B) \to (B \to {\sim}A)$ Contraposition
2. ${\sim}{\sim}A \to A$ Classical Double Negation
3. $(A \to {\sim}A) \to {\sim}A$ Reductio

Fact: $A \to {\sim}{\sim}A$ [Constructive Double Negation] follows easily from 1.
Substitute ${\sim}{\sim}A/A$, A/B.

For **B**

$A \vee {\sim}A$ Excluded Middle

${\sim}{\sim}A \to A$ Classical Double Negation

$A \to {\sim}B \vdash B \to {\sim}A$ Rule-form Contraposition

The sentential constant $\boldsymbol{t}$ can be added conservatively with the axioms

- $\boldsymbol{t}$
- $\boldsymbol{t} \to (A \to A)$.

For **R** this is equivalent to:

$A \to (\boldsymbol{t} \to A)$

$(\boldsymbol{t} \to A) \to A$.

This was key to the algebraization of **R** in my 1966 thesis . $\boldsymbol{t}$ corresponds to an identity element in a "De Morgan monoid."

If I was being careful I would use the notation $\mathbf{R}^t$ but … .

First algebraic treatments:

• Nuel D. Belnap and Joel H. Spencer, "Intensionally Complemented Distributive Lattices," *Portugalie Mathematica*, 25:99-104, 1966. Algebraic treatment of First Degree formulas (no nested implications) of the relevance logics **R** and **E** using De Morgan lattices with "truth filter" T that must be consistent and complete: a $\in$ T iff $\sim$a $\notin$ T. They show a De Morgan lattice has a truth filter iff for every element a, a $\neq$ $\sim$a.

• J. Michael Dunn, *The Algebra of Intensional Logics*, Ph. D. dissertation, University of Pittsburgh, 1966. Parts reprinted in A. R. Anderson and N. D. Belnap's Entailment, vol. 1, 1975. Algebraic treatment of First Degree Entailments (FDE) A $\rightarrow$ B (no $\rightarrow$ in A or B). Various representations of De Morgan lattices can be given various semantic interpretations. Also algebraic treatment of the whole of the system R using De Morgan lattice ordered commutative, square-increasing monoids – "De Morgan monoids."

Relevance Logic
Two important algebraic aspects

In Lindenbaum algebra of R:

1. First Degree Entailment fragment (FDE) is a De Morgan lattice.

2. Relevant implication is residuation.

1. De Morgan lattice

$(D, \leq, \wedge, \vee, \sim)$ is a De Morgan lattice iff

1) $(D, \leq, \wedge, \vee)$ is a *distributive lattice*, i.e.,
   a) $\leq$ is a partial order on D
   b) $a \wedge b = \text{glb}\ \{a, b\}$
   c) $a \vee b = \text{lub}\ \{a, b\}$
   d) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (Distribution)

   and

2) $\sim$ is a De Morgan complement, i.e.,

a) $\sim$ is a unary operation on A
b) $\sim\sim a = a$ (Period Two)
c) $a \leq b$ implies $\sim b \leq \sim a$ (Order Inversion)

Fact: $a \leq \sim b$ iff $b \leq \sim a$ (Galois connection)

Fact: Galois connection implies both b) and c)

Fact: $\sim(a \wedge b) = \sim a \vee \sim b$ (De Morgan Laws)
$\sim(a \vee b) = \sim a \wedge \sim b$

Antonio Monteiro (1960) used the term “De Morgan lattice” in honor of the 19th century British algebraic logician Augustus De Morgan.

De Morgan lattices were studied earlier under a variety of names:

Grigore Moisil (1935)
Białnycki-Birula and Helena Rasiowa (1957) “quasi-Boolean algebras”
John Kalman (1958) “distributive i-lattices” lattices with involution.

Sometimes they were required to have a top element 1 and a bottom element 0.

Białynicki-Birula & Rasiowa's (1957) Representation

An Involuted Frame is a pair (U, *), $U \neq \emptyset$, $* : U \to U$, s.t.
for all $\alpha \in U$, $\alpha^{**} = \alpha$ (period two, "involution")
Fact: * is 1-1, onto (permutation)

- For $X \subseteq U$, define: $X^* = \{\alpha^* : \alpha \in X\}$
  $\sim X = U - X^*$

A *quasi-field of sets* on U is a collection *Q*(U) of subsets of U closed under , $\cap$, $\cup$, $\sim$.

Fact: Every quasi-field of sets is a De Morgan lattice.

And conversely: (Theorem) Every De Morgan lattice is isomorphic to a quasi-field of sets.

* (not B-B and R's *g*) because this is the notation in the Routley-Meyer semantics for relevance logic.

2. Implication is residuation

$(A, \wedge, \vee)$ is a *lattice-ordered semi-group* [*l-semi-group*] iff $(A, \wedge, \vee)$ is a lattice , $\circ$ is an associative binary operation on A, and $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$. If it has an identity element e as well then it is a *lattice-ordered monoid*.

Fact: If $a \leq c$ and $b \leq d$, then $a \circ b \leq c \circ d$

An l-semi-group is right-residuated iff for every pair of elements a, b there exists an element $a \rightarrow b$ such that for all x,

$$a \circ x \leq b \text{ iff } x \leq a \rightarrow b.$$

An l-semi-group is left-residuated iff for every pair of elements a, b there exists an element $b \leftarrow a$ such that for all x,

$$x \circ a \leq b \text{ iff } x \leq b \leftarrow a.$$

Note: Residuation goes back implicitly to Dedekind, and was studied (among others) in the 1930'/40's by J. Certaine, G. Birkhoff, and most notably by Morgan Ward and Robert P. Dilworth, "Residuated lattices," *Trans. Amer. Math. Soc.* 45: 335-54, 1939.

OK, let's summarize. Meet corresponds to conjunction, join to disjunction, De Morgan complement corresponds to negation, and implication corresponds to the residual. But wait … the residual of what? What logical operation does $\circ$ correspond to?

OK, let's summarize. Meet corresponds to conjunction, join to disjunction, De Morgan complement corresponds to negation, and implication corresponds to the residual. But wait … the residual of what? What logical operation does $\circ$ correspond to?

The answer, for **R** anyway, is it corresponds to an operation that has variously been called co-tenability, consistency, intensional conjunction, or fusion (similar to Girard's later multiplicative conjunction in linear logic). It can be defined in **R** as: $A \circ B$ = ~($A \rightarrow$ ~B).

It can be conservatively added to $\mathbf{R}_{\rightarrow}$ and **R**+.

De Morgan Monoids

$(A, \wedge, \vee, \circ, \sim, e)$ is a *De Morgan monoid* iff

1. $(A, \wedge, \vee, \circ, e)$ is a distributive lattice ordered monoid,
2. $a \circ b = b \circ a$ [commutative]
3. $a \leq a \circ a$ [square-increasing]
4. $c \circ a \leq \sim b$ iff $b \circ c \leq \sim a$

Fact. When $\circ$ is commutative, then left and right residuals coincide. $a \rightarrow b = \sim (a \circ \sim b)$.

Fact: Set $c = e$, then $a \leq \sim b$ iff $c \leq \sim a$ (Galois Connection). So we have Period Two and Order Inversion, i.e., a De Morgan lattice.

Fact: $a \wedge b \leq a \circ b$

$a \wedge b \leq a$ and $a \wedge b \leq b$. So $a \wedge b \leq (a \wedge b) \circ (a \wedge b) \leq a \circ b$

Robert K. Meyer and Richard Routley, "Algebraic Analysis of Entailment 1," *Logique et Analyse,* 15: 407-428, 1972. Based on Robert K. Meyer's "Conservative Extension in Relevant Implication," *Notre Dame Journal of Formal Logic*, 31,:39-46, 1973.

Focus is on negation-free relevance logics.

- Implicational Ackermann groupoid $(G, \leq, \circ, \rightarrow, e)$: p.o. groupoid, $e \circ a = a$. $\rightarrow$ is left-residual, i.e., $a \circ b \leq c$ iff $a \leq b \rightarrow c$.
- Positive Ackermann groupoid $(G, \circ, \rightarrow, \wedge, \vee, e)$: Def. $a \leq b$ iff $a \vee b = b$. $(G, \circ, \leq, \rightarrow, e)$ is an implicational Ackermann groupoid. $(G, \wedge, \vee)$ is a distributive lattice. $\circ$ distributes over join in both directions.
- Church monoid $(C, \leq, \circ, \rightarrow, e)$: Implicational Ackermann groupoid where $\circ$ is associative, commutative, and square increasing ($a \leq a \circ a$)
- Dunn monoid $(D, \circ, \rightarrow, \wedge, \vee, e)$: $(D, \circ, \rightarrow, \wedge, \vee, e)$ is a positive Ackermann groupoid, and $\circ$, is associative, commutative, square increasing. Alternatively we have a distributive lattice ordered Church monoid.

- Implicational Ackermann groupoids correspond to the implicational fragment of **B**, $\mathbf{B}_{\rightarrow}$.
- Positive Ackermann groupoids correspond to the positive fragment of **B**, **B**+.
- Church monoids correspond to the implicational fragment of **R**, $\mathbf{R}_{\rightarrow}$.
- Dunn monoids correspond to positive **R** (no negation) **R**+.

We do not have time to get into the detail, but Meyer and Routley have a long list of conditions one might impose on these algebraic structures so as to get a correspondence to various relevance logics and their fragments (and they label these with corresponding combinators and the axioms they correspond to). E.g., for **E**+ take the set of positive Ackermann groupoids satisfying:

$(a \circ b) \circ c \leq a \circ (b \circ c)$	B'	$(A \to B) \to [(B \to C) \to (A \to C)]$
$a \circ b \leq (a \circ b) \circ b$	W	$[A \to (A \to B)] \to (A \to B)$
$a \leq a \circ e$	CI	$(\boldsymbol{t} \to A) \to A$

Another important thing is they give a correspondence between the conditions they list and so-called Routley-Meyer model structures. These are very roughly similar to the Kripke model structures for modal logic except they use a ternary relation of accessibility in place of a binary one. They claim that one can give soundness and completeness proofs for the logics relative to the corresponding model structures, and they illustrate this with **B**+. This implicitly contains representation theorems for the various classes of algebras.

Larisa Maksimova

Maksimova wrote her 1968 Ph. D. thesis *Logical Calculi of Rigorous Implication* under Anatolij Ivanovich Mal'tsev at Novosibirsk. Although its intended focus was on Wilhelm Ackermann, she ended up citing 7 papers by Anderson and Belnap. Her thesis was based on her five published papers:

1. On a "System of Axioms of the Calculus of Rigorous Implication," *Algebra i Logika*, 3:59-68, 1964.
2. "Formal Deductions in the Calculus of Rigorous Implication," *Algebra i Logika*, 5:33-39, 1966.
3. "Some Problems of the Ackermann Calculus," *Doklady AN SSSR*, 175:1222-1224, 1967.
4. "On Models of the Calculus E," *Algebra i Logika*, 6:5-20, 1967.
5. "On a Calculus of Rigorous Implication," *Algebra i Logika*, 7:55-75, 1968.

Algebraic models appear in these papers. We focus on her slightly later paper "Implication Lattices," *Algebra i Logica*, 12:445-467, 1973. She introduces the idea of a "strimpla" (strict implication lattice – note that "strict" here means "rigorous") as a structure (A, D, $\wedge$, $\vee$, $\rightarrow$) where (A, $\wedge$, $\vee$) is a distributive lattice, D $\subseteq$ A is a filter, $\rightarrow$ a binary operation such that

- $x \rightarrow y \in D$ iff $x \leq y$ (defined as $x = x \wedge y$)
- $x \in D$ implies $x \rightarrow y \leq y$
- $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$
- $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$
- $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow (y \wedge z)$
- $(x \rightarrow z) \wedge (y \rightarrow z) \leq (x \vee y) \rightarrow z$

She also introduces a "strimpla**na**" (strict implication lattice with **n**eg**a**tion) as a structure (A, D, $\wedge$,$\vee$, $\rightarrow$, ~), where (A, D, $\wedge$,$\vee$, $\rightarrow$) is a strimpla and ~ is a unary operation on A satisfying:

- $\sim \sim a = a$
- $a \rightarrow \sim b \leq b \rightarrow \sim a$
- $a \rightarrow \sim a \leq \sim a$

She relates her strimplas and strimplanas to a number of relevance logics, and gives representations of them using a ternary relation (she cites several of Routley and Meyer's papers in which they used a ternary relation similarly in their completeness theorems for relevance logic).

The following slide is the abstract she presented of a talk she gave in 1969, which contains, hidden away in algebraic code, arguably the first ternary relational semantics for relevance logic.

L. Maksimova. “An Interpretation of Systems with Rigorous Implication,” 10th All-Union Algebraic Colloquium (Abstracts), Novosibirsk, p.113, 1969.

Algebraic interpretations of models for calculi of strong implication **E**, **R** from [1] and **SE** from [2] are built.

An **E**- (**R**-, **SE**-) model $\mathfrak{A} = (A; D, \cup, \cap, -, \to)$ is isomorphic to an **E**- (**R**-, **SE**-) system of open–closed [clopen] subsets of a compact topological space *S* with an involution *g*, a partial order ≤ and a ternary relation τ. The operations ∪ and ∩ are defined as set-theoretical union and intersection, respectively. $X = S - \{ g(x) \mid \overline{x} \in X \}$, $X \to Y = \{ z \mid (\forall x\, y)(x \in X \;\&\; \tau(x, y, z) \Rightarrow y \in Y) \}$, D is a filter on the lattice A.

Characteristic axioms are presented for a class of systems $(S; g, \leq, \tau)$ such that $(A; D, \cup, \cap, -, \to)$ the system of their open–closed subsets of *S* is an **E**- (**R**-, **SE**-) model.

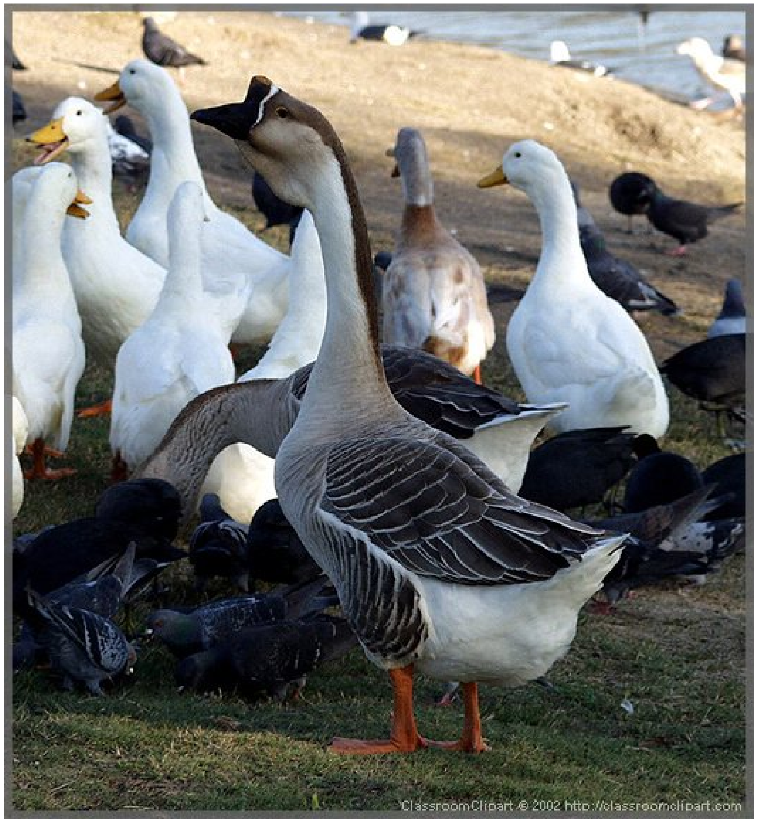
1. Belnap, N. D., Intensional models for first degree formulas, *Journal of Symbolic Logic*, 32(1) (1967), 1-22.

2. Maksimova, L. L. On the calculus of strong implication, *Algebra i logika*, 7(2) (1968), 55–76.

“Ggl” is the acronymn for “generalized galois logic.”

It is pronounced “gaggle.”

“Gaggle,” not “giggle”



Gaggles were inspired by work of

Bjarni Jónsson Alfred Tarski

"Boolean Algebras with Operators," Part I, *American Journal of Mathematics* ,73 (1951), 891-939, Part II, 74 (1952), 127-162.

Some Gaggle References

"Gaggle Theory, an Abstraction of Galois Connections and Residuation, with Applications to Negation, Implication, and Various Logical Operators," in *Logics in AI* (JELIA 1990, Amsterdam), ed. J. Van Eijck, Springer Verlag, pp. 31-51, 1990.

"Partial-Gaggles Applied to Logics with Restricted Structural Rules," in *Substructural Logics*, eds. P. Schroeder-Heister and K. Dosen, Oxford Press, pp. 63-108, 1993.

"Gaggle Theory Applied to Modal, Intuitionistic, and Relevance Logics," in *Logik und Mathematik: Frege-Kolloquium Jena*, eds. I. Max and W. Stelzner, de Gruyter, pp. 335-368, 1995.

Algebraic Methods in Philosophical Logic (with G. Hardgree), Oxford University Press, 2001.

"Symmetric Generalized Galois Logics" (with Katalin Bimbó), *Logica Universalis*, vol. 3, pp. 125-153, 2009.

Generalized Galois Logics. Relational Semantics of Nonclassical Logical Calculi (with K. Bimbó), CSLI Lecture Notes, University of Chicago Press, 2008.

Primer on Gaggle Theory

Consider a poset **D** =(D, $\leq$) that is a distributive lattice, with unary operations *f* and *g* f: P $\mapsto$ P g: P $\mapsto$ P.

We require “abstract residuation”, i.e., one of

- $p \leq fq$ *iff* $q \leq gp$ (galois connection)
- $fq \leq p$ *iff* $gp \leq q$ (dual galois connection)
  $p \geq fq$ $q \geq gp$
- $fq \leq p$ *iff* $q \leq gp$ (**residuation** or adjunction)
  $p \geq fq$
- $p \leq fq$ *iff* $gp \leq q$ (dual residuation)
  $q \geq gp$

It is easy to show that each of these pairs has a "distribution type (distributes or co-distributes over meet or join). Note for each pair the output is uniformly a meet, or else a join.

Residuation: $f(a \vee b) = f(a) \wedge f(b)$

$g(a \wedge b) = g(a) \wedge g(b)$

Galois connection: $f(a \wedge b) = f(a) \vee f(b)$

$g(a \wedge b) = g(a) \vee g(b)$

Dual Galois conn.: $f(a \vee b) = f(a) \wedge f(b)$

$g(a \vee b) = g(a) \wedge g(b)$

Dual Residuation: $f(a \wedge b) = f(a) \vee f(b)$

$g(a \vee b) = g(a) \vee g(b)$

Consider a binary case, a residuated groupoid $(S, \leq \circ, \leftarrow, \rightarrow)$:

$$a \leq c \leftarrow b \ \ \textit{iff} \ \ a \circ b \leq c \ \ \textit{iff} \ \ b \leq a \rightarrow c$$

It is part of the definition that $\circ$ distributes over join in each argument. It can be proven that:

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z) \quad \textit{co-distributes}$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \quad \textit{distributes}$$

Symmetrically for $\leftarrow$.

So distr. types. $\circ$: $(1, 1) \mapsto 1$
$\leftarrow$: $(1, 0) \mapsto 0$
$\rightarrow$: $(0, 1) \mapsto 0$

Note bene: again these are all contrapositives of each other.

How to state "abstract residuation" abstractly?

Start with "Distribution Types". Use 1 in place of join $\vee$, and 0 in place of meet $\wedge$, and view them as complements.

(Note that if we are dealing with only an underlying poset, we can add top $\top$ and bottom $\bot$, and then we get the concept of a trace, where 1 is in place of bottom -- $\bot$, and 0 in place of top -- $\top$)

1. *Residuation* implies that *f* distributes over join and *g* over meet, i.e., f has "distribution type" $1 \mapsto 1$, and *g* has type $0 \mapsto 1$.
2. *Galois connection* implies both *f* and *g* have distr. type $0 \mapsto 1$.
3. *Dual galois conn.* implies both *f* and *g* have distr. type $1 \mapsto 0$.
4. *Dual residuation* implies *f* has "distribution type" $0 \mapsto 0$, and *g* has type $1 \mapsto 0$.

Note that in each case *f* and *g* "contrapose" with each other.

A *distributoid* is a distributive lattice with operations that either distribute or co-distribute in each of their places.

Two operators f and g satisfy the Abstract Law of Residuation (in their i-th place) when f and g are "contrapositives" (with respect to their i-th place) and

$$f(a_1, \ldots, a_i, \ldots, a_n) \leq b \text{ iff } g(a_1, \ldots, b, \ldots, a_n) \leq a_i$$

Thank you!

Approximately 1965