Esakia duality for Sugihara monoids

Wesley Fussner
(Joint work with Nick Galatos)

University of Denver
Department of Mathematics
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- Kripke semantics is underwritten by Esakia duality for Heyting algebras

Motivating Question:

Can we construct an Esakia-style duality that lifts Dunn's semantics to a categorical equivalence?

Priestley and Esakia duality

Definitions:

- An ordered topological space $(X, \leq, \mathcal{T})$ is called a *Priestley space* if $(X, \mathcal{T})$ is compact, and whenever $x, y \in X$ with $x \not\leq y$ there exists a clopen up-set $U \subseteq X$ with $x \in U$, $y \notin U$.

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- A Priestley space is called an *Esakia space* if whenever U is clopen, the down-set $\downarrow U$ is also clopen.
- An *Esakia function* is a map $f : \mathbf{X} \to \mathbf{Y}$ between Esakia spaces that is continuous, order-preserving, and satisfies $\uparrow f(x) \subseteq f[\uparrow x]$ for every $x \in X$.

Priestley and Esakia duality

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The category of bounded distributive lattices (with morphisms the algebraic homomorphisms) is dually equivalent to the category of Priestley spaces with morphisms the continuous, isotone maps.

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The Esakia duality is a restricted form of the Priestley duality.

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- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, t)$ is a commutative monoid, and
- for all $a, b, c \in A$,

$$a \cdot b \leq c \iff a \leq b \rightarrow c$$

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Definition:

The expansion of a CRL by a unary operation $\neg$ satisfying $\neg\neg a = a$ and $a \to \neg b = b \to \neg a$ is called an *involutive* CRL. A distributive, idempotent, involutive CRL is called a *Sugihara monoid*.

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For simplicity, we consider expansions of Sugihara monoids by universal bounds $\bot$ and $\top$.

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- $\neg$ satisfies the identities
  - $\neg\neg a = a$,
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  - $a \wedge \neg a \leq b \vee \neg b$.

Proposition:

The $(\wedge, \vee, \neg, \bot, \top)$-reduct of every bounded Sugihara monoid is a Kleene algebra.

Kleene algebras

Example:

We may define a Kleene algebra $\mathbf{K} = (\{-1, 0, 1\}, \wedge, \vee, \neg, -1, 1)$, where $-1 < 0 < 1$ and $\neg$ is the additive inversion $-$.

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This suggests that a duality for Sugihara monoids could be obtained by restricting a natural duality for their reducts, just as in the case of Heyting algebra.

Davey-Werner duality

Definition:

A structure $\mathbf{X} = (X, \leq, Q, X_0, \mathcal{T})$ is called a *Kleene space* if $(X, \leq, \mathcal{T})$ is a Priestley space, Q is a closed binary relation on X, X_0 is a closed subspace, and for all $x, y, z \in X$,

- xQx,
- xQy and $x \in X_0 \implies x \leq y$,
- xQy and $y \leq z \implies zQx$.

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Example:

We may define a Kleene space $\underset{\sim}{\mathbf{K}} = (\{-1, 0, 1\}, \sqsubseteq, Q, K_0, \mathcal{T})$, where $-1 \sqsubseteq 0$ and $1 \sqsubseteq 0$, -1 and 1 are incomparable, $K_0 = \{-1, 1\}$, and Q is the relation of comparability with respect to the order.

Davey-Werner duality

Proposition (Davey-Werner, 1983):

The category of Kleene algebras K is dually equivalent to the category of Kleene spaces KS (with morphisms the continuous, structure-preserving maps in the signature $(\leq, Q, X_0)$).

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Given a Kleene algebra $\mathbf{A}$, let $D(\mathbf{A})$ be the collection of Kleene algebra homomorphisms $\mathbf{A} \to \mathbf{K}$ endowed with structure inherited pointwise from $\underset{\sim}{\mathbf{K}}$. For a morphism $h\colon \mathbf{A} \to \mathbf{B}$ in K, let $D(h)\colon D(\mathbf{B}) \to D(\mathbf{A})$ be defined by $D(h)(x) = x \circ h$.

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Given a Kleene space $\mathbf{X}$, let $E(\mathbf{X})$ be the collection of Kleene space morphisms $\mathbf{X} \to \underset{\sim}{\mathbf{K}}$ endowed with structure inherited pointwise from $\mathbf{K}$. For a morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ in KS, let $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$ be defined by $E(\varphi)(\alpha) = \alpha \circ \varphi$.

Sugihara spaces

We restrict the Davey-Werner duality to obtain a duality for the Sugihara monoids.

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Definition:

We call a Kleene space $\mathbf{X} = (X, \leq, Q, X_0, \mathcal{T})$ a *Sugihara space* if

- $(X, \leq, \mathcal{T})$ is an Esakia space,
- X_0 is open,
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- $(X, \leq, \mathcal{T})$ is an Esakia space,
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We define operations on the dual of a Sugihara space $\mathbf{X}$ in order to make this structure into a Sugihara monoid.

Kleene space morphisms

Given a Kleene space $\mathbf{X}$, a morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{K}}$ may be represented uniquely as a map of the form

$$\varphi_{(U,V)}(x) = \begin{cases} 1, & \text{if } x \notin V \\ 0, & \text{if } x \in U \cap V \\ -1, & \text{if } x \notin U \end{cases}$$

where U, V are clopen up-sets of $\mathbf{X}$ satisfying $U \cup V = X$, $U \cap V \subseteq X - X_0$, and $((X - U) \times (X - V)) \cap Q = \emptyset$.

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Let $\mathbf{X}$ be a Sugihara space. We define binary operations $\cdot$ and $\rightsquigarrow$ on the collection of Kleene space morphisms from $\mathbf{X}$ to $\underset{\sim}{\mathbf{K}}$ as follows.

New operations

The clopen up-sets of **X** form a Heyting algebra with $\to$ defined by

$$U \to V = \{x \in X : \uparrow x \cap U \subseteq V\}$$

Define also $U \Rightarrow V = [U \cap (X - X_0)] \to V$, and
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We define a multiplication $\cdot$ on pairs $\langle U_1, V_1\rangle, \langle U_2, V_2\rangle$ of clopen up-sets of **X** by $\langle U_1, V_1\rangle \cdot \langle U_2, V_2\rangle = \langle U_3, V_3\rangle$, where

$$U_3 = [(U_1 \Rightarrow V_2) \cap (U_2 \Rightarrow V_1)] \to (U_1 \cap U_2),$$

and

$$\begin{aligned} V_3 = &[(U_1 \Rightarrow V_2) \cap (U_2 \Rightarrow V_1)] \\ &\cap [((U_1 \Rightarrow V_2) \cap (U_2 \Rightarrow V_1)) \to (U_1 \cap U_2)]^* \end{aligned}$$

New operations

Define also a new implication $\rightsquigarrow$ on pairs $\langle U_1, V_1\rangle, \langle U_2, V_2\rangle$ of clopen up-sets by $\langle U_1, V_1\rangle \rightsquigarrow \langle U_2, V_2\rangle = \langle U_3, V_3\rangle$, where

$$U_3 = (U_1 \to U_2) \cap (V_2 \Rightarrow V_1),$$

and

$$\begin{aligned} V_3 =& [((U_1 \to U_2) \cap (V_2 \Rightarrow V_1)) \Rightarrow (U_1 \cap (X \Rightarrow V_2)] \\ & \cap [(U_1 \to U_2) \cap (V_2 \Rightarrow V_1)]^* \end{aligned}$$

New operations

Finally, for morphisms $\varphi_{(U_1,V_1)}, \varphi_{(U_2,V_2)} \colon \mathbf{X} \to \underset{\sim}{\mathbf{K}}$, define

$$\varphi_{(U_1,V_1)} \cdot \varphi_{(U_2,V_2)} = \varphi_{(U_1,V_1)\cdot(U_2,V_2)}$$

and

$$\varphi_{(U_1,V_1)} \rightsquigarrow \varphi_{(U_2,V_2)} = \varphi_{(U_1,V_1)\rightsquigarrow(U_2,V_2)}$$

The equivalence

Let SM denote the category of bounded Sugihara monoids with morphisms the algebraic homomorphisms. Let SS denote the category whose objects are Sugihara spaces $(X, \leq, \leq \cup \geq, X_0, \mathcal{T})$, and whose morphisms are Esakia functions preserving X_0.

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Define a functor $F \colon \mathsf{SM} \to \mathsf{SS}$ as follows. For a Sugihara monoid $\mathbf{A}$, let $F(\mathbf{A})$ be the Davey-Werner dual of the Kleene algebra reduct of $\mathbf{A}$. For a morphism $h \colon \mathbf{A} \to \mathbf{B}$, let $F(h) \colon F(\mathbf{B}) \to F(\mathbf{A})$ be defined by $F(h)(x) = x \circ h$.

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Define a functor $G\colon \mathrm{SS} \to \mathrm{SM}$ as follows. For a Sugihara space $\mathbf{X}$, let $G(\mathbf{X})$ be the Davey-Werner dual of $\mathbf{X}$ endowed with the additional binary operations $\cdot$ and $\rightsquigarrow$, and the nullary operation $\varphi_{(X, X - X_0)}$. For a morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ of SS, define $G(\varphi)\colon G(\mathbf{Y}) \to G(\mathbf{X})$ by $G(\varphi)(\alpha) = \alpha \circ \varphi$.

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Main Theorem:

The functors F and G witness a dual equivalence of categories between SM and SS.

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A Sugihara space $\mathbf{X}$ is, *inter alia*, an Esakia space. What happens if instead of taking its Kleene dual, we take its Esakia dual?

We obtain the Heyting algebra that forms the negative cone of the Sugihara monoid $F(\mathbf{X})$.

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Along these lines, the choice of taking the Kleene dual of such an Esakia space can be thought of as a topological construction of the twist product used in paraconsistent logics.

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Along these lines, the choice of taking the Kleene dual of such an Esakia space can be thought of as a topological construction of the twist product used in paraconsistent logics.

This reflects recent results by Galatos and Raftery that the Sugihara monoids are (covariantly) equivalent to their negative cones.

Thank you!

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