On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices

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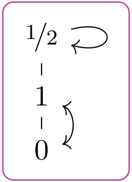

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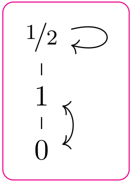

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$$\Gamma \vDash_{\mathbf{PWK}} \alpha \iff \text{for every } v, \quad v[\Gamma] \subseteq \{1, 1/2\} \Rightarrow v(\alpha) \in \{1, 1/2\}$$

Theorem (Ciuni, Carrrara)

For all Γ *and* α*, the following are equivalent:*

- $\Gamma \vDash_{\mathbf{PWK}} \alpha$
- *there is a subset* $\Delta \subseteq \Gamma$ *s.t.* $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\alpha)$ *and* $\Delta \vdash_{\mathrm{CL}} \alpha$*.*

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- Hilbert system for **PWK**:
  - any set of axioms for Classical Logic and
  - the rule:

$$[\text{RMP}]\ \frac{\alpha \qquad \alpha \to \beta}{\beta} \qquad \text{provided that } \mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$$

Involutive Bisemilattices

Definition

An involutive bisemilattice is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1 \rangle$

I1. $x \vee x \approx x$;

I2. $x \vee y \approx y \vee x$;

I3. $x \vee (y \vee z) \approx (x \vee y) \vee z$;

I4. $\neg\neg x \approx x$;

I5. $x \wedge y \approx \neg(\neg x \vee \neg y)$;

I6. $x \wedge (\neg x \vee y) \approx x \wedge y$;

I7. $0 \vee x \approx x$;

I8. $1 \approx \neg 0$.

Thus, the class of involutive bisemilattices is an equational class, which we denote by $\mathcal{IBSL}$.

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Theorem

The only nontrivial subdirectly irreducible bisemilattices are $\mathbf{WK}$, $\mathbf{S}_2$, *and* $\mathbf{B}_2$, *up to isomorphism.*

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$\mathcal{IBSL}$ *is the variety generated by* $\mathbf{WK}$.

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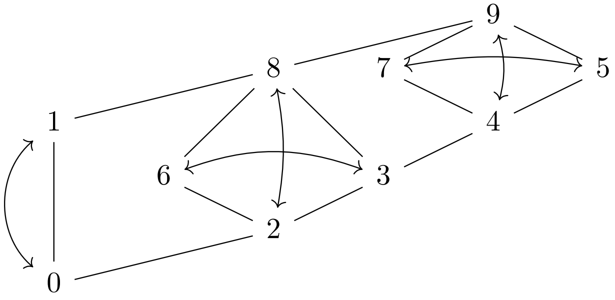
The only nontrivial proper subvarieties of $\mathcal{IBSL}$ *are the disjoint varieties* $\mathcal{BA}$ *of Boolean algebras and* $\mathcal{SL}$ *of semilattices with zero.*

Example



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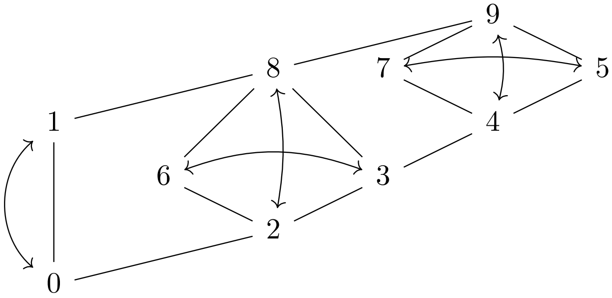

- Positive elements: $P(\mathbf{B}) = \{x \in B : 1 \leqslant x\}$.

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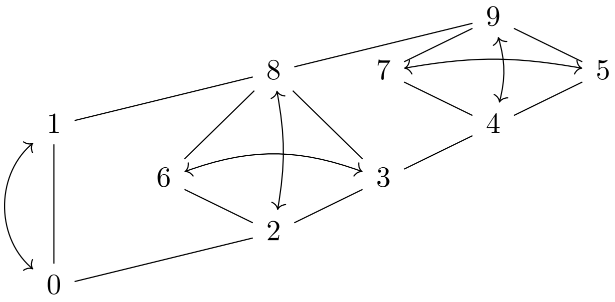

- Positive elements: $P(\mathbf{B}) = \{x \in B : 1 \leqslant x\}$.
- Fibers: $[\neg c, c]$ for every $c \in P(\mathbf{B})$. They are Boolean algebras.

Płonka sums

A direct system of algebras: $\mathtt{T} = \langle (\varphi_{ij} : i \leqslant j), \mathbf{I} \rangle$ such that:

- $\mathbf{I} = \langle I, \leqslant \rangle$ is a join semilattice;
- $\varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ is a homomorphism, for each $i \leqslant j$,

  φ_{ii} is the identity in $\mathbf{A}_i$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$;

- If $i, j \in I$ are different, then $\mathbf{A}_i$ and $\mathbf{A}_j$ are disjoint.

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Płonka sum over $\mathtt{T}$ is the algebra $\mathbf{T} = \langle \bigcup_I A_i, \{g^{\mathbf{T}} : g \in \nu\} \rangle$,

- for every n-ary $g \in \nu$, and $a_1, \ldots, a_n \in T$, where $n \geqslant 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \vee \cdots \vee i_n$ and define

$$g^{\mathbf{T}}(a_1, \ldots, a_n) = g^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n));$$

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- if $g \in \nu$ is a constant, we assume that $\mathbf{I}$ has a least element $\bot$, and $g^{\mathbf{T}} = g^{\mathbf{A}_\bot}$.

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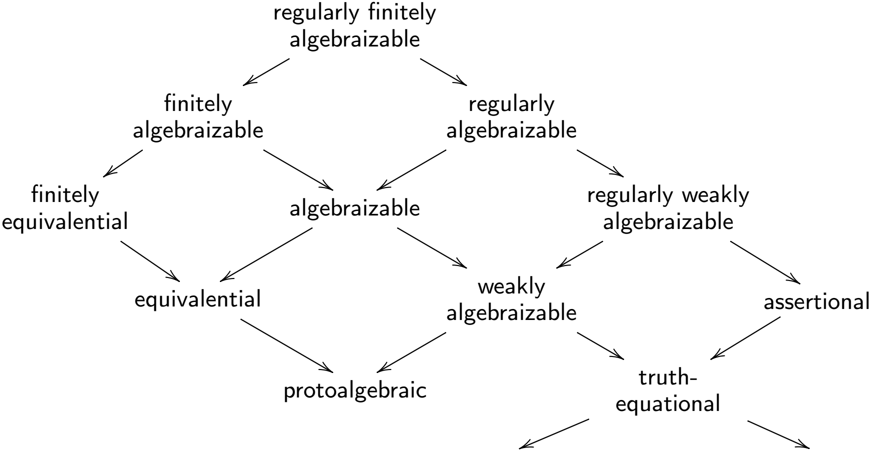
Corollary

$\mathcal{IBSL}$ is the variety satisfying exactly the regular equations satisfied by Boolean algebras.

Leibniz Hierarchy



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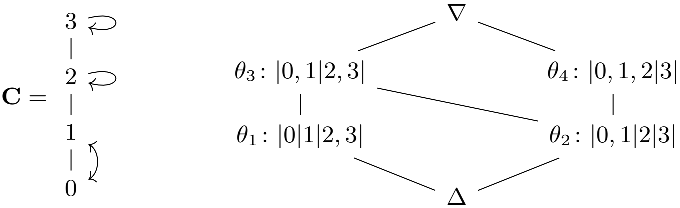

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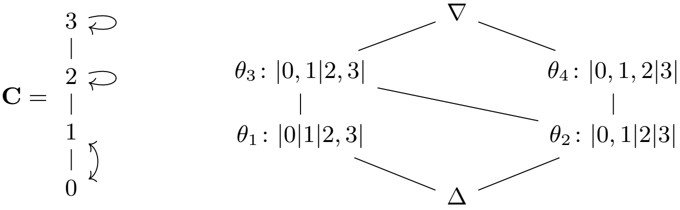

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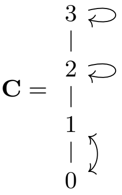

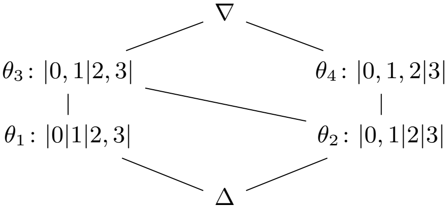

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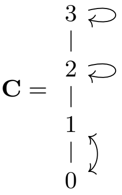

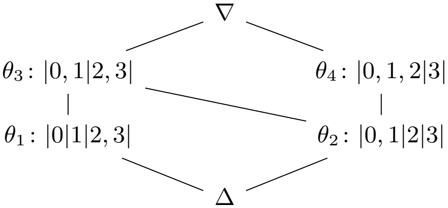

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- It follows that $\emptyset$ is also an L-filter, L is purely inferential, and this leads to a contradiction.

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- Thus, $v[\{p\} \cup p \Rightarrow q] = \{1/2\}$, while $v(q) = 0$, which is a contradiction.

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- Notice that $\neg p \vee p \dashv\vDash_{\mathrm{PWK}} \neg q \vee q$.
- Consider the valuation v on $\mathbf{WK}$: $v(p) = 1/2$, $v(q) = 0$.
  - $v(\neg(\neg p \vee p)) = 1/2$.
  - $v(\neg(\neg q \vee q)) = 0$.
- Therefore $\neg(\neg p \vee p) \dashv\vDash_{\mathrm{PWK}} \neg(\neg q \vee q)$, does not hold. That is, $\dashv\vDash_{\mathrm{PWK}}$ is not a congruence.

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$\mathsf{Mod}^*(\mathrm{L})$ = class of reduced models of L.

$\mathsf{Alg}^*(\mathrm{L}) = \{\mathbf{A} : \text{there is a reduced model } \langle \mathbf{A}, F \rangle \text{ of } \mathrm{L}\}$.

Lemma

If $\mathbf{A}$ *is an algebra of type of* $\mathcal{IBSL}$ *and* $F \in \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\,\mathbf{A}$, *then for every* $a, b \in A$, $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}\, F$ *if and only if for every* $c \in A$,

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Theorem

$\mathrm{Alg}^*(\mathrm{PWK}) \subseteq \mathcal{IBSL}$.

PWK in the Leibniz Hierarchy

Theorem

Moreover, $\langle \mathbf{B}, F \rangle \in \mathsf{Mod}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B} \in \mathcal{IBSL}$*, for every* $a < b$ *positive elements, there is* $c \in B$ *such that*

$$1 \leqslant \neg b \vee c \quad \textit{but} \quad 1 \nleqslant \neg a \vee c$$

and $F = P(\mathbf{B})$ *the set of positive elements of* $\mathbf{B}$*, which is given by:*

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Corollary

PWK *is truth-equational but it is not assertional.*

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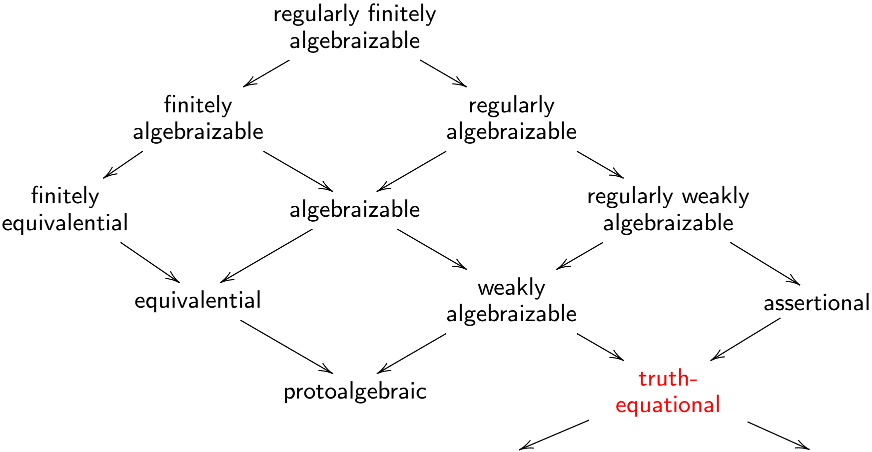

- A g-matrix is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ s.t. $\mathcal{C}$ is a closure system on A.
- A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a g-model of a logic L if $\boxed{\mathcal{C} \subseteq \mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A}}$.
- The Tarski congruence of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is

$$\boxed{\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} \mathcal{C} = \bigcap_{T \in \mathcal{C}} \boldsymbol{\Omega}^{\mathbf{A}}\, T}.$$

- $\langle \mathbf{A}, \mathcal{C} \rangle$ is reduced if $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} \mathcal{C} = Id_{\mathbf{A}}$.

$$\mathsf{Alg}(\mathrm{L}) = \{\mathbf{A} : \langle \mathbf{A}, \mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A} \rangle \text{ is a reduced g-model of } \mathrm{L}\}.$$

In general,

$$\mathsf{Alg}^*(\mathrm{L}) \subseteq \mathsf{Alg}(\mathrm{L}) \subseteq \mathbb{V}(\mathrm{L}).$$

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