Cube Term Blockers Without Finiteness

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AMS Fall Western Sectional Meeting
University of Denver
October 8, 2016

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Or $\mathbb{A}$ has a near-unanimity term:

$$M\begin{pmatrix} y & x & x & \cdots & x \\ x & y & x & & x \\ x & x & y & & x \\ \vdots & & & \ddots & \\ x & x & x & & y \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \\ \vdots \\ x \end{pmatrix}.$$

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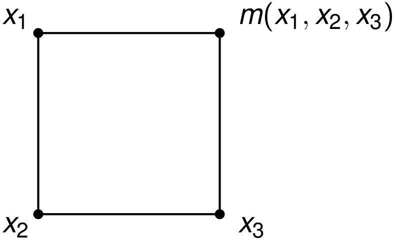

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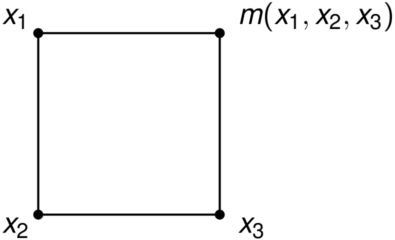

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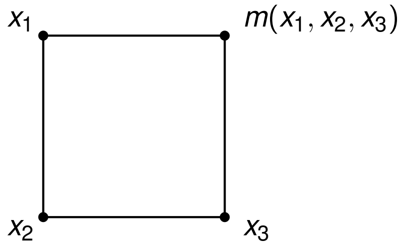

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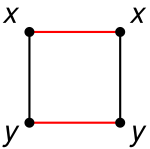

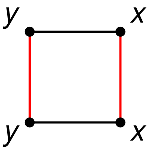

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Cube terms are equivalent to parallelogram terms

$$
\mathrm{P}\left(\begin{array}{ccc|ccccccccc}
x & y & y & z & x & \cdots & x & x & \cdots & x & x \\
x & y & y & x & z & & x & x & & x & x \\
 & \vdots & & \vdots & & \ddots & & & & & \vdots \\
x & y & y & x & x & & z & x & & x & x \\
\cline{1-3}
y & y & x & x & x & & x & z & & x & x \\
 & \vdots & & \vdots & & & & & \ddots & & \vdots \\
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In 2005, Berman, Idziak, Markovic, McKenzie, Valeriote and Willard proved that a finite algebra $\mathbb{A}$ has “few compatible relations” in the sense that the number of compatible n-ary relations of $\mathbb{A}$ is $2^{O(n^d)}$ iff it has a d-dimensional cube term.

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- (Aich-Mayr-McK) A finite algebra with a cube term has a finitely related clone. There are countably many such clones on a finite set.
- (Barto) If a finite algebra $\mathbb{A}$ generates a congruence modular variety and has a finitely related clone, then $\mathbb{A}$ must have a cube term.

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- (K-Kiss-Sz) If $\mathbb{A}$ has a d-dimensional cube term and $\mathbb{A}^d$ is finitely generated, then $\mathbb{A}^n$ is finitely generated for all n, and the minimal number of generators for $\mathbb{A}^n$ is $O(n)$.

Crosses

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The d**-ary cross** on F with bases $U_1, \ldots, U_d$ ($\emptyset \subsetneq U_i \subsetneq F$) is

$$\begin{aligned}\mathrm{Cross}(U_1, \ldots, U_d) \quad = \quad & U_1 \times F \times \cdots \times F \\ & \cup\, F \times U_2 \times \cdots \times F \\ & \vdots \\ & \cup\, F \times F \times \cdots \times U_d\end{aligned}$$

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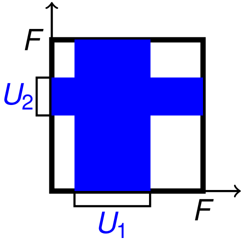

$\mathrm{Cross}(U_1, U_2)$

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Theorem

1. *If* $\mathbb{A}$ *has a compatible* d*-ary cross, then* $\mathbb{A}$ *has no* d*-dimensional cube term.*
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$$\begin{bmatrix} a \\ b \end{bmatrix} = m\left(\begin{bmatrix} u_1 \\ b \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} a \\ u_2 \end{bmatrix}\right).$$

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The results are proved by employing Hall's Marriage Theorem.

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Let Γ and Δ be sets of identities in disjoint languages, both of which entail that all operations involved are idempotent. If every model of $\Gamma \cup \Delta$ has a d-cube term, then either (i) every model of Γ already has a d-cube term or (ii) every model of Δ already has a d-cube term.

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Proving this involves showing that if an algebra in a variety has a compatible d-ary cross, then some infinite algebra in the variety has a compatible d-ary “generic” cross.