There is no finite-variable equational axiomatization of representable relation algebras over weakly representable relation algebras

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Dedication: to Bjarni Jónsson (1920–2016)

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8:30 a.m.–8:50 a.m., Room 453, Sturm Hall

- **Theorem** Any equational basis that defines representable relation algebras (RRA) over weakly representable relation algebras must contain infinitely many variables.
- **Proof** For every positive integer n, there is a finite relation algebra
  - that has a weak representation (on a finite set BTW),
  - whose n-generated subalgebras are representable (all n-variable equations true in RRA are satisfied),
  - that is not representable.
- **Example** The smallest (7 atoms) known non-representable relation algebra that has a weak representation on a finite set.

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Relation algebras—definition

$\mathfrak{A} = \langle A, +, \cdot, ^{-}, ;, ^{\smile}, 1' \rangle$ is a **relation algebra** if

$\langle A, +, \cdot, ^{-} \rangle$ is a Boolean algebra (3 equations)

$$(a;b);c = a;(b;c)$$

$$(a+b);c = a;c + b;c$$

$$1';a = a = a;1'$$

$$\breve{\breve{a}} = a$$

$$(a;b)^{\smile} = \breve{b};\breve{a}$$

$$(a+b)^{\smile} = \breve{a} + \breve{b}$$

$$\breve{a};\overline{a;b} + b = b$$

Representable and weakly representable

- A relation algebra is **representable** if it is isomorphic to an algebra whose elements are binary relations and whose operations $+$, $\cdot$, $^-$, $;$, $\breve{}$, and $1'$ are union, intersection, complementation, composition, converse, and the identity relation, resp. (JT52)
- To change **representable** to **weakly representable**, delete "$+$", "$^-$", "union", and "complemention". (J59)
- A relation algebra is **weakly representable** if it is isomorphic to an algebra whose elements are binary relations and whose operations $\cdot$, $;$, $\breve{}$, and $1'$ are intersection, composition, converse, and the identity relation, resp.

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The RA case—by Lyndon, Monk, Jónsson, and Tarski

- RA = relation algebras
- RRA = representable relation algebras
- RRA ⊆ RA (because the axioms hold: composition is associative,...)
- RA has a 3-variable equational basis (10 equations)
- RRA has an equational basis (T55) but no finite one (M64)
- **Theorem** (J91,T74) RRA has no finite-variable basis (over RA)
- **Proof** For every n, there is an algebra in RA ∼ RRA whose n-generated subalgebras are in RRA, namely
  - the Lyndon algebra $\mathfrak{L}$ of a large_n projective line that does not lie in any projective plane (hence $\mathfrak{L} \notin$ RRA, L61)
  - such a line exists by the Bruck-Ryser Theorem (no projective plane of order k if $k \equiv 1$ or $2 \pmod 4$ and k is not the sum of two squares, e.g. k is 6, 14, 21, 22, 30, 33, 38, 42, 46, 54, 57, 62, 66, 69, 70, 77, 78, 86, 93, 94, 102, 105, 110, 114, 118, 126, 129, 133, 134, 138, 141, 142, 150, 154, ...)
  - atoms of $\mathfrak{L}$ = points $a, b, \ldots$ on the line plus new atom 1'
  - $\breve{a} = a \qquad a;1' = a = 1';a \qquad a;a = 1' + a$
    $a;b = a + b + \cdots = \overline{1'}$

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Two Lyndon algebras

$\mathfrak{L}(3,0)$ is the Lyndon algebra of a 4-point line (a line of order 3)
$\mathfrak{L}(3,0)$ is representable on 9 points

	1'	a	b	c	d
1'	1'	a	b	c	d
a	a	1' $+ a$	$c + d$	$b + d$	$b + c$
b	b	$c + d$	1' $+ b$	$a + d$	$a + c$
c	c	$b + d$	$a + d$	1' $+ c$	$a + b$
d	d	$b + c$	$a + c$	$a + b$	1' $+ d$

$\mathfrak{L}(6,0) \notin$ RRA is the Lyndon algebra of a 7-point line (a line of order 6)

Key facts about the class of Lyndon algebras of lines

- larger algebras require more generators
- proper subalgebras occur in all larger algebras
- there are arbitrarily large representable algebras (by finite fields)
- hence proper subalgebras are representable
- there are arbitrarily large non-representable algebras (by Bruck-Ryser)

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- Are all the Lyndon algebras of lines weakly representable? If so, use them, otherwise
- Does the Bruck-Ryser Theorem also show there are arbitrarily large algebras that are not even weakly representable?

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The wRRA case

- wRRA = weakly representable relation algebras
- RRA $\subseteq$ wRRA ($\subseteq$ RA)
- wRRA has an equational basis (P09), no finite basis (HH01),
- RRA has an equational basis, but
- **Theorem** RRA has no finite-variable basis over wRRA
- **Proof** For every n, there is an algebra in wRRA $\sim$ RRA whose n-generated subalgebras are in RRA
- start with the Lyndon algebra of a large$_n$ projective line (with $p+1$ points) that does lie in a projective plane (e.g. any big$_n$ prime p), add $t = (p+1)/2$ new atoms, get algebra $\mathfrak{L}(p,t)$
  - the n-generated subalgebras of $\mathfrak{L}(p,t)$ are in RRA
  - $\mathfrak{L}(p,t)$ is not in RRA (t is too big)
  - $\mathfrak{L}(p,t)$ is in wRRA (on a finite set!)

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- **Theorem** RRA has no finite-variable basis over wRRA
- **Proof** For every n, there is an algebra in wRRA $\sim$ RRA whose n-generated subalgebras are in RRA
- start with the Lyndon algebra of a large$_n$ projective line (with $p+1$ points) that does lie in a projective plane (e.g. any big$_n$ prime p), add $t = (p+1)/2$ new atoms, get algebra $\mathfrak{L}(p, t)$
  - the n-generated subalgebras of $\mathfrak{L}(p, t)$ are in RRA
  - $\mathfrak{L}(p, t)$ is not in RRA (t is too big)
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The wRRA case

- wRRA = weakly representable relation algebras
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Key facts about the class of $\mathfrak{L}(p,t)$ algebras

- larger algebras require more generators
- proper subalgebras occur in larger algebras
- $p < 2t$ implies $\mathfrak{L}(p,t) \notin$ RRA
- prime power p implies $\mathfrak{L}(p,t) \in$ wRRA (on finite sets!)
  - prime power p, hence $\mathfrak{L}(p,0)$ has a (unique) representation Q on p^2 points
  - the kth direct power of Q is a weak representation of $\mathfrak{L}(p,0)$ on $(p^2)^k$ points (key observation!)
  - take two copies of a big$_t$ direct power of Q
  - randomly assign edges between the copies to the t new atoms
  - non-zero probability of a weak representation on $2(p^2)^k$ points

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$\mathfrak{L}(3,0)$ is the Lyndon algebra of a 4-point line

	1’	a	b	c	d
1’	1’	a	b	c	d
a	a	$1' + a$	$c + d$	$b + d$	$b + c$
b	b	$c + d$	$1' + b$	$a + d$	$a + c$
c	c	$b + d$	$a + d$	$1' + c$	$a + b$
d	d	$b + c$	$a + c$	$a + b$	$1' + d$

	1’	a	b	c	d
1’	1’	a	b	c	d
a	a	$1'a$	cd	bd	bc
b	b	cd	$1'b$	ad	ac
c	c	bd	ad	$1'c$	ab
d	d	bc	ac	ab	$1'd$

Tables for $\mathfrak{L}(3,1) \in \mathsf{RRA}$

$\mathfrak{L}(3,1)$ is the Lyndon algebra of a 4-point line, plus one atom r
$\mathfrak{L}(3,1)$ is representable on 18 points

	1’	a	b	c	d	r
1’	1’	a	b	c	d	r
a	a	1’ $+ a$	$c + d$	$b + d$	$b + c$	r
b	b	$c + d$	1’ $+ b$	$a + d$	$a + c$	r
c	c	$b + d$	$a + d$	1’ $+ c$	$a + b$	r
d	d	$b + c$	$a + c$	$a + b$	1’ $+ d$	r
r	r	r	r	r	r	1’ $+ a + b + c + d$

	1’	a	b	c	d	r
1’	1’	a	b	c	d	r
a	a	1’a	cd	bd	bc	r
b	b	cd	1’b	ad	ac	r
c	c	bd	ad	1’c	ab	r
d	d	bc	ac	ab	1’d	r
r	r	r	r	r	r	1’$abcd$

Tables for $\mathfrak{L}(3,2) \in$ wRRA $\sim$ RRA

$\mathfrak{L}(3,2)$ is the Lyndon algebra of a 4-point line, plus atoms r,s
Is $\mathfrak{L}(3,2)$ weakly representable on $2 \cdot 9^2 = 162$ points? (unlikely)
on $2 \cdot 9^3 = 1458$? (maybe) on $2 \cdot 9^4 = 13122$? (probably)

	1'	a	b	c	d	r	s
1'	1'	a	b	c	d	r	s
a	a	1' $+ a$	$c+d$	$b+d$	$b+c$	$r+s$	$r+s$
b	b	$c+d$	1' $+ b$	$a+d$	$a+c$	$r+s$	$r+s$
c	c	$b+d$	$a+d$	1' $+ c$	$a+b$	$r+s$	$r+s$
d	d	$b+c$	$a+c$	$a+b$	1' $+ d$	$r+s$	$r+s$
r	r	$r+s$	$r+s$	$r+s$	$r+s$	1' $+ a+b+c+d$	$a+b+c+d$
s	r	$r+s$	$r+s$	$r+s$	$r+s$	$a+b+c+d$	1' $+ a+b+c+d$

	1'	a	b	c	d	r	s
1'	1'	a	b	c	d	r	s
a	a	1'a	cd	bd	bc	rs	rs
b	b	cd	1'b	ad	ac	rs	rs
c	c	bd	ad	1'c	ab	rs	rs
d	d	bc	ac	ab	1'd	rs	rs
r	r	rs	rs	rs	rs	1'$abcd$	$abcd$
s	r	rs	rs	rs	rs	$abcd$	1'$abcd$

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