Separability for lattice-ordered Abelian groups and MV-algebras: a characterisation theorem

Vincenzo Marra
vincenzo.marra@unimi.it

Dipartimento di Matematica *Federigo Enriques*
Università degli Studi di Milano
Italy

Joint work with Matías Menni, CONICET, Argentina

AMS Fall Western Sectional Meeting — Special session on Algebraic Logic
University of Denver, Colorado, USA

October 8–9, 2016

Commutative algebra: Given field k, a k-algebra A is separable if for every field extension $K \supseteq k$, the algebra $A \otimes_k K$ is semisimple (=has trivial radical).

Generalisation to categories: A. Carboni, G. Janelidze, S. Lack, W. Lawvere, S. Schanuel, R. Walters et al.

Preliminary question: for which categories C can we think of C^{op} as a category of spaces?

Commutative algebra: Given field k, a k-algebra A is separable if for every field extension $K \supseteq k$, the algebra $A \otimes_k K$ is semisimple (=has trivial radical).

Generalisation to categories: A. Carboni, G. Janelidze, S. Lack, W. Lawvere, S. Schanuel, R. Walters et al.

Preliminary question: for which categories C can we think of C^{op} as a category of spaces?

> A space is just an object in the category of spaces!
>
> F. W. Lawvere, *What is a space?*,
> Seminario di Matematica, 18 giugno 2010.
> Scuola Normale Superiore Pisa

Commutative algebra: Given field k, a k-algebra A is separable if for every field extension $K \supseteq k$, the algebra $A \otimes_k K$ is semisimple (=has trivial radical).

Generalisation to categories: A. Carboni, G. Janelidze, S. Lack, W. Lawvere, S. Schanuel, R. Walters et al.

Preliminary question: for which categories C can we think of C^{op} as a category of spaces?

> A space is just an object in the category of spaces! The implied further question is given a very general answer involving **lextensivity**, as well as a much more structured answer involving [. . .] toposes. [. . .]
>
> F. W. Lawvere, *What is a space?*,
> Seminario di Matematica, 18 giugno 2010.
> Scuola Normale Superiore Pisa

Commutative algebra: Given field k, a k-algebra A is separable if for every field extension $K \supseteq k$, the algebra $A \otimes_k K$ is semisimple (=has trivial radical).

Generalisation to categories: A. Carboni, G. Janelidze, S. Lack, W. Lawvere, S. Schanuel, R. Walters et al.

Preliminary question: for which categories C can we think of C^{op} as a category of spaces?

Minimal requirement: C must be **co-extensive**, or equivalently, C must be **extensive** (=has well-behaved sums, see below).

Blanket assumption. “Lextensive” means “Left exact (=with finite limits) and extensive”. From now on C is a variety, so C is complete and co-complete. Far too strong an assumption for the general theory, but convenient here.

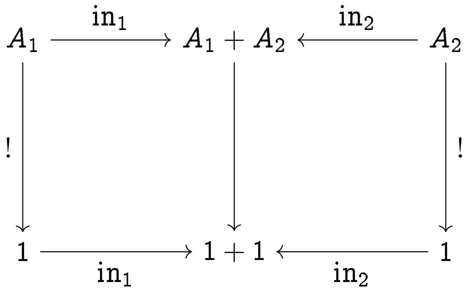
Definition (Extensive category)

The category C^{op} is **extensive** if for each pair of objects A_1, A_2 the commutative diagram below comprises a pair of pullback squares.

$$
\begin{array}{ccccc}
A_1 & \xrightarrow{\mathrm{in}_1} & A_1 + A_2 & \xleftarrow{\mathrm{in}_2} & A_2 \\
\downarrow{!} & & \downarrow & & \downarrow{!} \\
1 & \xrightarrow[\mathrm{in}_1]{} & 1 + 1 & \xleftarrow[\mathrm{in}_2]{} & 1
\end{array}
$$

Definition (Extensive category)

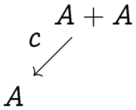
The category C^{op} is **extensive** if for each pair of objects A_1, A_2 the commutative diagram below comprises a pair of pullback squares.



Examples (Picture!). The categories of topological spaces, of posets, and of Priestley spaces are extensive. The opposite of the category of finitely generated k-algebras (=affine schemes) is extensive. The opposite of the category of groups is not extensive.

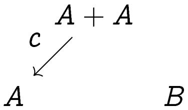
Definition (Separable algebra)

An object A in C is **separable** if there exists a morphism $b\colon A + A \to B$ such that the morphism $i\colon A + A \to A \times B$ induced by the co-diagonal map $c\colon A + A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



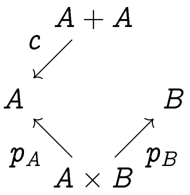
Definition (Separable algebra)

An object A in C is separable if there exists a morphism $b\colon A + A \to B$ such that the morphism $i\colon A + A \to A \times B$ induced by the co-diagonal map $c\colon A + A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



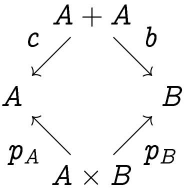
Definition (Separable algebra)

An object A in C is **separable** if there exists a morphism $b\colon A + A \to B$ such that the morphism $i\colon A + A \to A \times B$ induced by the co-diagonal map $c\colon A + A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



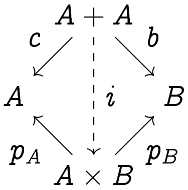
Definition (Separable algebra)

An object A in C is **separable** if there exists a morphism $b\colon A+A \to B$ such that the morphism $i\colon A+A \to A \times B$ induced by the co-diagonal map $c\colon A+A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



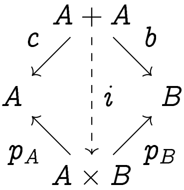
Definition (Separable algebra)

An object A in C is **separable** if there exists a morphism $b\colon A + A \to B$ such that the morphism $i\colon A + A \to A \times B$ induced by the co-diagonal map $c\colon A + A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



Definition (Separable algebra)

An object A in C is **separable** if there exists a morphism $b\colon A + A \to B$ such that the morphism $i\colon A + A \to A \times B$ induced by the co-diagonal map $c\colon A + A \to A$, by the projections p_A, p_B of the product $A \times B$, and by b, is an isomorphism.



Decidability

The formally opposite property is called **decidability** (from topos theory).

Example: Decidability in KHaus (or in Top).

Consider the Stone-Yosida-Gelfand duality between vector lattices (or rings) of continuous functions $\mathrm{C}(X)$, X compact Hausdorff, and compact Hausdorff spaces.

Trivially (definition!):

$$\mathrm{C}(X) \text{ is separable} \iff X \text{ is decidable.}$$

Example: Decidability in KHaus (or in Top).

Consider the Stone-Yosida-Gelfand duality between vector lattices (or rings) of continuous functions $\mathrm{C}(X)$, X compact Hausdorff, and compact Hausdorff spaces.

Trivially (definition!):

$$\mathrm{C}\,(X) \text{ is separable} \iff X \text{ is decidable.}$$

Easy observation

A space KHaus *is decidable precisely when it is finite and discrete.*

Proof: Picture!

Hence, TFAE:

1. $\mathrm{C}\,(X)$ is separable.
2. $\mathrm{C}\,(X)$ is a finite product of copies of $\mathbb{R}$.

Theorem (VM, M. Menni, 2016)

For any MV-algebra A, the following are equivalent.

1. *A is separable.*
2. *A is a finite product of subalgebras of $[0,1] \cap \mathbb{Q}$.*

Remark

For Abelian ℓ-groups with unit: (G, u) is separable if, and only if, (G, u) is a finite direct product of subgroups of $(\mathbb{Q}, 1)$.

First deal with $1 \Rightarrow 2$: *separable* $\Rightarrow$ *finite product of subalgebras of* $[0,1] \cap \mathbb{Q}$. Key steps in the proof. If A is **separable**, then:

1. Use spectral functor Max to show A must be finite product of local algebras (=with precisely maximal ideal).

First deal with $1 \Rightarrow 2$: *separable* $\Rightarrow$ *finite product of subalgebras of* $[0,1] \cap \mathbb{Q}$. Key steps in the proof. If A is **separable**, then:

1. Use spectral functor Max to show A must be finite product of local algebras (=with precisely maximal ideal).
2. Use general theory of extensive categories to show it suffices to characterise separable local algebras. If A is **separable and local**, then:

First deal with $1 \Rightarrow 2$: *separable* $\Rightarrow$ *finite product of subalgebras of* $[0,1] \cap \mathbb{Q}$. Key steps in the proof. If A is **separable**, then:

1. Use spectral functor Max to show A must be finite product of local algebras (=with precisely maximal ideal).
2. Use general theory of extensive categories to show it suffices to characterise separable local algebras. If A is **separable and local**, then:
3. Use behaviour of non-trivial radical under co-products to show A must have trivial radical, and hence be simple.

First deal with $1 \Rightarrow 2$: *separable* $\Rightarrow$ *finite product of subalgebras of* $[0,1] \cap \mathbb{Q}$. Key steps in the proof. If A is **separable**, then:

1. Use spectral functor Max to show A must be finite product of local algebras (=with precisely maximal ideal).
2. Use general theory of extensive categories to show it suffices to characterise separable local algebras. If A is **separable and local**, then:
3. Use behaviour of non-trivial radical under co-products to show A must have trivial radical, and hence be simple.
4. Use Hölder's Theorem to identify simple algebras with subalgebras of $[0,1]$.

First deal with $1 \Rightarrow 2$: *separable* $\Rightarrow$ *finite product of subalgebras of* $[0,1] \cap \mathbb{Q}$. Key steps in the proof. If A is **separable**, then:

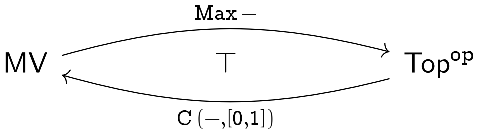
1. Use spectral functor Max to show A must be finite product of local algebras (=with precisely maximal ideal).
2. Use general theory of extensive categories to show it suffices to characterise separable local algebras. If A is **separable and local**, then:
3. Use behaviour of non-trivial radical under co-products to show A must have trivial radical, and hence be simple.
4. Use Hölder's Theorem to identify simple algebras with subalgebras of $[0,1]$.
5. Use behaviour of simple algebras with irrational elements under co-products to show A must be subalgebra of $[0,1] \cap \mathbb{Q}$.

The converse implication $2 \Rightarrow 1$: *finite product of subalgebras of* $[0,1] \cap \mathbb{Q} \Rightarrow$ *separable* amounts to a (non-trivial) co-product computation.

Step 1: The Max functor

Basic spectral adjunction:

$$\mathrm{C} \dashv \mathrm{Max}$$



Step 1: The Max functor

Basic spectral adjunction:

$$\mathrm{C} \dashv \operatorname{Max}$$

$$\mathsf{MV} \underset{\mathrm{C}\,(-,[0,1])}{\overset{\operatorname{Max} -}{\underset{\top}{\rightleftarrows}}} \mathsf{Top}^{\mathsf{op}}$$

Max preserves all existing limits, and in particular products. That is,

$$\operatorname{Max} \prod_{i \in I} A_i = \sum_{i \in I} \operatorname{Max} A_i .$$

Basic spectral adjunction:

$$\mathrm{C} \dashv \mathrm{Max}$$

In analogy with spectra of commutative rings, we can also prove:

Lemma

Max *preserves finite co-products. That is,*

$$\mathrm{Max}\, A_1 + \cdots + A_n = \mathrm{Max}\, A_1 \times \cdots \times \mathrm{Max}\, A_n.$$

For example, $\mathrm{Max}\,\mathrm{F}(1) = [0,1]$, so that

$$\mathrm{Max}\,\mathrm{F}(n) = [0,1]^n.$$

By the preservation properties of the Max functor above, we obtain:

Lemma

If A is separable then Max A *is a decidable space (in* Top, *or in* KHaus*).*

By the preservation properties of the Max functor above, we obtain:

Lemma

If A is separable then Max A *is a decidable space (in* Top*, or in* KHaus*).*

Next note: by definition, A is local $\Leftrightarrow$ Max A is a point.

Easy observation

A space in KHaus *is decidable precisely when it is finite and discrete.*

By the preservation properties of the Max functor above, we obtain:

Lemma

If A is separable then $\mathrm{Max}\, A$ *is a decidable space (in* Top*, or in* KHaus*).*

Next note: by definition, A is local $\Leftrightarrow$ $\mathrm{Max}\, A$ is a point.

Easy observation

A space in KHaus *is decidable precisely when it is finite and discrete.*

Corollary

If A is separable, it is a finite product of local algebras.

Hence we can consider $A = A_1 \times \cdots \times A_n$ separable with each A_i local.

Step 2: Extensivity

Lemma (A. Carboni and G. Janelidze, 1996)

In any co-extensive variety, for any finite family of objects $A_1, \ldots, A_n$ the following are equivalent.

1. $A_1 \times \cdots \times A_n$ *is separable.*
2. *Each* A_i *is separable,* $i = 1, \ldots, n$.

Step 2: Extensivity

Lemma (A. Carboni and G. Janelidze, 1996)

In any co-extensive variety, for any finite family of objects $A_1, \ldots, A_n$ the following are equivalent.

1. $A_1 \times \cdots \times A_n$ *is separable.*
2. *Each* A_i *is separable,* $i = 1, \ldots, n$.

Relatively easy lemma

The category of MV-algebras is co-extensive.

(Holds for the same reason that rings are co-extensive: direct product splittings are induced by idempotents. For MV, idempotents are known as Boolean elements. In the literature on ℓ-groups with unit, idempotents are known as components of the unit.)

Step 2: Extensivity

Lemma (A. Carboni and G. Janelidze, 1996)

In any co-extensive variety, for any finite family of objects $A_1, \ldots, A_n$ the following are equivalent.

1. $A_1 \times \cdots \times A_n$ *is separable.*
2. *Each* A_i *is separable,* $i = 1, \ldots, n$.

Relatively easy lemma

The category of MV-algebras is co-extensive.

(Holds for the same reason that rings are co-extensive: direct product splittings are induced by idempotents. For MV, idempotents are known as Boolean elements. In the literature on ℓ-groups with unit, idempotents are known as components of the unit.)

Now it is enough to identify which local algebras are separable!

Step 3: From local to simple

Intuition. If A is local and has non-trivial radical, its dual spectral space (primes included!) is a point equipped with (at least one) infinitesimal displacement.

Step 3: From local to simple

Intuition. If A is local and has non-trivial radical, its dual spectral space (primes included!) is a point equipped with (at least one) infinitesimal displacement.

Lemma

If A is separable and local, $\operatorname{Rad} A = \{0\}$ — *hence A is simple.*

Proof: Picture!

Step 4: From simple to real

Theorem (O. Hölder, 1901)

If A is a non-trivial, local MV-algebra, there is exactly one homomorphism

$$A \xrightarrow{h_A} [0,1].$$

Furthermore, h_A is injective if, and only if, A is simple.

In particular, simple MV-algebras can be identified **in one and only one way** with subalgebras of $[0,1]$. So, for example, it makes perfect sense to say that an element $a \in A$ of a simple MV-algebra is rational, irrational, transcendental etc.

Step 5: From real to rational

Intuition. If $A \subseteq [0,1]$ contains an irrational number, its dual space is an ordinary point p with no infinitesimal displacements. However, this point should be thought of as "irrational", and different in nature from the dual of a rational subalgebra of $[0,1]$. Indeed, due to its "irrationality", the co-product $A+A$ is **no longer simple**! That is, the product $p \times p$ is again a "point with infinitesimal displacements"! Then A can't be separable. This phenomenon is the analogue for MV/ℓ-groups of ramification in algebraic geometry and Galois theory. It doesn't happen if A is rational (converse implication!).

Step 5: From real to rational

Intuition. If $A \subseteq [0,1]$ contains an irrational number, its dual space is an ordinary point p with no infinitesimal displacements. However, this point should be thought of as "irrational", and different in nature from the dual of a rational subalgebra of $[0,1]$. Indeed, due to its "irrationality", the co-product $A + A$ is **no longer simple**! That is, the product $p \times p$ is again a "point with infinitesimal displacements"! Then A can't be separable. This phenomenon is the analogue for MV/ℓ-groups of ramification in algebraic geometry and Galois theory. It doesn't happen if A is rational (converse implication!).

Lemma

If $A \subseteq [0,1]$ is separable, then $A \subseteq [0,1] \cap \mathbb{Q}$.

Proof: Picture!

Epilogue: Motivation

Theorem (VM, M. Menni, 2016)

For any MV-algebra A, the following are equivalent.

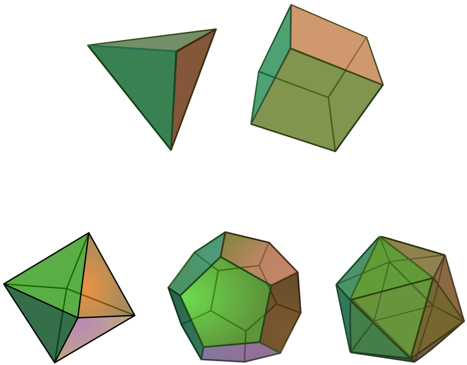
1. *A is separable.*
2. *A is a finite product of subalgebras of $[0, 1] \cap \mathbb{Q}$.*

Remark

For Abelian ℓ-groups with unit: (G, u) is separable if, and only if, (G, u) is a finite direct product of subgroups of $(\mathbb{Q}, 1)$.

In particular:

- Simple separable MV-algebras are precisely the subalgebras of $[0, 1] \cap \mathbb{Q}$, i.e. the extensions of $\{0, 1\}$ by rational numbers.
- Simple separable Abelian ℓ-groups with unit are precisely the unital ℓ-sugroups $(\mathbb{Q}, 1)$, i.e. the extensions of $(\mathbb{Z}, 1)$ by rational numbers.



The Five Platonic Solids

(*In Plato's Timaeus, ca. 350 B.C., after his friend mathematician Theaetetus.*)

Thank you for your attention.