Partial right orders on free groups and the word problem for free ℓ-groups

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Two Decision Problems

1. Does a given finitely generated partial right order on a free group extend to a total right order?

2. Is a given equation valid in all lattice-ordered groups?

Partial Right Orders

A **partial right order** on a group G is a partial order $\leq$ satisfying

$$a \leq b \implies ac \leq bc.$$

Its positive cone $\{a \in G : a > \mathrm{e}\}$ is a subsemigroup of G omitting e. Conversely, any such subsemigroup P produces a partial right order

$$a \leq b \iff ba^{-1} \in P \cup \{\mathrm{e}\}.$$

So we can identify partial right orders with subsemigroups omitting e.

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Right Orders

A partial right order P on a group G is a (total) **right order** on G if

$$P \cup P^{-1} \cup \{e\} = G.$$

Question: When does a partial right order extend to a right order?

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Extending Partial Right Orders

Let $\langle S \rangle$ denote the subsemigroup generated by a subset S of G.

Theorem (Kopytov and Medvedev 1994)

The following are equivalent for any partial right order P on a group G:

(1) *P extends to a right order of G.*

(2) *For all $a_1, \ldots, a_n \in G \setminus \{e\}$, there exist $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ so that*

$$e \notin \langle \{a_1^{\delta_1}, \ldots, a_n^{\delta_n}\} \cup P \rangle.$$

Problem: Can we *check* if a partial right order extends to a right order?

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Extending Partial Right Orders on Free Groups

Let F denote a **finitely generated free group** of rank at least 2, writing t for both an element of F and the reduced term of length $|t|$, and let F_N denote the set of all elements of F of length $\leq N$.

We call $S \subseteq F$ an **N-truncated right order** on F if $S = \langle S \rangle \cap F_N$, $e \notin S$, and $S \cup S^{-1} \cup \{e\} = F_N$.

Theorem (Smith 2005, Clay and Smith 2009)

The following are equivalent for any finite subset S of F:

(1) *S extends to a right order of F.*

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A Decidability Result

Corollary

Checking if a given finitely generated partial right order of a free group extends to a right order is decidable.

An Example

Consider the partial right order $\langle S_1 \rangle$ on the 2-generated free group for

$$S_1 = \{x^2, xy, yx^{-1}\}.$$

We add all products in F_2 of members of S_1, producing

$$S_2 = \{x^2, xy, yx^{-1}, yx, y^2\}.$$

We choose a sign δ for each of the remaining members of F_2 to obtain

$$S_3 = \{x^2, xy, yx^{-1}, yx, y^2, x, y, x^{-1}y\}.$$

Then S_3 is a 2-truncated right order on F and, using the previous theorem, the partial right order $\langle S_1 \rangle$ extends to a right order of F.

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Lattice-Ordered Groups

A **lattice-ordered group** (or ℓ**-group**) is an algebraic structure

$$\langle L, \wedge, \vee, \cdot, ^{-1}, \mathrm{e}\rangle$$

satisfying the following conditions:

- $\langle L, \wedge, \vee\rangle$ is a lattice
- $\langle L, \cdot, ^{-1}, \mathrm{e}\,\rangle$ is a group
- $a(b \vee c)d = abd \vee acd$ for all $a, b, c, d \in L$.

It follows also that L is distributive and satisfies $\mathrm{e} \leq a \vee a^{-1}$.

Automorphism ℓ-Groups

The order-preserving bijections on a chain Ω with function composition and inverse form a group $Aut(\Omega)$ lattice-ordered by

$$f \leq g \qquad \Longleftrightarrow \qquad f(a) \leq g(a) \text{ for all } a \in \Omega.$$

Theorem (Holland 1963)

Every ℓ-group embeds into $Aut(\Omega)$ for some chain Ω.

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The variety $\mathcal{LG}$ of ℓ-groups is generated by $Aut(\mathbb{R})$.

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Rewriting Equations

For any term t, there exist I, J_i ($i \in I$) and group terms t_{ij} such that

$$\mathcal{LG} \models t \approx \bigwedge_{i \in I} \bigvee_{j \in J_i} t_{ij}.$$

It follows easily that for checking the validity of ℓ-group equations, it suffices to be able to check for group terms $t_1, \ldots, t_n$ whether or not

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A First Lemma

Let $t_1, \ldots, t_n$ be members (terms) of a finitely generated free group F.

Lemma

If $\{t_1, \ldots, t_n\}$ extends to a right order on F, then $\mathcal{LG} \not\models e \leq t_1 \vee \ldots \vee t_n$.

Proof.

Suppose that $\{t_1, \ldots, t_n\}$ extends to a right order on F. Then we obtain also a right order of F where $t_1, \ldots, t_n$ are negative. Consider the ℓ-group $Aut(F)$ and evaluate each variable x by the map $s \mapsto sx$. Then each t_i maps e to $t_i <$ e, and $t_1 \vee \ldots \vee t_n$ maps e to some $t_j <$ e. So e $\not\leq t_1 \vee \ldots \vee t_n$ in $Aut(F)$, and $\mathcal{LG} \not\models e \leq t_1 \vee \ldots \vee t_n$. □

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If $\mathcal{LG} \not\models \mathrm{e} \leq t_1 \vee \ldots \vee t_n$*, then* $\{t_1, \ldots, t_n\}$ *extends to a right order on* F*.*

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Suppose that $\{t_1, \ldots, t_n\}$ does not extend to a right order on F. Then there exist $s_1, \ldots, s_m \in F \setminus \{\mathrm{e}\}$ such that for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$

$$\mathrm{e} \in \langle \{t_1, \ldots, t_n, s_1^{\delta_1}, \ldots, s_m^{\delta_m}\} \rangle.$$

We prove $\mathcal{LG} \models \mathrm{e} \leq t_1 \vee \ldots \vee t_n$ by induction on m.

Base case: We have $\mathrm{e} \in \langle \{t_1, \ldots, t_n\} \rangle$ and the result follows easily.

Inductive step: We use the (non-trivial) fact that for any $s \in F \setminus \{\mathrm{e}\}$ and join t of elements from F,

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Suppose that $\{t_1, \ldots, t_n\}$ does not extend to a right order on F. Then there exist $s_1, \ldots, s_m \in F \setminus \{\mathrm{e}\}$ such that for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$

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We prove $\mathcal{LG} \models \mathrm{e} \leq t_1 \vee \ldots \vee t_n$ by induction on m.

Base case: We have $\mathrm{e} \in \langle \{t_1, \ldots, t_n\} \rangle$ and the result follows easily.

Inductive step: We use the (non-trivial) fact that for any $s \in F \setminus \{\mathrm{e}\}$ and join t of elements from F,

$$\mathcal{LG} \models \mathrm{e} \leq t \vee s \quad \text{and} \quad \mathcal{LG} \models \mathrm{e} \leq t \vee s^{-1} \quad \Longrightarrow \quad \mathcal{LG} \models \mathrm{e} \leq t. \qquad \square$$

A Second Lemma

Lemma

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Combining the Lemmas

Theorem

Exactly one of the following holds:

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Checking the validity of an equation in all ℓ*-groups is decidable.*

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The Holland-McCleary Theorem Revisited

Given $S \subseteq F$, let $\mathcal{I}(S)$ denote the set of all $s^{-1}t \in F$ such that s and t are **initial subterms** of terms in S.

Theorem (Holland and McCleary 1979)

The following are equivalent:

(1) $\mathcal{LG} \models e \leq t_1 \vee \ldots \vee t_n$.

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Right Partial Orders Again

Fix some $T \subseteq F$. Then we call $S \subseteq F$ a T**-truncated right order** on F if $S = \langle S \rangle \cap T$, $e \notin S$, and $S \cup S^{-1} \cup \{e\} = T$.

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The following are equivalent for any finite subset $S \subseteq F$:

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Concluding Remarks

- Extending partial right orders to right orders on free groups can also be used to develop a proof theory for ℓ-groups.

  Proof Theory for Lattice-Ordered Groups. N. Galatos and G. Metcalfe. *Annals of Pure and Applied Logic* 167 (2016), 707–724.

- Checking validity of equations in ℓ-groups is co-NP-complete; the extending partial right orders on free groups problem is in NP.
- Extending partial bi-orders on free groups to total bi-orders corresponds to validity in representable ℓ-groups (or o-groups).
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