Linear logic properly displayed

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The wider picture

Multi-type algebraic proof theory

- constructive canonical extensions — algebra, formal topology
- unified correspondence theory — duality
- proper display calculi — structural proof theory

Proof calculi with a uniform metatheory:

- supporting an **inferential theory of meaning**
- canonical **cut elimination** and **subformula property**
- **soundness**, **completeness**, **conservativity**

Range

- DEL, PDL, Logic of Resources and Capabilities...
- normal DLEs and their analytic inductive axiomatic extensions
- Inquisitive logic
- Linear logic
- Lattice logic
- basic LEs and their analytic inductive axiomatic extensions

Starting point: Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\begin{array}{c} Y \vdash X > Z \\ \hline X\, ;\, Y \vdash Z \\ \hline Y\, ;\, X \vdash Z \\ \hline X \vdash Y > Z \end{array}$$

  display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

Cut elimination metatheorem (Belnap 82, Wansing 98)

Theorem

Cut elimination and subformula property hold for any **proper display calculus**.

Definition

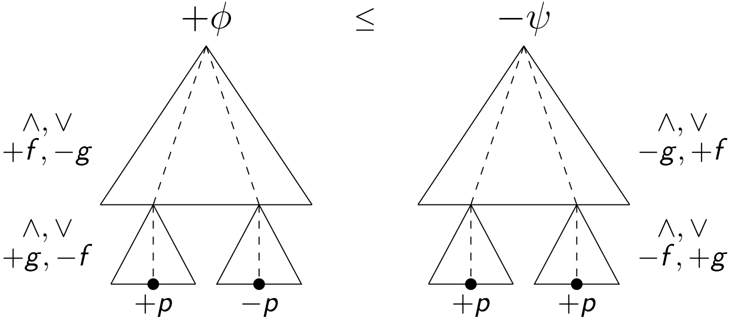
A **proper display calculus** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation:
    - same shape, same position, non-proliferation;
3. **principal** = **displayed**
4. rules are closed under **uniform substitution** of congruent parameters (Properness!);
5. **reduction strategy** exists when both cut formulas are principal.

Which logics are properly displayable?

Complete characterization (Ciabattoni et al. 15, Greco et al. 16):

1. the logics of any **basic** normal DLE;
2. axiomatic extensions of these with **analytic inductive inequalities**: ⇝ unified correspondence



Analytic inductive ⇒ Inductive ⇒ Canonical

Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheoem + ALBA-technology.

For many... but not for all.

- The characterization theorem sets **hard boundaries** to the scope of proper display calculi.
- Interesting logics are **left out**.

Can we **extend the scope** of proper display calculi?

Yes: proper display calculi $\rightsquigarrow$ proper **multi-type** calculi

The case of Linear Logic

(Belnap 92): **not** a **proper** display calculus:

$$\frac{Y \vdash A}{Y \vdash\, !A} \qquad \frac{A \vdash X}{!A \vdash X}$$

$$\frac{X \vdash A}{X \vdash\, ?A} \qquad \frac{A \vdash Z}{?A \vdash Z}$$

Y and Z not arbitrary but *exponentially restricted*.

$!!A = !A$

$!A \leq A$

$A \vdash B$ implies $!A \vdash !B$

$!\top = 1$

$!(A \& B) = !A \otimes !B$ analytic?

Related case: Lattice Logic

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B} \qquad \frac{A \vdash X}{A \wedge B \vdash X} \qquad \frac{B \vdash X}{A \wedge B \vdash X}$$

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X} \qquad \frac{X \vdash A}{X \vdash A \vee B} \qquad \frac{X \vdash A}{X \vdash A \vee B}$$

In general lattices, $\wedge$ and $\vee$ are adjoins but not residuals.
Belnap's approach: no structural counterparts.
Hence: no structural rules capturing interaction between $\wedge$ and $\vee$ and other connectives...

Linear logic: algebraic analysis

$$\begin{array}{ll} !!a = !a & !\top = 1 \\ !a \leq a & !(a \& b) = !a \otimes !b \\ a \leq b \text{ implies } !a \leq !b & \end{array}$$

$! : \mathbb{L} \to \mathbb{L}$ interior operator. Then $! = e \circ \iota$, where

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook, "e"'] & \mathbb{L} \arrow[l, bend right=50, "\iota"'] \arrow[l, phantom, bend right=25, "\top"] \end{tikzcd}$$

Fact: Range(!) $:= \mathbb{A}$ has natural BA/HA-structure.

Upshot: natural semantics for the following **multi-type** language:

$$\text{Kernel} \ni \alpha ::= \iota A \mid \mathtt{t} \mid \mathtt{f} \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

$$\text{Linear} \ni A ::= p \mid e\alpha \mid 1 \mid \bot \mid A \otimes A \mid A \parr A \mid A \multimap A \mid$$
$$\top \mid 0 \mid A \,\&\, A \mid A \oplus A$$

Reverse-engineering linear logic - Part 1

$$\frac{E\alpha \vdash X}{e\alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{E\Gamma \vdash e\alpha} \qquad \frac{\Gamma \vdash IA}{\Gamma \vdash \iota A} \qquad \frac{A \vdash X}{\iota A \vdash IX}$$

$$\begin{array}{c} \Gamma \vdash IY \\ \hline\hline E\Gamma \vdash Y \end{array}$$

Interior operator axioms/rule recaptured:

$$\begin{array}{c} A \vdash A \\ \hline \iota A \vdash IA \\ \hline E\iota A \vdash A \\ \hline e\iota A \vdash A \\ \hline\hline !A \vdash A \end{array} \qquad \begin{array}{c} A \vdash A \\ \hline \iota A \vdash IA \\ \hline \iota A \vdash \iota A \\ \hline E\iota A \vdash e\iota A \\ \hline \iota A \vdash Ie\iota A \\ \hline \iota A \vdash \iota e\iota A \\ \hline E\iota A \vdash e\iota e\iota A \\ \hline e\iota A \vdash e\iota e\iota A \\ \hline\hline !A \vdash !!A \end{array} \qquad \begin{array}{c} A \vdash B \\ \hline \iota A \vdash IB \\ \hline \iota A \vdash \iota B \\ \hline E\iota A \vdash e\iota B \\ \hline e\iota A \vdash e\iota B \\ \hline\hline !A \vdash !B \end{array}$$

Reverse-engineering linear logic - Part 2

Problem: the following axiom is **non-analytic**.

$$!(A \,\&\, B) = !A \otimes !B \quad \rightsquigarrow \quad e\iota(A \,\&\, B) = e\iota A \otimes e\iota B$$

Solution: ι surjective and finitely meet-preserving $\Rightarrow$ axioms above semantically equivalent to the following analytic identity:

$$e(\alpha \wedge \beta) = e\alpha \otimes e\beta$$

corresponding to the following analytic rules:

$$\frac{E(\Gamma, \Delta) \vdash X}{E\Gamma ; E\Delta \vdash X} \text{ reg/co-reg}$$

Deriving $!(A \,\&\, B) = !A \otimes !B$

$$\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\iota A \vdash IA$}
\UnaryInfC{$E\iota A \vdash A$}
\LeftLabel{W_m}
\UnaryInfC{$E\iota A \,;\, E\iota B \vdash A$}
\AxiomC{$B \vdash B$}
\UnaryInfC{$\iota B \vdash IB$}
\UnaryInfC{$E\iota B \vdash B$}
\LeftLabel{W_m}
\UnaryInfC{$E\iota A \,;\, E\iota B \vdash B$}
\BinaryInfC{$(E\iota A \,;\, E\iota B) \cdot (E\iota A \,;\, E\iota B) \vdash A \,\&\, B$}
\LeftLabel{C_A}
\UnaryInfC{$E\iota A \,;\, E\iota B \vdash A \,\&\, B$}
\LeftLabel{reg}
\UnaryInfC{$E(\iota A, \iota B) \vdash A \,\&\, B$}
\UnaryInfC{$\iota A, \iota B \vdash I(A \,\&\, B)$}
\UnaryInfC{$\iota A, \iota B \vdash \iota(A \,\&\, B)$}
\UnaryInfC{$E(\iota A, \iota B) \vdash e\iota(A \,\&\, B)$}
\LeftLabel{co-reg}
\UnaryInfC{$E\iota A \,;\, E\iota B \vdash e\iota(A \,\&\, B)$}
\dashedLine
\UnaryInfC{$e\iota A \otimes e\iota B \vdash e\iota(A \,\&\, B)$}
\doubleLine
\UnaryInfC{$!A \otimes !B \vdash !(A \,\&\, B)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$A \vdash A$}
\LeftLabel{W_a}
\UnaryInfC{$A \cdot B \vdash A$}
\UnaryInfC{$A \,\&\, B \vdash A$}
\UnaryInfC{$\iota(A \,\&\, B) \vdash IA$}
\UnaryInfC{$\iota(A \,\&\, B) \vdash \iota A$}
\UnaryInfC{$E\iota(A \,\&\, B) \vdash e\iota A$}
\AxiomC{$B \vdash B$}
\LeftLabel{W_a}
\UnaryInfC{$A \cdot B \vdash B$}
\UnaryInfC{$A \,\&\, B \vdash B$}
\UnaryInfC{$\iota(A \,\&\, B) \vdash IB$}
\UnaryInfC{$\iota(A \,\&\, B) \vdash \iota B$}
\UnaryInfC{$E\iota(A \,\&\, B) \vdash e\iota B$}
\BinaryInfC{$E\iota(A \,\&\, B) \,;\, E\iota(A \,\&\, B) \vdash e\iota A \otimes e\iota B$}
\LeftLabel{reg}
\UnaryInfC{$E(\iota(A \,\&\, B), \iota(A \,\&\, B)) \vdash e\iota A \otimes e\iota B$}
\UnaryInfC{$\iota(A \,\&\, B), \iota(A \,\&\, B) \vdash I(e\iota A \otimes e\iota B)$}
\LeftLabel{C_K}
\UnaryInfC{$\iota(A \,\&\, B) \vdash I(e\iota A \otimes e\iota B)$}
\UnaryInfC{$E\iota(A \,\&\, B) \vdash e\iota A \otimes e\iota B$}
\UnaryInfC{$e\iota(A \,\&\, B) \vdash e\iota A \otimes e\iota B$}
\doubleLine
\UnaryInfC{$!(A \,\&\, B) \vdash !A \otimes !B$}
\end{prooftree}$$

Conclusions

Proper display calculi $\rightsquigarrow$ Proper **multi-type** calculi

- The same order-theoretic principles underlying **Sahlqvist**-type correspondence and canonicity also underlie the metatheory of proper **multi-type** calculi;
- **Uniform route** to soundness, completeness, cut-elimination, subformula property, conservativity;
- **scope** of proper display calculi **enlarged** (linear logic as a case study);
- **multi-type algebraic proof theory**: from substructural logics to the logics for social behaviour.

Next developments:

Logics, Decisions, and Interactions
Lorentz Center, Leiden 24-28 October 2016

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