Undecidability for certain subvarieties of commutative residuated lattices.

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Outline

Residuated Lattices

Definition

A (commutative) **residuated lattice** is a structure $\mathbf{R} = \{R, \cdot, \vee, \wedge, \backslash, /, 1\}$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y,$$

  where $\leq$ is the lattice order.

We denote the variety of (commutative) residuated lattices by $(\mathcal{CRL})$ $\mathcal{RL}$.
If (r) is a a rule (axiom), then $(\mathcal{C})\mathcal{RL}_{\mathrm{r}} := (\mathcal{C})\mathcal{RL} + (\mathrm{r})$.

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(k_n^m)	For $n \neq m$, $x^n \leq x^m$	$\dfrac{\Gamma}{X, Z_1, ..., Z_n, Y \Rightarrow C}$ $[\mathrm{k}_n^m]$

Some known results

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- We will take an algebraic, rather than proof-theoretic, approach via the theory of residuated lattices.
- We will only inspect $\{\vee, \cdot, 1\}$-equations in $\mathcal{CRL}$.
  ∘ Undecidability results for many $\{\vee, \cdot, 1\}$-equations in $\mathcal{RL}$ are consequences of [Chvalovský & Horčík 2016].

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- For any $n \geq 1$ and $m \geq 0$,

$$(\forall z)\ z^n \leq z^m \iff x_1 \cdots x_n \leq (x_1 \vee ... \vee x_n)^m\ ,\ (\forall x_1, ..., x_n)$$

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Thus, any equation $s = t$ in the signature $\{\vee, \cdot, 1\}$ is equivalent to some conjunction of **simple rules**, i.e. linear inequations of the form:

$$x_1 \cdots x_n \leq \bigvee \{x_1^{a_1} \cdots x_n^{a_n} : (a_i)_{i=1}^{n} \in A\}\ ,$$

for some finite set $A \subset \mathbb{N}^n$.

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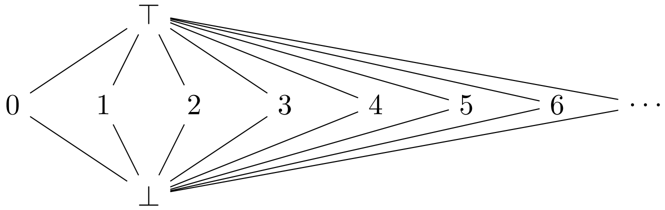
Example:

$$x \leq x^2 \vee 1$$

Characterization of $\mathcal{D}$

Consider the cancelative monoid $(\mathbb{N}, +)$. By adjoining bounds $\bot, \top$, we form the residuated lattice $\mathbf{M}_{\mathbb{N}}$:



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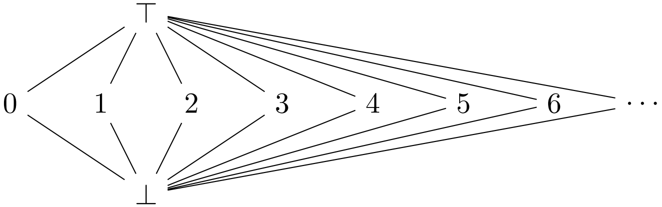

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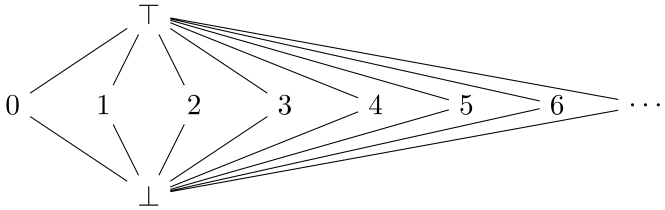

$$\mathbf{M}_{\mathbb{N}} \models (x^n \leq x^m) \iff \mathbf{M}_{\mathbb{N}} \models (\forall x)\ nx \leq mx \iff n = m.$$

Therefore, $\mathbf{M}_{\mathbb{N}}$ satisfies no knotted rules.

$$\mathbf{M}_{\mathbb{N}} \models (\mathrm{r}) \implies (\mathrm{r}) \in \mathcal{D}.$$

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Hence,
- (d) is a **d**-rule $\iff$ $\mathbf{M}_{\mathbb{N}} \models$ (d) $\iff$ no single-variable substitution instance of (d) yields a knotted rule.

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Define the collection $D_q \subset \mathcal{D}$ by

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$$(\mathrm{d}) \in D_e \iff \mathcal{CRL}_\mathrm{d} \models x^n \leq \bigvee_{i=1}^{k} x^{n+c_i},$$

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Main Result

Theorem

Let (d) $\in D_q$. *Then there exists* $\mathbf{R}_{\mathrm{d}} \in \mathcal{CRL}_{\mathrm{d}}$ *such that for every variety* $\mathcal{V}$,

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Corollary

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3. Following Chvalovský & Horčík (2016), we use the theory of *residuated frames* [Galatos & Jipsen 2013] to encode the halting problem for M as a decision problem in $\mathcal{CRL}_{\mathrm{d}}$, for a given (d)$\in D_q$

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  - There are no instructions of the form $q_F \cdots$.
- A **configuration** $C \in \text{Conf}(M) := Q \times \mathbb{N}^k$ is a tuple $\langle q; n_1, ..., n_k \rangle$, where $n_i = |r_i|$ for each $i = 1, ..., k$.
- M has a designated **final** (or *halting*) configuration C_F.

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Theorem

There exists a 2-CM M such that membership in set of terminating configurations of M is undecidable.

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The algebra $\mathbf{A}_M$

Let $M = (R_k, Q, P)$ be a k-CM, and let $Z = \{z_1, ..., z_k, q_f\}$ be a set of $(k+1)$-many fresh states.

Let $\mathbf{A}_M = (A_M, \vee, \cdot, \bot, 1)$ to be the commutative idempotent semiring generated by $R_k \cup Q \cup Z \cup \{\bot, 1\}$, where

- $(A_M, \vee, \bot)$ is a $\vee$-semilattice with bottom element $\bot$ (i.e. it is a commutative idempotent monoid with the additive identity $\bot$), and
- $(A_M, \cdot, 1)$ is a commutative monoid with the multiplicative identity 1.

Note that $x(y \vee z) = xy \vee xz$ for all $x, y, z \in A_M$.

Instructions in $\mathbf{A}_M$

Let $\theta : \mathrm{Conf}(M) \to A_M$ be the map defined by

$$\langle q; n_1, ..., n_k \rangle \overset{\theta}{\mapsto} q r_1^{n_1} \cdots r_k^{n_k}$$

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Increment and decrement instructions can naturally be simulated, for all $x \in R_k^*$, by

$$\begin{array}{lllll} p: & q + r_i q' & \Longrightarrow & qx & <_M^p & q' r_i x \\ p: & q - r_i q' & \Longrightarrow & q r_i x & <_M^p & q' x. \end{array}$$

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\begin{aligned}
p: \quad q\ 0\ r_i q' \quad &\Longrightarrow \quad qx <_M^p q'x \vee z_i x \\
q = z_i \quad &\Longrightarrow \quad z_i xy <_M^{z_i} q_f x \\
q = q_f \quad &\Longrightarrow \quad \theta(C_F) <_M^{q_f} q_f,
\end{aligned}
$$

for all $i \in \{1, ..., k\}$, $x \in R_k^*$, and $y \in (R_3 \setminus \{r_i\})^*$.

Note

We need $qx \vee z_i x <_M q_f \implies qx <_M q_f$ and $z_i x <_M q_f$.

The relation $<_M$ on A_M^θ

We define the set $A_M^\theta \subset A_M$ by,

$$u \in A_M^\theta \iff u = \bigvee_{i=1}^{m} q_i x_i, \quad m \geq 1, q_i \in Q \cup Z, x_i \in R_k^*$$

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We construct the **computation** relation $<_M$ on A_M^θ by

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Proposition (Lincoln & Mitchell 1992)

For all configurations $C \in \mathrm{Conf}(M)$,

$$C \rightsquigarrow_M C_F \iff \theta(C) <_M q_f.$$

Residuated frames

Definition [Galatos & Jipsen 2013]

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!\!/, 1)$, s.t.

- $(W \circ, 1)$ is a monoid and W' is a set.
- $N \subseteq W \times W'$, called the *Galois relation*, and
- $\backslash\!\backslash : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$ such that
- N is a **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
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$X^{\triangleright} = \{y \in W' : \forall x \in X,\ xNy\}$ and
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$Y^{\triangleleft} = \{x \in W : \forall y \in Y,\ xNy\}$, for each $X \subseteq W$ and $Y \subseteq W'$.
Then $(^{\triangleright}, ^{\triangleleft})$ is a Galois connection.
So $X \overset{\gamma_N}{\longmapsto} X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(W)$.

Residuated frames cont.

Fact [Galatos & Jipsen 2013]

$\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash\!\backslash, /\!/, \gamma_N(\{1\}))$,

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y),$$

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Then N is nuclear with $\backslash\!\!\backslash = /\!\!/$ since $\mathbf{A}_M$ is commutative.

Fact

$\mathbf{W}_M = (A_M, A_M, N, \cdot, \backslash\!\!\backslash, \{1\})$ is a residuated frame and $\mathbf{W}_M^+ \in \mathcal{CRL}$

As a consequence of this construction,

$$u <_M q_f \iff \mathcal{CRL} \models \left(\mathop{\&}_{x \in P \cup Z} \theta(x) \Rightarrow u \leq q_f \right),$$

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where we view $R_k \cup Q \cup Z$ as variables in the $\mathcal{CRL}$, and

$$\begin{array}{rlclrcl} p: & q + r_i q' & \Longrightarrow & \theta(p): & q & \leq & q' r_i \\ p: & q - r_i q' & \Longrightarrow & \theta(p): & q r_i & \leq & q' \\ p: & q 0 r_i q' & \Longrightarrow & \theta(p): & q & \leq & q' \vee z_i \\ & q = z_i & \Longrightarrow & \theta(q): & z_i r_j & \leq & z_i \\ & & & & \& \ z_i & \leq & q_f \\ & q = q_f & \Longrightarrow & \theta(q): & z_i & \leq & q_f, \end{array}$$

for each $i \in \{1, ..., k\}$ and $j \neq i$.

The effect of **d**-rules on the encoding

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Then, in general,

$$u <_M q_f \not\Leftrightarrow \mathcal{CRL}_{\mathrm{d}} \models \left(\mathop{\&}_{x \in P \cup Z} \theta(x) \Rightarrow u \leq q_f \right).$$

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Let (d)$\in D_q$.

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  - M_K can "detect" instances of (d) over $<_{M_K}$.
- Construct a new relation $<_{\mathrm{d}(M_K)}$ with enough instances of (d) so that:
  - For all $u \in A^\theta_{M_K}$, $u <_{M_K} q_f$ iff $u <_{\mathrm{d}(M_K)} q_f$, and
  - $W^+_{\mathrm{d}(M_K)} \models$ (d).

In this way, we can show for each (d)$\in D_q$, there exists a machine M_K such that

$$u <_{\mathrm{d}(M_K)} q_f \iff \mathcal{CRL}_{\mathrm{d}} \models \left(\mathop{\&}_{x \in P \cup Z} \theta(x) \Rightarrow u \leq q_f \right),$$

In this way, we can show for each (d)$\in D_q$, there exists a machine M_K such that

$$u <_{\mathrm{d}(M_K)} q_f \iff \mathcal{CRL}_{\mathrm{d}} \models \left(\mathop{\&}_{x \in P \cup Z} \theta(x) \ \Rightarrow u \leq q_f \right),$$

and if (d)$\in Q_e$,

$$u <_{\mathrm{d}(M_K)} q_f \iff \mathcal{CRL}_{\mathrm{d}} \models u \cdot \theta \leq q_f,$$

for a term $\theta \leq 1$ that encodes the machine instructions.

Questions

- For (d)$\in D_q \setminus D_e$, is the equational theory of $\mathcal{CRL}_{\mathrm{d}}$ (un)decidable?
- For (d)$\in \setminus D_d$:
  - Is the quasi-equational theory of $\mathcal{CRL}_{\mathrm{d}}$ (un)decidable?
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e.g.

How does a rule (r) such as

$$x \leq x^2 \vee 1$$

effect decidability in $\mathcal{CRL}_{\mathrm{r}}$?

Thank You!

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