The Complexity of Comparing Subalgebras Given by Generators

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Joint work with A. Bulatov and P. Mayr

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A polynomial time equivalent problem:

Subpower Membership Problem for $\mathcal{K}$, denoted $\mathrm{SMP}(\mathcal{K})$:

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  - $\exists$ 10-element semigroup $\mathbf{S}$ and a 9-element homomorphic image $\bar{\mathbf{S}}$ of $\mathbf{S}$ such that SMP($\mathbf{S}$) $\in$ P while SMP($\bar{\mathbf{S}}$) is NP-complete [Steindl, ∼2016]

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Problem. Is $\mathrm{SMP}(\mathbf{A}) \in \mathsf{P}$ whenever $\mathcal{V}(\mathbf{A})$ has a Mal'tsev/cube term?
[Willard, 2007]/[IMMVW, 2010]

Cube Terms

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  $\diamond$ $\log_2 |\mathrm{Sub}(\mathbf{A}^n)| \leq \text{const} \cdot n^k$ for some k

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  - Generalizes [Dalmau, Jeavons, 2003] and [Bulatov, Chen, Dalmau, 2007]
- $\mathrm{SMP}(\mathbf{A}) \in \mathsf{P}$ would yield a more direct aproach (and cleaner proof).

Main Results

Theorem

If $\mathcal{V}$ has a cube term, then for every finite $\mathcal{K} \subseteq \mathcal{V}_{\text{fin}}$ the following problems are all polynomial time equivalent, and are in NP*:*

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Proof uses structure theorem for subalgebras of products [Kearnes–Sz, 2012].

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Idea of Proof of 2nd Theorem

INPUT: $b_1, \ldots, b_k, c \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ $(\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K} \subseteq \mathcal{V}_{\text{fin}})$

Let $\mathbf{B} := \langle b_1, \ldots, b_k \rangle \leq_{\text{sd}} \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$ $(\mathbf{B}_i \leq \mathbf{A}_i)$

QUESTION: Is $c \in \mathbf{B}$?

May assume:

- $\mathcal{V}$ has a d-cube term;
- $\mathbb{H}\mathbb{S}\mathcal{K} \subseteq \mathcal{K}$;
- $c|_I \in \mathbf{B}|_I = \langle b_1|_I, \ldots, b_k|_I \rangle$ for all $I \in \binom{[n]}{d}$;
- in particular, $c \in \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$;
- $\mathbf{B}_1, \ldots, \mathbf{B}_n$ are subdirectly irreducible.

Idea of Proof (Cont'd)

Structure Theorem $\Rightarrow$

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- we have an equivalence relation $\sim$ on $[n] = \{1, \ldots, n\}$ (indexing the coordinates) such that
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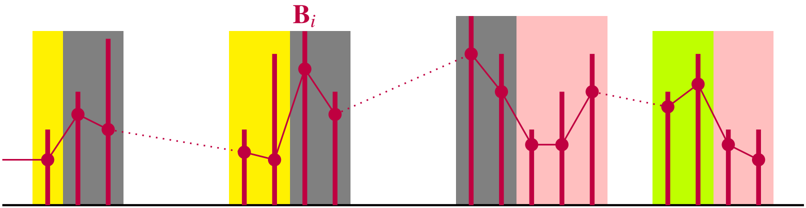

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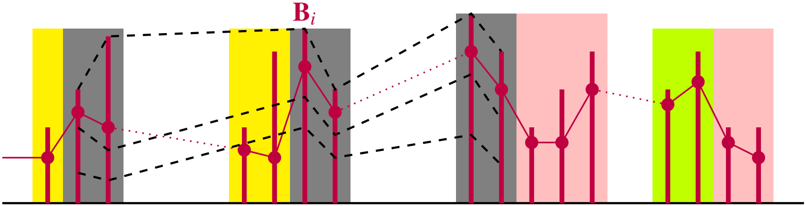

Idea of Proof (Cont'd)

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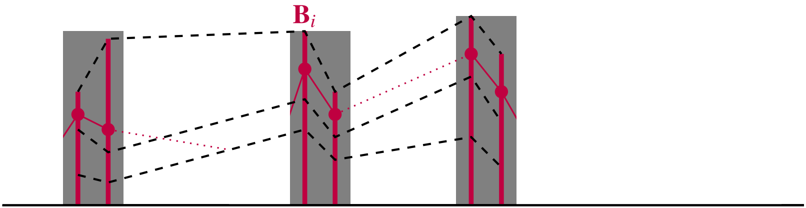

- $c \in \mathbf{B}$ iff $c|_U \in \mathbf{B}|_U$ for all blocks U $(\subseteq [n])$ of $\sim$ of size $|U| \geq \max\{d, 3\}$.

Idea of Proof (Cont'd)

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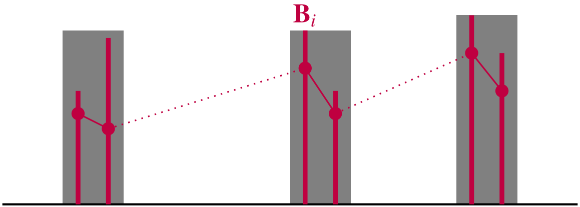

Idea of Proof (Cont'd)

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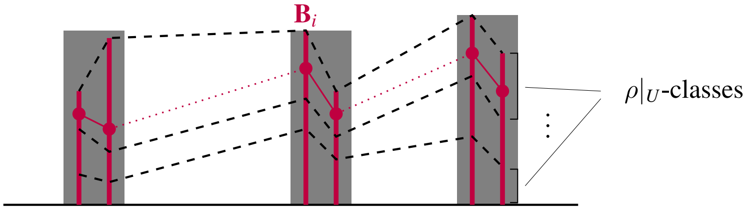

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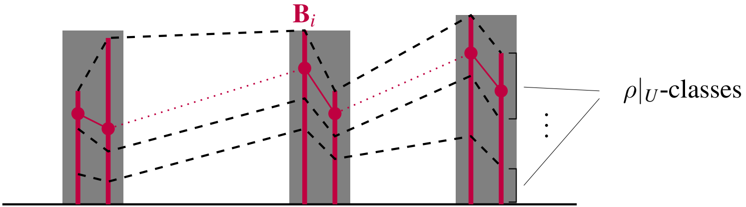

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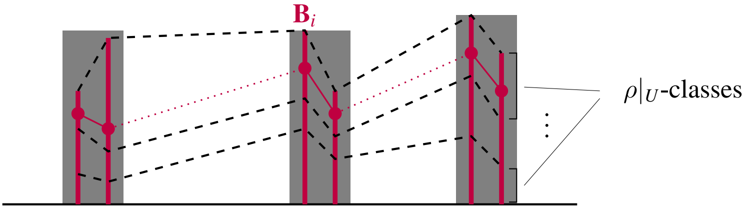

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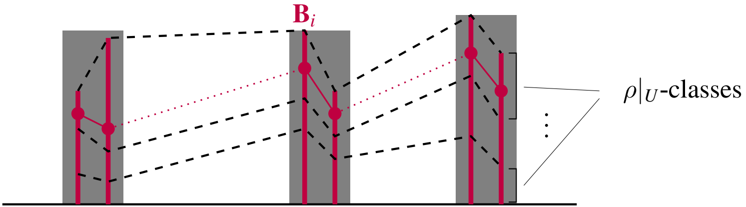

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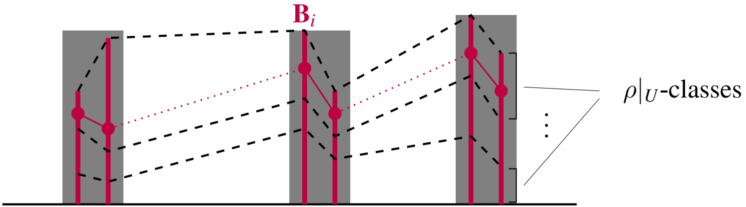

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- $\mathrm{SMP}({}_R\mathbf{M}) \in \mathsf{P} \;\Rightarrow\; \mathrm{SMP}(\mathcal{K}) \in \mathsf{P}$.