The Geometry of Relevant Implication

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The Logic KR

KR results by adding *ex falso quodlibet* to **R**, that is, the axiom scheme $(A \wedge \neg A) \to B$. Surprisingly, this does not cause a collapse into classical logic – far from it! We get the model theory for **KR** from the ternary relational semantics for **R** by adding the postulate $x^* = x$, so that the truth condition for negation is classical:

$$x \models \neg A \Leftrightarrow x \not\models A.$$

The condition $x^* = x$ has a notable effect on the ternary accessibility relation. The postulates for an **R** model structure include the following implication:

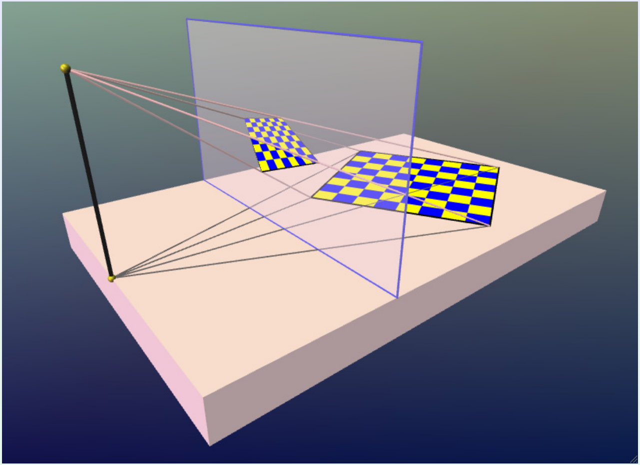
$$Rxyz \Rightarrow (Ryxz \,\&\, Rxz^*y^*).$$

The result of the identification of x and x^* is that the ternary relation in a **KR** model structure (KRms) is *totally symmetric*.

In detail, a KRms $\mathcal{K} = \langle S, R, 0 \rangle$ is a 3-place relation R on a set containing a distinguished element 0, and satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$;
2. $Raaa$;
3. $Rabc \Rightarrow (Rbac \mathbin{\&} Racb)$ (total symmetry);
4. $(Rabc \mathbin{\&} Rcde) \Rightarrow \exists f(Radf \mathbin{\&} Rfbe)$ (Pasch's postulate).

A Puzzling Question: How to construct such weird models?



Answer: Projective Geometry!

Projective Spaces

Definition

Let A be a set and L a collection of subsets of A. We call the members of A points and those of L lines. For $p, q \in A$, $p \neq q$, let $p + q$ denote the unique line containing p and q; if $p = q$, set $p + q = \{p\}$. The pair $\langle A, L \rangle$ is a projective space iff the following properties hold:

1. If p and q are two points, then there is exactly one line on both p and q.
2. If L is a line, then there are at least three points on L.
3. If a, b, d, e are four points such that the lines $a + b$ and $d + e$ meet, then lines $a + d$ and $b + e$ also meet.

Apart from degenerate cases, the Pasch Postulate states that if a line $b + e$ intersects two sides, $a + c$ and $c + d$ of the triangle $\{a, c, d\}$, then it intersects the third side, $a + d$.

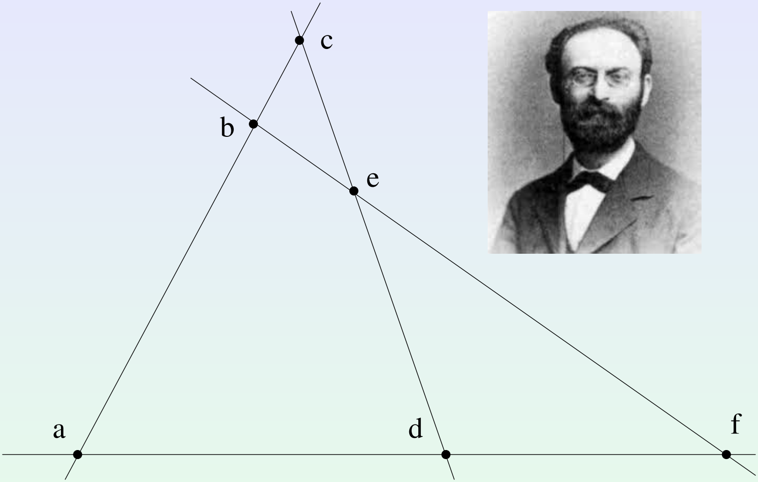


Figure: The Pasch Postulate

Constructing Frames from Geometries

Definition

Let $\mathcal{S} = \langle \mathcal{P}, \mathcal{L}, I \rangle$ be a projective space and 0 an element distinct from all the points in $\mathcal{P}$. Then $Frame(\mathcal{S})$ is defined to be the relational structure $\langle S, R, 0 \rangle$, where $S = \mathcal{P} \cup \{0\}$, and R is the smallest totally symmetric three-place relation satisfying the conditions:

1. $R0aa$ for all $a \in P$;
2. $Raaa$ for all $a \in S$;
3. $Rabc$ where a, b, c are three distinct collinear points

Theorem

Let $\mathcal{S}$ be a projective space in which there are at least four points on every line. Then $Frame(\mathcal{S})$ is a KR-frame.

The Algebra of **KR**

Given a **KR** model structure $\mathcal{K} = \langle S, R, 0 \rangle$, we can define an algebra $\mathfrak{A}(\mathcal{K})$ as follows:

Definition

The algebra $\mathfrak{A}(\mathcal{K}) = \langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot, t, \circ \rangle$ is defined on the Boolean algebra $\langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot \rangle$ of all subsets of S, where $\top = S, \bot = \emptyset, t = \{0\}$, and the operator $A \circ B$ is defined by

$$A \circ B = \{c \mid \exists a \in A, b \in B(Rabc)\}.$$

The algebra $\mathfrak{A}(\mathcal{K})$ is a De Morgan monoid in which $A \cap \neg A = \bot$. Hence the fusion operator $A \circ B$ is associative, commutative, and monotone. In addition, it satisfies the square-increasing property, and t is the monoid identity:

Constructing Geometries from Frames

Definition

Let $\mathcal{K} = \langle S, R, 0 \rangle$ be a **KR** model structure. The family $\mathcal{L}(\mathcal{K})$ is defined to be the elements of $\mathfrak{A}(\mathcal{K})$ that are $\geq t$ and idempotent, that is to say, $A \in \mathcal{L}(\mathcal{K})$ if and only if $A \circ A = A$ and $t \leq A$.

If $\mathcal{K} = \langle S, R, 0 \rangle$ is a **KR** model structure, then a subset A of S is a *linear subspace* if it satisfies the condition

$$(a, b \in A \wedge Rabc) \Rightarrow c \in A.$$

A lattice is modular if it satisfies the implication

$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

Theorem

If $\mathcal{K} = \langle S, R, 0 \rangle$ is a **KR** *model structure, then:*

1. *The elements of $\mathcal{L}(\mathcal{K})$ are the non-empty linear subspaces of $\mathcal{K}$;*
2. *$\mathcal{L}(\mathcal{K})$, ordered by containment, forms a modular lattice, with least element t, and the lattice operations of join and meet defined by $A \wedge B = A \cap B$ and $A \vee B = A \circ B$.*

If $\mathcal{S} = \langle \mathcal{P}, \mathcal{L}, I \rangle$ is a projective space, a subset X of $\mathcal{P}$ is a linear subspace if $a, b \in X \Rightarrow a + b \subseteq X$. If $Frame(\mathcal{S})$ is the frame constructed from $\mathcal{S}$, then $\mathcal{L}(\mathcal{K})$ is isomorphic to the lattice of linear subsets of $\mathcal{S}$ because a set of points X is a linear subset of $\mathcal{S}$ if and only if $X \cup \{0\}$ is a linear subset of $Frame(\mathcal{S})$.

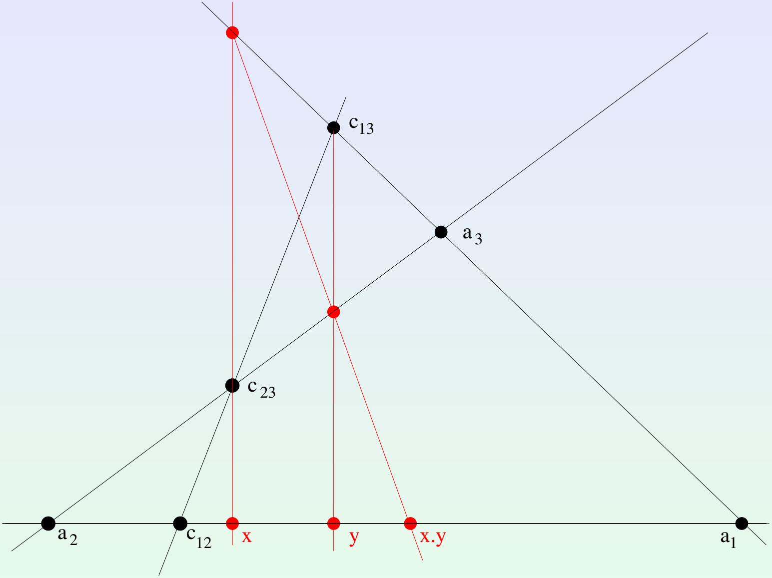


Figure: Multiplication on a line in real projective space

If we assume Desargues's law, then the geometrical multiplication defined in this way is associative.

In a two-dimensional projective space, however, we cannot assume the Desargues law in general, because of the existence of non-Arguesian projective planes. If we add a third dimension to our coordinate frame, however, then we can prove enough of Desargues's law to prove associativity of $x \cdot y$ with appropriate assumptions. This is the construction that proves undecidability for a wide family of relevance logics.

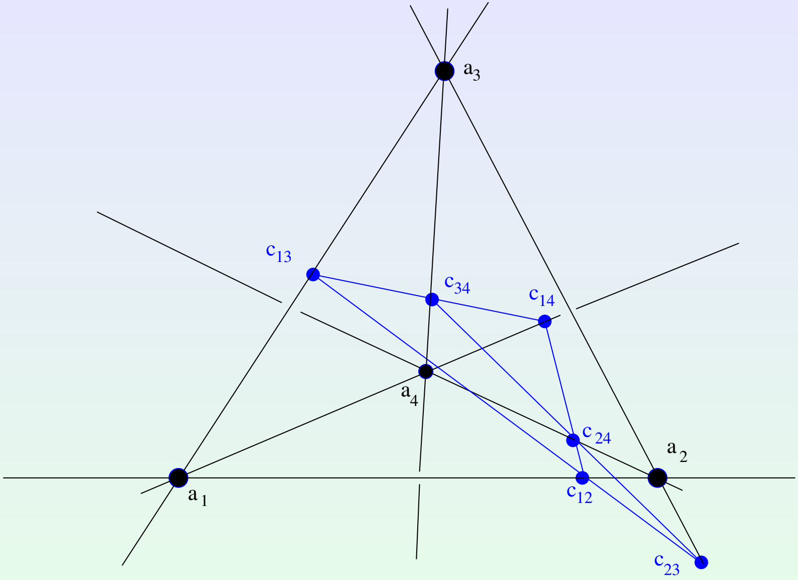


Figure: A 4-frame in real projective space

If L is a lattice with least element 0, then $a \in L$ is an *atom* if $h(a) = 1$. An element a of a complete lattice L is *compact* if and only if $a \leq \bigvee X$ for some $X \subseteq L$ implies that $a \leq \bigvee Y$ for some finite $Y \subseteq X$.

Definition

A lattice L is a *modular geometric lattice* iff L is complete, every element of L is a join of atoms, all atoms are compact, and L is modular.

A subset X of the set of atoms of a projective space is a *linear subspace* iff $p + q \subseteq X$ whenever $p, q \in X$.

Theorem

The linear subspaces of a projective space form a modular geometric lattice, where $A \wedge B = A \cap B$ *and*

$$A \vee B = \bigcup \{a + b \mid a \in A, b \in B\}.$$

The projective space construction can only represent modular geometric lattices. What is worse, it does not even cover all projective spaces, since projective spaces based on the two-element field are not included. For example, the Fano plane is not representable in this way.

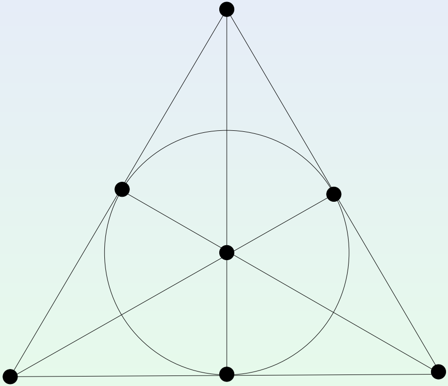


Figure: The Fano Plane

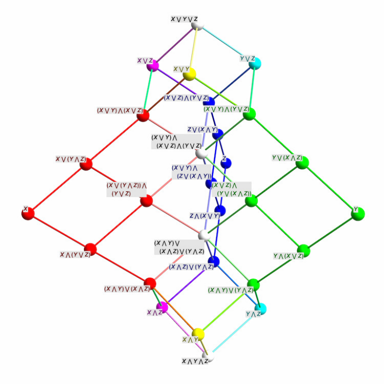


Figure: The free modular lattice on 3 generators

More generally, which modular lattices are representable in **KR** frames?

A New Construction

Definition

Let L be a modular lattice with least element 0. Define a ternary relation R on the elements of L by:

$$Rabc \Leftrightarrow a \vee b = b \vee c = a \vee c,$$

and let $\mathcal{K}(L)$ be $\langle L, R, 0 \rangle$.

Theorem

$\mathcal{K}(L)$ *is a* **KR** *model structure.*

Definition

If L is a lattice, then an *ideal* of L is a non-empty subset I of L such that

1. If $a, b \in I$ then $a \vee b \in I$;
2. If $b \in I$ and $a \leq b$, then $a \in I$.

Theorem

Let L be a modular lattice with least element 0 , and $\mathcal{K}(L) = \langle L, R, 0\rangle$ the **KR** *model structure constructed from L. Then $\mathcal{L}(\mathcal{K}(L))$ is identical with the lattice of ideals of L.*

Corollary

Any modular lattice of finite height (hence any finite modular lattice) is representable as $\mathcal{L}(\mathcal{K})$ for some **KR** *model structure $\mathcal{K}$. In addition, any modular lattice is representable as a sublattice of $\mathcal{L}(\mathcal{K})$ for some* **KR** *model structure $\mathcal{K}$.*

Problem

Can we use this construction to refute Beth's theorem for the logic **KR**?

Idea: Adapt Ralph Freese's 1979 proof that modular lattice epimorphisms need not be onto.


That's all Folks!