Series-parallel posets having a near-unanimity polymorphism

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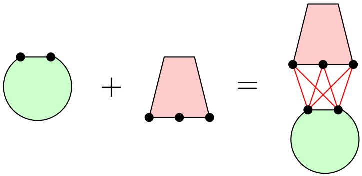
All posets are finite.

If $\mathbf{P}, \mathbf{Q}$ are posets, then $\mathbf{P} + \mathbf{Q}$ is their **ordinal sum**:



$\mathbf{P} \mathbin{\dot\cup} \mathbf{Q}$ is their disjoint union.

$\mathbf{1} \;=\; \bullet$ $\qquad$ $\mathbf{1} \mathbin{\dot\cup} \mathbf{1} \;=\; \bullet\;\bullet \;=\; \mathbf{2}$ $\qquad$ $\mathbf{1} + \mathbf{1} \;=$ 

Definition. Let **P** be a poset.

A function $f : P^n \to P$ is a **near unanimity (NU) polymorphism** of **P** if

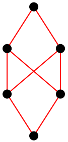
- $n \geq 3$.
- $\forall\ 1 \leq i \leq n,\ \ \forall\ a, b \in P$,

$$f(a, a, \ldots, a, \underset{\substack{\uparrow \\ i}}{b}, a, \ldots, a) \quad = \quad a$$

- f is monotone in each variable.

Clone theorists (last century) and CSPers (this century) care about which posets have an NU polymorphism.

Key examples

$\mathbf{T}_2 =$ 

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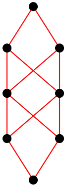
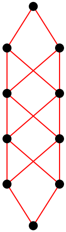
$\mathbf{T}_3 =$ 

$\mathbf{T}_4 =$ 

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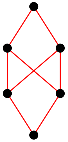

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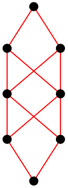

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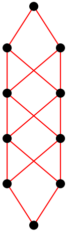

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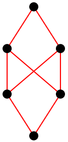
$\mathbf{T}_2 =$ 

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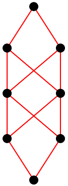
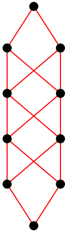
$\mathbf{T}_3 =$ 

$\mathbf{T}_4 =$ 

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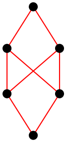
$\mathbf{T}_2 =$ 

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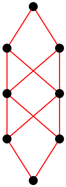
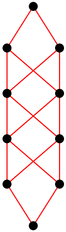
$\mathbf{T}_3 =$ 

$\mathbf{T}_4 =$ 

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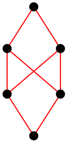
$\mathbf{T}_2 =$ 

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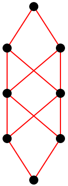
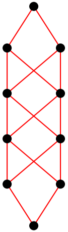
$\mathbf{T}_3 =$ 

$\mathbf{T}_4 =$ 

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$\mathbf{T}_4$: Does not even have "weaker" polymorphisms (Dem. & Rónyai, 1989).

$\mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, \ldots$ are examples of **series-parallel posets**.

Definition

A poset is **series-parallel** if it can be constructed from (copies of) $\mathbf{1}$ by finitely many applications of $+$ and $\dot\cup$.

Equivalently (Valdes, Tarjan, Lawler 1982), a poset is series-parallel iff

 does not embed into it.

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- By "forbidden retracts" (list of 5, including $\mathbf{T}_4$, $\mathbf{2}+\mathbf{2}$, and $\mathbf{2}+\mathbf{2}+\mathbf{2}$).
- By an internal characterization, easily checkable in polynomial time.

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Our main result: We can do something similar for NU polymorphisms.

New operations: $\mathbf{P} \between \mathbf{Q}$, $\mathbf{P} \triangle \mathbf{Q}$, $\mathbf{P} \bigtriangledown \mathbf{Q}$, and $\mathbf{P} \Diamond \mathbf{Q}$

$\mathbf{P} \between \mathbf{Q}$: defined when $\mathbf{P}$ has 1 and $\mathbf{Q}$ has 0.

$\mathbf{P} \triangle \mathbf{Q}$: defined when both $\mathbf{P}$ and $\mathbf{Q}$ have 1.

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New operations: **P** ⧖ **Q**, **P** $\triangle$ **Q**, **P** $\triangledown$ **Q**, and **P** $\lozenge$ **Q**

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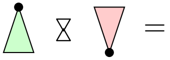

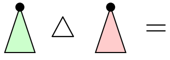

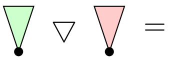

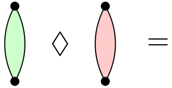

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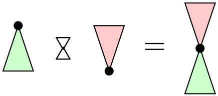

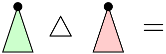

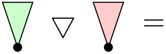

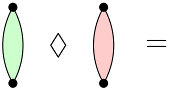

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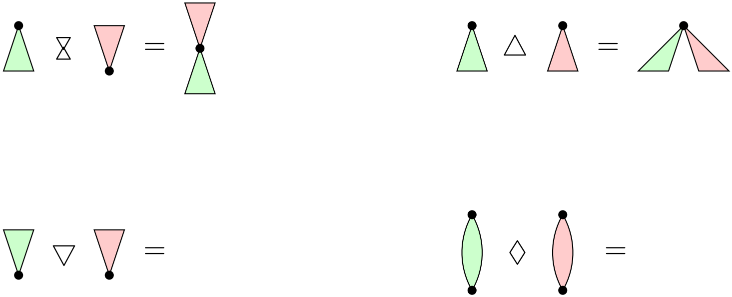

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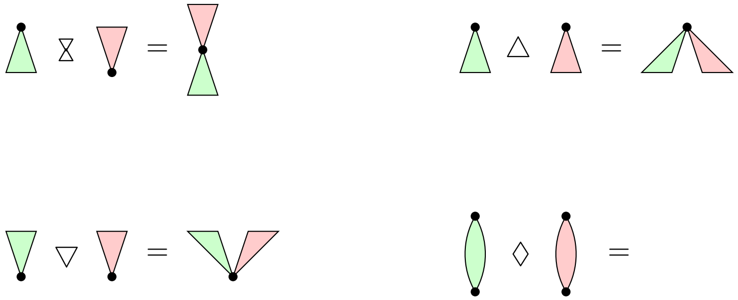

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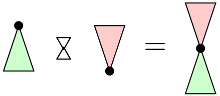

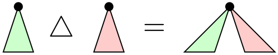

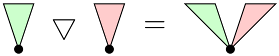

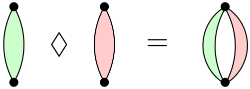

Here is our result.

Theorem

Let $\mathbf{P}$ *be a series-parallel poset. TFAE:*

1. $\mathbf{P}$ *has an NU polymorphism.*
2. $\mathbf{P}$ *does not retract onto* $\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{1}$ *or its dual, or* $\mathbf{T}_3$.
3. *Each connected component of* $\mathbf{P}$ *having more than one element is in the closure of* $\{\mathbf{1}+\mathbf{1}\}$ *under* $+, \between, \triangle, \bigtriangledown, \Diamond$.

About the proof:

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Problem

For fixed $k \geq 3$, characterize the series-parallel posets which have a k-ary NU polymorphism.