A Survey of Partially Ordered Algebras

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DRAFT version:

math.chapman.edu/~jipsen/Survey-of-po-algebras-DRAFT.pdf

Version 0.3, 2021 November 14, Chapman University

Contents

CHAPTER 1

Introduction

Disclaimer: This project is currently in a DRAFT stage. For some classes of algebras it may contain incomplete and/or incorrect information. In particular, the introduction needs to be (re)written.

This survey of partially ordered algebras contains definitions and descriptions of many algebraic categories. The most general classes of algebraic structures covered here are partially ordered sets with finitary operations that preserve or reverse the partial order in each argument. These structures are known as *po-algebras*, and they form a category with morphisms that are order-preserving homomorphisms. While po-algebras are not purely algebraic, their (in)equational theory is a relatively straight forward extension of universal algebra. The details can be found in Pigozzi [2004], but we (will eventually) also cover the main points below.

Chapter 2 contains the main classes of po-algebras. Every class has a definition with quasi-inequalities that indicate for each argument of each fundamental operation whether it is

$$\begin{aligned} \text{order-preserving:} \quad & x \leq y \implies f(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \leq f(z_1, \ldots, z_{i-1}, y, z_{i+1} \ldots, z_n) \text{ or} \\ \text{order-reversing:} \quad & x \leq y \implies f(z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n) \geq f(z_1, \ldots, z_{i-1}, y, z_{i+1} \ldots, z_n). \end{aligned}$$

If the operation has (left/right) residuals this behaviour can also be inferred from the residuation property. In Chapter 3 we cover classes of join-semilattice ordered algebras, followed by classes of meet-semilattice ordered algebras in Chapter 4. Since joins and meets can both be used to define the partial order by an equation, these classes are purely algebraic and are entirely within the realm of universal algebra. However, we now also record if an argument of a fundamental operation is join/meet-preserving, and we continue to use the perspective of po-algebras since it captures the close connections between proof theory and inequational logic. Chapter 5 contains lattice-ordered algebras, with fundamental operations that preserve join and/or meets, or reverse joins and/or meets in each argument. To say that an n-ary operation f *reverses joins* in the ith argument means that

$$f(z_1, \ldots, z_{i-1}, x \vee y, z_{i+1}, \ldots, z_n) = f(z_1, \ldots, z_{i-1}, x, z_{i+1} \ldots, z_n) \wedge f(z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_n)$$

and dually for *reversing meets*. Any operation that preserves all joins or all meets in an argument is automatically order-preserving in that argument, and likewise any operation that reverses all joins or all meets in an argument is automatically order-reversing in that argument.

The following diagram shows the highest level of our classification of categories of partially ordered algebras, numbered by the corresponding chapter number.

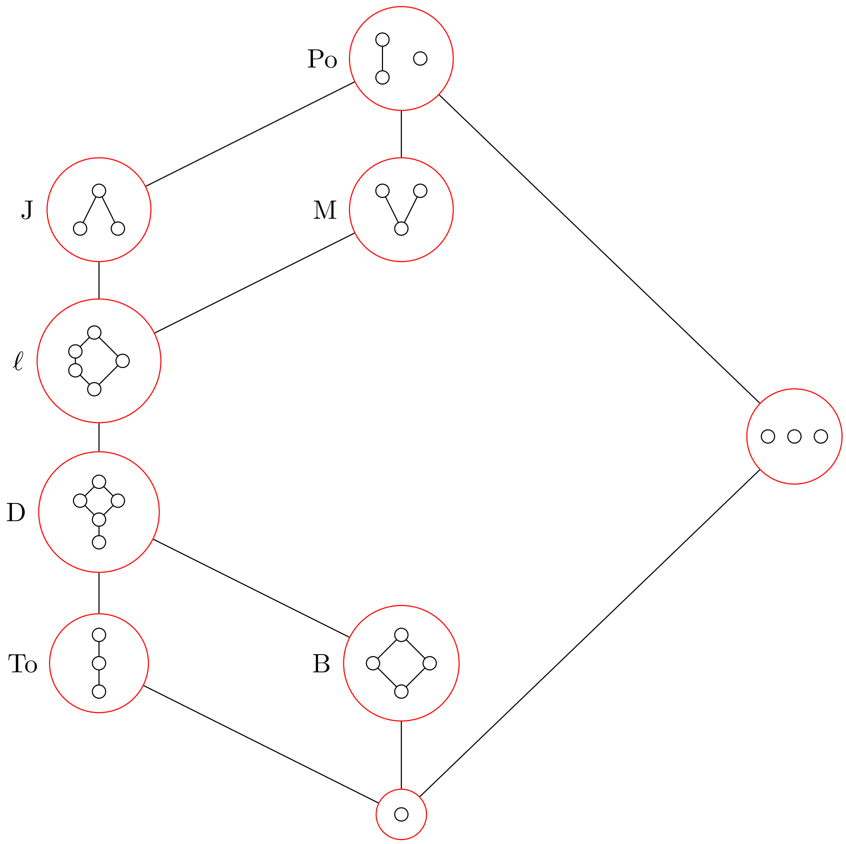
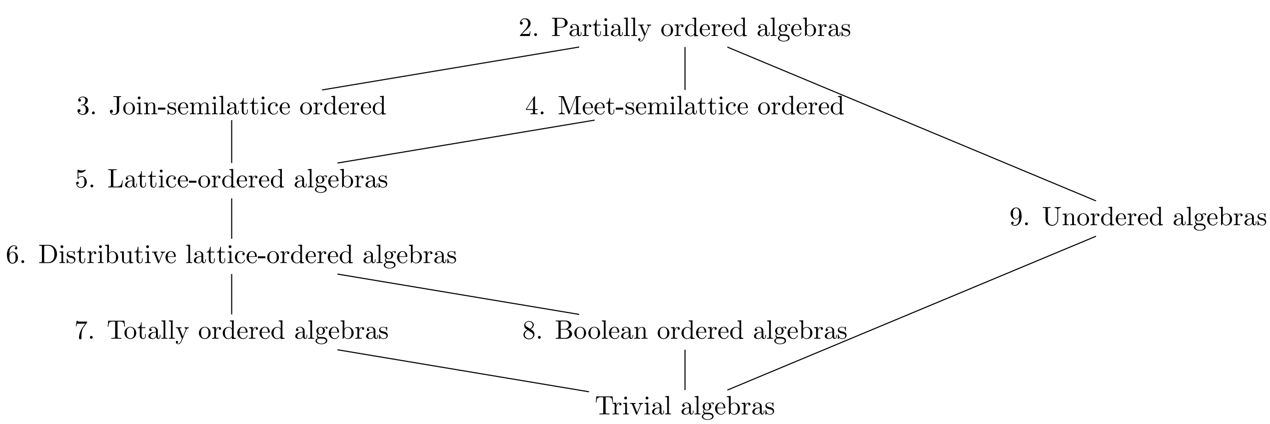


FIGURE 1. Examples of a small poset from each chapter



Many of the algebras we consider have a binary operation $\cdot$, and the next level of classification is based on whether this operation is commutative. Categories that contain noncommutative algebras precede the commutative ones.

The third level of classification is along an axis of residuation for the operation $\cdot$ in the order: nonresiduated, left-residuated, residuated, involutive, and cyclic involutive.

The fourth level considers whether $\cdot$ is nonassociative, associative, unital, integral, and/or idempotent. Combinations of these properties produce a framework of roughly 50 categories in each of the eight chapters, which are then augmented by several other standard categories that satisfy additional properties. Altogether the survey currently contains (some very basic) information about ~500 categories, with links in the pdf-file that are useful for browsing and comparing closely related categories.

Symbols	arity	order type
$\cdot, \odot, \circ, ;$	2	join-preserving, join-preserving
$+, \oplus$	2	meet-preserving, meet-preserving
$\to, \backslash$	2	join-reversing, meet-preserving
$/$	2	meet-preserving, join-reversing
$\dot{-}$	2	meet-reversing, join-preserving
$f, \Diamond, {}^\smile$	1	join-preserving
$g, \Box$	1	meet-preserving
$\sim, -$	1	join-reversing
${}^{-1}$	1	join-and-meet-reversing

FIGURE 2. Order types of operation symbols

Recall that the fine spectrum of a class of algebras is a sequence of natural numbers f_n such that up to isomorphism there are exactly f_n many algebras of size n in the class. One of the features of this survey is that for most classes the fine spectrum has been calculated (usually only up to a small value of n). In particular for the linearly ordered algebras this sequence is sometimes (related to) a sequence in the Online Encyclopedia of Integer Sequences (`OEIS.org`), in which case the entry in the OEIS can lead to additional references and combinatorial results relevant to these algebras.

The github page for this survey also contains some Jupyter notebooks with short Python programs that can extract and check information about the categories. It is likely that the survey will be updated from time-to-time, with the latest version (and a record of the changes) available on github.

The starting point for this survey was an online collection of web pages about classes of mathematical structures that can still be found at `math.chapman.edu/~jipsen/mathstructures`. In this pdf-file we concentrate on finitely axiomatized classes of partially ordered algebras and also provide some lists of finite algebras that separate many of these classes.

The signature of po-algebras in this survey mostly uses operations symbols from a fixed set, with arity ≤ 2 and with a specific order type for each argument. The convention is given in Table 2.

In addition we adopt the following convention: **If a po-algebra does not have a join operation $\vee$ then any operation join-preserving order type defaults to order-preserving, and the join-reversing order type defaults to order-reversing.**

E.g., a jsl-semigroup and ℓ-semigroup have a binary operation $\cdot$ that is join-preserving in both arguments, but in an msl-semigroup or a po-semigroup the operation $\cdot$ is only order-preserving in each argument.

An operation $\cdot$ on a poset is *residuated* if there exist binary operations $\backslash, /$ such that

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

It is worth noting that such a residuated operation $\cdot$ automatically preserves all existing joins, hence is order-preserving. Its left and right residual $\backslash, /$ preserve all existing meets in the numerator and reverse all existing joins to meets in the denominator.

1. Universal algebra and category theory

1.1. Algebras and subalgebras. An n-ary *operation* on a nonempty set A is a function $f : A^n \to A$. Each n-ary function on A has a corresponding arity (or rank): nullary operations have arity 0 and are *constants* (fixed elements of A), unary operations have arity 1, binary operations have arity 2, and so on.

An *algebra* $\mathbf{A} = (A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots)$ is a set A with operations $f_i^{\mathbf{A}}$ of arity $n_i \in \mathbb{N}$. The *signature* of an algebra is its list of operation arities $(n_1, n_2, \ldots)$. Operations are usually listed in descending order of their arity.

Let f be an operation on a set A and g an operation of the same arity on a subset B of A. Then g *is the restriction of* f *to* B, written $g = f|_B$, if for all $b_i \in B$, $\quad g(b_1, \ldots, b_n) = f(b_1, \ldots, b_n)$.

An algebra $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B \quad$ (for all i). In other words, B is closed under all operations of $\mathbf{A}$.

1.2. Homomorphisms and isomorphisms. Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same signature. A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$ such that for all i

$$h(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_i})).$$

As usual, h is *surjective* or *onto* if $h[A] = \{h(a) \mid a \in A\} = B$. In this case $\mathbf{B} = h[\mathbf{A}]$ is called a *homomorphic image* of $\mathbf{A}$.

A homomorphism h is *one-to-one* if for all $x, y \in A$, $\quad x \neq y$ implies $h(x) \neq h(y)$, and h is an *isomorphism* if h is a one-to-one and onto homomorphism. In this case $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$.

1.3. Products and HSP. Products of algebras can combine multiple algebras into one larger algebra. The *cartesian product* of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ is defined as the set $A_1 \times A_2$ with $a_j \in A_1$ and $a'_j \in A_2$ with an operation f such that $f^{\mathbf{A}_1 \times \mathbf{A}_2}(\langle a_1, a'_1\rangle, ..., \langle a_n, a'_n\rangle) = \langle f^{\mathbf{A}_1}(a_1, ..., a_n), f^{\mathbf{A}_2}(a'_1, ..., a'_n)\rangle$ for $1 \leq j \leq n$.

The *direct product* of algebras $\mathbf{A}_j$ $(j \in J)$ is $\mathbf{A} = \prod_{j \in J} \mathbf{A}_j$ where $A = \prod_{j \in J} A_j$ and $f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})(j) = f_i^{\mathbf{A}_j}(a_1(j), \ldots, a_{n_i}(j))$ for all $j \in J$.

Let $\mathcal{K}$ be a class of algebras of the same signature.

- $H(\mathcal{K})$ is the class of *homomorphic images* of members of $\mathcal{K}$.
- $S(\mathcal{K})$ is the class of algebras isomorphic to *subalgebras* of members of $\mathcal{K}$.
- $P(\mathcal{K})$ is the class of algebras isomorphic to *direct products* of members of $\mathcal{K}$.

$\mathcal{K}$ is a *variety* if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K} \quad (\iff HSP(\mathcal{K}) = \mathcal{K})$ Tarski [1946].

1.4. Term algebras and equational classes. For a fixed signature, the *set of terms with variables from a set* X is the smallest set $T(X)$ such that $X \subseteq T(X)$ and

$$\text{if } t_1, \ldots, t_{n_i} \in T(X) \text{ then “}f_i(t_1, \ldots, t_{n_i})\text{”} \in T(X) \text{ for all } i.$$

The *term algebra over a set* X is $\mathbf{T}(X) = (T(X), f_1^{\mathbf{T}}, f_2^{\mathbf{T}}, \ldots)$ with

$$f_i^{\mathbf{T}}(t_1, \ldots, t_{n_i}) = \text{“}f_i(t_1, \ldots, t_{n_i})\text{”} \quad \text{for all } i \text{ and } t_1, \ldots, t_{n_i} \in T(X).$$

An *equation* is a pair of terms (s, t), written $s{=}t$. An *assignment* into an algebra $\mathbf{A}$ is a homomorphism $h : \mathbf{T}(X) \to \mathbf{A}$. An algebra $\mathbf{A}$ *satisfies* $s{=}t$ if $h(s) = h(t)$ for all assignments into $\mathbf{A}$. For a set E of equations, $\mathrm{Mod}(E) = \{\mathbf{A} \mid \mathbf{A} \text{ satisfies } s{=}t \text{ for all } s{=}t \in E\}$. An *equational class* is of the form $\mathrm{Mod}(E)$ for some set of equations E.

1.5. Varieties and equational logic. HSP “preserves” equations, so every equational class is a variety.

Conversely,

Theorem 1 (Birkhoff 1935). *Every variety is an equational class*

An *equational theory* for some class of algebras $\mathcal{K}$ is of the form $\mathrm{Eq}(\mathcal{K})$, where $\mathrm{Eq}(\mathcal{K}) = \{s{=}t \mid \mathbf{A} \text{ satisfies } s{=}t \text{ for all } \mathbf{A} \in \mathcal{K}\}$.

Theorem 2 (Birkhoff 1935). *E is an equational theory if and only if for all terms q, r, s, t*
$t{=}t \in E;\quad s{=}t \in E \implies t{=}s \in E;\quad r{=}s,\ s{=}t \in E \implies r{=}t \in E$
and $\quad q{=}r,\ s{=}t \in E \implies s[x{\mapsto}q]{=}t[x{\mapsto}r] \in E$

1.6. Equivalence relations and congruences. Let $\mathbf{A}$ be an algebra and θ a binary relation on A. Then θ is an *equivalence relation* if it is reflexive, symmetric and transitive. A binary relation θ is a *congruence* on $\mathbf{A}$ if it is an equivalence relation and

$$x\theta y \text{ implies } f_i^{\mathbf{A}}(a_1, \ldots, x, \ldots, a_{n_i})\,\theta\, f_i^{\mathbf{A}}(a_1, \ldots, y, \ldots, a_{n_i}) \quad (\text{for all } 1 \leq j \leq n_i \text{ and all } i).$$

A *congruence class* or *block* is a set of the form $[a]_\theta = \{x \mid a\theta x\}$.

A family of sets $\{C_i : i \in I\}$ is a *partition* of A if $A = \bigcup_{i \in I} C_i$ and $C_i \cap C_j = \emptyset$ or $C_i = C_j$. The set $A/\theta = \{[a]_\theta \mid a \in A\}$ of *all congruence classes* is a partition of A.

1.7. Homomorphic images and quotient algebras. The *quotient algebra* $\mathbf{A}/\theta = (A/\theta, f_1, f_2, \ldots)$ is defined by

$$f_i([a_1]_\theta, \ldots, [a_{n_i}]_\theta) = [f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})]_\theta.$$

Note that f_i is *well-defined* if and only if θ is a *congruence.*

For a homomorphism $h : \mathbf{A} \to \mathbf{B}$, define the *kernel* $\ker h = \{(x, y) \mid h(x) = h(y)\}$. Then $\ker h$ is a congruence on $\mathbf{A}$ and the *natural map* $[\_]_\theta : \mathbf{A} \to \mathbf{A}/\theta$ is a homomorphism.

Theorem 3 (First Isomorphism Theorem). *The map $k : \mathbf{A}/\mathsf{ker}h \to h[\mathbf{A}]$ defined by $k([a]_{\mathsf{ker}h}) = h(a)$ is an isomorphism.*

Theorem 4 (Second Isomorphism Theorem). *If $\theta \subseteq \psi$ are congruences on $\mathbf{A}$ and $\varphi = \{([a]_\theta, [b]_\theta) \mid a\psi b\}$ then $T \in Con(\mathbf{A}/\theta)$ and $(\mathbf{A}/\theta)/\varphi \cong \mathbf{A}/\psi$.*

Theorem 5 (Correspondence Theorem).

1.8. Subdirectly irreducible algebras. Let $\theta_j \in \mathrm{Con}(\mathbf{A})$ and define $h : \mathbf{A} \to \prod_{j\in J} \mathbf{A}/\theta_j$ by $h(a)(j) = [a]_{\theta_j}$. Then h is one-to-one if and only if $\bigcap_{j\in J} \theta_j = id_A$. In this case h is called a *subdirect decomposition* of $\mathbf{A}$.

An element c in a lattice is *completely meet irreducible* if $c \neq \bigwedge\{x \mid c < x\}$ (note that such meets always exist).

An algebra $\mathbf{A}$ is *subdirectly irreducible* if id_A is completely meet irreducible in $\mathrm{Con}(\mathbf{A})$.

Theorem 6 (Birkhoff [1944]). *Every algebra $\mathbf{A}$ has a subdirect decomposition using only subdirectly irreducible homomorphic images of $\mathbf{A}$*

Let $\mathcal{K}_{SI}$ be the class of subdirectly irreducible members of $\mathcal{K}$. Birkhoff's Theorem says that every algebra is a subalgebra of a product of subdirectly irreducible algebras (s.i. algebras for short). So, the s.i. algebras are building blocks of varieties:

$$\mathcal{V} = SP(\mathcal{V}_{SI})$$

For any class of algebras $\mathcal{K}$, the *variety generated by* $\mathcal{K}$ is $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$. It is the smallest variety containing $\mathcal{K}$.

2. Partially-ordered universal algebra

Here we repeat the definitions from the previous section, but suitably modified to cover the partially-ordered aspect of this theory. We closely follow the presentation in Pigozzi [2004].

A *partially ordered algebra* or *po-algebra* $\mathbf{A} = (A, \leq^{\mathbf{A}}, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots)$ is a poset $(A, \leq)$ with operations $f_i^{\mathbf{A}}$ of arity $n_i \in \mathbb{N}$ that are order-preserving (isotone) or order-reversing (antitone) in each argument. The *order-type* τ_f of an n-ary operation f is an n-tuple with entries from {i,a,c,n}, which abbreviate i=isotone, a=antitone, c=constant on components, n=none. Note that if a function is both isotone and antitone for some argument then it maps all elements in a connected component of the poset to the same element (in that argument), so its order-type is c. The *signature* of a po-algebra is a list of the order-types of all its fundamental operations.

A po-algebra $\mathbf{B}$ is a *subalgebra* of a po-algebra $\mathbf{A}$ if $\leq^{\mathbf{B}}=\leq^{\mathbf{A}} \cap B^2$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ (for all i). In other words, $(B, \leq^{\mathbf{B}})$ is a subposet of $(A, \leq^{\mathbf{A}})$ *with the induced partial order* and B is closed under all operations of $\mathbf{A}$.

2.1. Homomorphisms and isomorphisms. Let $\mathbf{A}$, $\mathbf{B}$ be po-algebras of the same signature. A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is an *order-preserving* function $h : A \to B$ (i.e., $h[\leq^{\mathbf{A}}] \subseteq \leq^{\mathbf{B}}$) and for all i

$$h(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_i})).$$

As usual, h is *surjective* or *onto* if $h[A] = \{h(a) \mid a \in A\} = B$. In this case $\mathbf{B} = h[\mathbf{A}]$ is called a *homomorphic image* of $\mathbf{A}$.

A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is an *embedding* if it is one-to-one and *order-reflecting*, i.e., $h^{-1}[\leq^{\mathbf{B}}] \subseteq \leq^{\mathbf{A}}$, or equivalently $h(x) \leq^{\mathbf{B}} h(y) \implies x \leq y$.

A homomorphism h is an *isomorphism* if h is a surjective embedding. In this case $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$, and it is easy to check that h^{-1} is an isomorphism as well.

The concept of congruence needs to be generalized to work well with po-algebras. Recall that a *preorder* is a reflexive and transitive binary relation. A *precongruence* on a po-algebra $\mathbf{A}$ is a preorder α on A that contains $\le^{\mathbf{A}}$ and is *compatible*: $x\alpha y \implies f^{\mathbf{A}}(z_1,\dots,z_{i-1},x,z_{i+1},\dots,z_n)\alpha f^{\mathbf{A}}(z_1,\dots,z_{i-1},y,z_{i+1},\dots,z_n)$ if $\sigma(f)=1$ and $x\alpha y \implies f^{\mathbf{A}}(z_1,\dots,z_{i-1},y,z_{i+1},\dots,z_n)\alpha f^{\mathbf{A}}(z_1,\dots,z_{i-1},x,z_{i+1},\dots,z_n)$ if $\sigma(f)=\partial$ for all $i\in\{1,\dots,n\}$ and all fundamental operations f of $\mathbf{A}$.

The set of all precongruences of $\mathbf{A}$ is denoted by $\mathrm{Pcon}(\mathbf{A})$. Every precongruence α contains a largest congruence $\hat\alpha=\alpha\cap\alpha^{-1}$. However, $\hat\alpha$ may not contain $\le^{\mathbf{A}}$, so in general $\hat\alpha$ is not in $\mathrm{Pcon}(\mathbf{A})$.

The *quotient algebra* $\mathbf{A}/\alpha$ of a po-algebra $\mathbf{A}$ modulo a precongruence α is given by $(A/\hat\alpha,\alpha/\hat\alpha,f_1^{\mathbf{A}/\alpha},f_2^{\mathbf{A}/\alpha},\dots)$, where $\alpha/\hat\alpha$ is the *partial* order given by $[x]_{\hat\alpha}\le^{\mathbf{A}/\alpha}[y]_{\hat\alpha}\iff x\alpha y$.

With these definitions it is a good exercise to prove the isomorphism theorems and correspondence theorem for po-algebras.

2.2. Products and HSP. The product $\prod_{i\in I}\mathbf{A}_i$ of a family $\{\mathbf{A}_i\mid i\in I\}$ of po-algebras is defined as for ordinary algebras, but with the product partial order given by the pointwise order: $a\le b\iff a(i)\le^{\mathbf{A}_i} b(i)$ for all $i\in I$.

3. Definitions of properties

This section defines the terms found in the **Properties** tables.

Classtype: The classtype of a class of structures describes the "behavior" of the structure. It is chosen from the list of classtypes below:

- *variety*: A variety is a class of algebras of the same signature that is defined by a set of identities, i.e., universally quantified equations. Varieties are also called equational classes.
- *po-variety*: A partial order variety is a class of po-algebras that is defined by a set of (in)equations.
- *quasivariety*: A quasivariety is a class of algebras of the same signature that is defined by a set of quasi-identities.
- *universal class*: A class of first-order structures of the same signature is universal if it can be defined by first-order formulas that contain only universal quantifiers when written in prenex form.
- *first-order class*: A class of first-order structures of the same signature defined by a set of first-order formulas.

Equational theory: The equational theory of a class of (po-)algebras is the set of (in)equations that hold in all members of the class. For a class of algebras, this is simply the collection of all equations that hold in all members of the class.

The *decision problem* for the equational theory of a class of structures is the problem with input: an (in)equation of length n and output: "true" if the (in)equation holds in all members of the class, and "false" otherwise. The equational theory is decidable if there is an algorithm that solves the decision problem, otherwise it is undecidable. The complexity of the decision problem (if known) is one of PTIME (polynomial time), NPTIME (nondeterministic polynomial time), PSPACE (polynomial space), or EXPTIME (exponential polynomial time). While there are many other complexity classes, this survey only considers these particular ones.

G. Birkhoff showed that for classes of algebras, equational theories are precisely the sets of equations that are closed under the standard rules of equational logic, see Burris and Sankappanavar [1981] .

Quasiequational theory: A quasiequation is a universal formula of the form

$$\phi_1 \text{ and } \phi_2 \text{ and } \cdots \text{ and } \phi_m \implies \phi_0,$$

where the ϕ_i are (in)equations. Note that for a purely algebraic language, the ϕ_i are simply equations. For $m=0$, a quasiequation is just a single (in)equation. The quasiequational theory of a class of po-algebras is the set of quasiequations that hold in all members of the class.

The *decision problem* for the quasiequational theory of a class of po-algebras is the problem with input: a quasiequation of length n (as a string) and output: "true" if the quasiequation holds in all members of the class, and "false" otherwise. The quasiequational theory is decidable if there is an algorithm that solves the decision problem, otherwise it is undecidable. The complexity of the decision problem (if known) is one of PTIME, NPTIME, PSPACE, or EXPTIME.

A complete deductive system for quasiequations is given in Selman [1972]. Additional information on quasiequations can be found in Burris and Sankappanavar [1981].

Universal theory:

First-order theory: A first-order formula is an expression constructed from atomic formulas combined with logical connectives $\text{not}, \text{and}, \text{or}, \Longrightarrow, \iff$ and quantifiers $\forall$, $\exists$ followed by variables. The first-order theory of a class of structures is the set of first-order formulas that hold in all members of the class.

The decision problem for the first-order theory of a class of structures is the problem with input: a first-order formula of length n (as a string) and output: "true" if the formula holds in all members of the class, and "false" otherwise. A first-order theory is decidable if there is an algorithm that solves the decision problem, otherwise it is undecidable. A first-order theory is hereditarily undecidable if every consistent subtheory is undecidable. The complexity of the decision problem (if known) is one of PTIME, NPTIME, PSPACE, or EXPTIME.

Locally finite: An algebraic structure is locally finite if every finitely generated substructure is finite. A class of algebraic structures is locally finite if each member is locally finite.

Residual size: The residual size of a class of algebraic structures is the least upper bound (supremum) of the cardinalities of the subdirectly irreducible members of the class. If there is no bound on the size of the subdirectly irreducible members, the residual size is said to be unbounded. In this case the class is said to be residually large, otherwise it is residually small. If all subdirectly irreducible members are finite, the class is residually finite.

Congruence distributive: An algebra is congruence distributive (or CD for short) if its lattice of congruence relations is a distributive lattice. A class of algebras is congruence distributive if each of its members is congruence distributive.

Congruence distributivity has many structural consequences. The most striking one is perhaps Jónsson's Lemma Jonsson [1967] which implies that a finitely generated CD variety is residually finite. Congruence modularity is implied by congruence distributivity. Moreover, if an algebra has equationally definable principal congruences, then it is congruence distributive.

Congruence modular: An algebra is congruence modular (or CM for short) if its lattice of congruence relations is modular. A class of algebras is congruence modular if each of its members is congruence modular.

A Mal'cev condition (with 4-ary terms) for congruence modularity is given by Day [1969]. Another Mal'cev condition (with ternary terms) for congruence modularity is given by Gumm [1981]. Several further characterizations are given by Tschantz [1985].

If an algebra is congruence n-permutable for $n = 2$ or $n = 3$ or it is congruence distributive, then it is congruence modular.

Congruence n-permutable: An algebra is congruence n-permutable if for all congruence relations θ, ϕ of the algebra

$$\theta \circ \phi \circ \theta \circ \phi \circ ... = \phi \circ \theta \circ \phi \circ \theta \circ ...,$$

where n congruences appear on each side of the equation. A class of algebras is congruence n-permutable if each of its members is congruence n-permutable. The term congruence permutable is short for congruence 2-permutable, i.e. $\theta \circ \phi = \phi \circ \theta$.

Congruence n-permutability implies congruence $n+1$-permutability. Congruence 3-permutability implies congruence modularity Jónsson [1953].

Congruence regular: An algebra is congruence regular if each congruence relation of the algebra is determined by any one of its congruence classes, i.e. $\forall a, b\ [a]_\theta = [b]_\psi \Longrightarrow \theta = \psi$. A class of algebras is congruence regular if each of its members is congruence regular.

Congruence uniform: An algebra is congruence uniform if for all congruence relations θ of the algebra it holds that all congruence classes of θ have the same cardinality. A class of algebras is congruence uniform if each of its members is congruence uniform.

Congruence types: A minimal algebra is a finite nontrivial algebra in which every unary polynomial is either constant or a permutation. Peter P. Pálfy Pálfy [1984] shows that if $\mathbf{M}$ is a minimal algebra then $\mathbf{M}$ is polynomially equivalent to one of the following:

* a unary algebra in which each basic operation is a permutation * a vector space * the 2-element Boolean algebra * the 2-element lattice * a 2-element semilattice.

The type of a minimal algebra $\mathbf{M}$ is defined to be permutational (1), abelian (2), Boolean (3), lattice (4), or semilattice (5) accordingly.

The type set of a finite algebra is defined and analyzed extensively in the groundbreaking book Hobby and McKenzie [1988]. With each two-element interval $\{\theta, \psi\}$ in the congruence lattice of a finite algebra the authors associate a collection of minimal algebras of one of the 5 types, and this defines the value of typ(θ, ψ). For a finite algebra $\mathbf{A}$, typ$(\mathbf{A})$ is the union of the sets typ(θ, ψ) where $\{\theta, \psi\}$ ranges over all two-element intervals in the congruence lattice of $\mathbf{A}$. For a class $\mathcal{K}$ of algebras, typ$(\mathcal{K}) = \{\text{typ}(\mathbf{A}) : \mathbf{A}$ is a finite algebra in $\mathcal{K}\}$.

Congruence extension property: An algebraic structure $\mathbf{A}$ has the congruence extension property (CEP) if for any algebraic substructure $\mathbf{B} \le \mathbf{A}$ and any congruence relation θ on $\mathbf{B}$ there exists a congruence relation ψ on $\mathbf{A}$ such that $\psi \cap (B \times B) = \theta$. A class of algebraic structures has the congruence extension property if each of its members has the congruence extension property.

For a class $\mathcal{K}$ of algebraic structures, a congruence θ on an algebra $\mathbf{B}$ is a $\mathcal{K}$-congruence if $\mathbf{B}//\theta \in \mathcal{K}$. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, we say that a $\mathcal{K}$-congruence θ of $\mathbf{B}$ can be extended to $\mathbf{A}$ if there is a $\mathcal{K}$-congruence ψ on $\mathbf{A}$ such that $\psi \cap (B \times B) = \theta$. Note that if $\mathcal{K}$ is a variety and $B \in \mathcal{K}$ then every congruence of $\mathbf{B}$ is a $\mathcal{K}$-congruence.

Definable principal congruences: A (quasi)variety $\mathcal{K}$ of algebraic structures has first-order definable principal (relative) congruences (DP(R)C) if there is a first-order formula $\phi(u, v, x, y)$ such that for all $\mathbf{A} \in \mathcal{K}$ we have $\langle x, y\rangle \in \text{Cg}_{\mathcal{K}}(u, v) \iff \mathbf{A} \models \phi(u, v, x, y)$. Here $\theta = \text{Cg}_{\mathcal{K}}(u, v)$ denotes the smallest (relative) congruence that identifies the elements u, v, where "relative" means that $\mathbf{A}//\theta \in \mathcal{K}$.

If an algebra has equationally definable principal (relative) congruences, then it has definable principal congruences.

Equationally def. pr. cong.: A (quasi)variety $\mathcal{K}$ of algebraic structures has equationally definable principal (relative) congruences (EDP(R)C) if there is a finite conjunction of atomic formulas $\phi(u, v, x, y)$ such that for all algebraic structures $\mathbf{A} \in \mathcal{K}$ we have $\langle x, y\rangle \in \text{Cg}_{\mathcal{K}}(u, v) \iff \mathbf{A} \models \phi(u, v, x, y)$. Here $\theta = \text{Cg}_{\mathcal{K}}(u, v)$ denotes the smallest (relative) congruence that identifies the elements u, v, where "relative" means that $\mathbf{A}//\theta \in \mathcal{K}$. Note that when the structures are algebras then the atomic formulas are simply equations. Blok and Pigozzi [1994]

Amalgamation property: An amalgam is a tuple $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are structures of the same signature, and $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ are embeddings (injective morphisms).

A class $\mathcal{K}$ of structures is said to have the amalgamation property if for every amalgam $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $A \neq \emptyset$ there exists a structure $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

Strong amalgamation property: A class $\mathcal{K}$ of structures is said to have the strong amalgamation property, or SAP for short, if for every amalgam $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $A \neq \emptyset$ there exists a structure $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$ and $\text{Im}(f') \cap \text{Im}(g') = \text{Im}(f' \circ f)$, where $\text{Im}(f') = \{f'(x) | x \in B\}$.

If an algebra has the amalgamation property or its epimorphisms are surjective, then it has the strong amalgamation property. If an algebra has the strong amalgamation property, then it has the amalgamation property.

Epimorphisms are surjective: A morphism h in a category is an epimorphism if it is right-cancellative, i.e. for all morphisms f, g in the category $f \circ h = g \circ h$ implies $f = g$.

Epimorphisms are surjective in a (concrete) category of structures if the underlying function of every epimorphism is surjective.

If a concrete category has the amalgamation property and all epimorphisms are surjective, then it has the strong amalgamation property Kiss et al. [1983].

4. Comments, questions and open problems

A *proper* po-algebra is one where the partial order $\le$ is not equationally definable so, in particular, neither a join-semilattice nor a meet-semilattice.

The most interesting po-algebras in this survey are the proper ones with some operation(s) that are order-reversing is some coordinate(s) since they have not been studied much, especially from an algebraic point of view (with the notable exception of po-groups Glass [1999]).

Some simple results are included here, and while they may be well known, we are not aware of references to them in the literature.

LEMMA 7. *For any po-algebra the equivalence relation corresponding to the partition of the poset into connected components is a congruence.*

LEMMA 8. *If po-algebra has a residuated binary operation then the connected components of the poset are both up and down directed. Hence in the finite case each connected component is bounded.*

The class of posets has several subclasses that could be of interest:

The class of (lower/upper) bounded posets $\mathrm{Pos}_\bot$, $\mathrm{Pos}_\top$, $\mathrm{Pos}_{\bot\top}$.

The class of forests: $x, y \le z \implies x \le y$ or $y \le x$

The class of root systems (dual forests).

The class of posets that are both forests and root systems. (Prove this is equivalent to having all components linearly ordered.)

The class of (up/down)-directed posets (but these are not universal classes).

The class of posets with bounded components. (Is this a first-order class?)

The class Pos_m of posets with m constants that are maximal elements (for fixed m). This should not be a po-quasivariety.

The class of posets with n constants that are minimal elements (for fixed n).

Here are some (very naive) questions:

- Can a finite proper po-algebra support a residuated binary operation?
- Can Jónsson's lemma be generalized to po-algebras?
- Can the Malcev condition for congruence distributivity be generalized? How about all Malcev conditions from universal algebra? Do they transfer?
- Is there a congruence distributive po-variety that includes proper po-algebras?

5. Naming of classes

There are many conventions for naming particular categories and classes of structures. Long names usually contain several adjectives followed by a name for a (large) class. To avoid too many different names for the same class, the adjectives are usually listed in alphabetical order.

Most adjectives and prefixes refer to properties that restrict a larger class, but *pseudo*, *generalized*, *semi*, *noncommutative*, etc. remove certain properties. In this setting, the prefix *non* is usually nonexclusive, so e. g., the class of noncommutative rings includes all commutative rings (and probably should have been called *not necessarily commutative* rings).

The conventions for abbreviated names far less standardized. Here we mostly follow conventions from Galatos et al. [2007], extended with many well known abbreviations.

List of prefixes used in the unique names for (most) classes. They are usually added in alphabetical order.

- Ab = abelian $xy = yx$
- B = Boolean
- b = bounded $\bot \le x \le \top$
- C = commutative $xy = yx$
- $_c$ = contraction $x \le xx$
- Can = cancellative $xz = yz$ or $zx = zy \implies x = y$
- Cy = cyclic $\sim x = -x$
- D = distributive $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- Dm = De Morgan $-(x \wedge y) = -x \vee -y$, $-(x \vee y) = -x \wedge -y$
- $d\ell$ = distributive lattice-ordered
- $_e$ = exchange = commutative
- G = generalized (noncommutative and no bottom constant)
- H = Heyting
- I = integral $x \le 1$, or $xy \le x, y$
- Id = idempotent $xx = x$
- In = involutive $-\sim x = x = \sim - x$
- J = join-semilattice-ordered
- K = Kleene
- L = lattice-ordered
- lb = lower bounded $\bot \le x$
- Lr = left residuated $xy \le z \iff y \le x\backslash z$

- Lt = left
- M = meet-semilattice-ordered
- Mod = modular
- N = negated
- Nl = nilpotent
- p = pointed $c = c$
- ps = pseudo
- q = quasi
- Po = partially-ordered
- Reg = regular
- R = residuated = Lr and $xy \leq z \iff x \leq z/y$
- Rt = right
- Sl = semilinear
- Sqd = square decreasing $xx \leq x$
- Sqi = square increasing $x \leq xx$
- To = totally-ordered $x \leq y$ or $y \leq x$
- $_w$ = weakening = integral and lower bounded

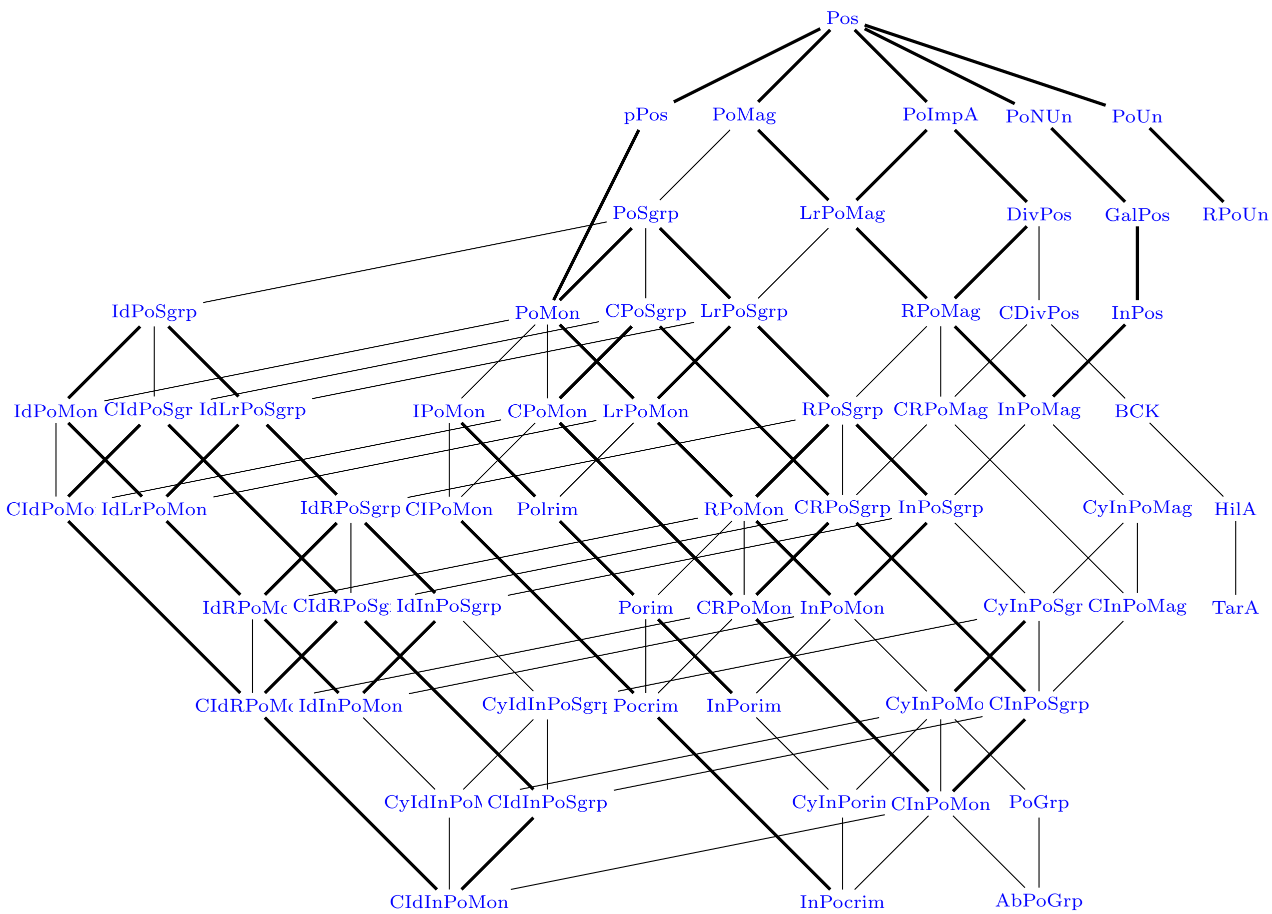
List of abbreviations used at the end of the unique names for (most) classes:

- Alg = A = algebras
- BL = basic logic algebras
- Bnd = bands
- Chn = chains = totally ordered sets
- Dom = domain
- Grp = groups
- FL = full Lambek algebras
- Fld = fields
- Hp = hoops
- IMTL = involutive MTL-algebras
- Jslat = join-semilattices
- Lat = lattices
- Lp = loops
- Mag = magmas
- Mon = monoids
- Mslat = meet-semilattices
- MTLA = monoidal t-norm logic algebras
- MV = many-valued logic algebras
- Pos = posets
- Qgrp = quasigroup
- RA = relation algebras
- RL = residuated lattices
- Rng = rings
- Set = sets
- Sgrp = semigroups
- Srng = semirings
- Un = Unar = set with a unary operation

CHAPTER 2

Partially ordered algebras

Thick lines mean that new operations or constants are added. Standard lines mean only new (quasi)(in)equational axioms are added.



RPoUn = residuated po-unar

Pregroups (Lambek 2000)

Abelian pregroups (Cesari 1989 https://doi.org/10.1016/S0049-237X(08)70269-6)

Quantum B-algebras (Rump 2013 https://doi.org/10.2478/s11533-013-0302-0 Def 1.2)

1. Pos: Partially ordered sets

Definition

A *partially ordered set* (also called *ordered set* or *poset* for short) is a po-algebra $\mathbf{P} = \langle P, \leq \rangle$ with no operations such that P is a set and $\leq$ is a binary relation on P that is

reflexive: $x \leq x$,

transitive: $x \leq y$ and $y \leq z \implies x \leq z$ and antiymmetric: $x \le y$ and $y \le x \implies x = y$.

Definition

A *strict partial order* is a po-algebra $\langle P, < \rangle$ such that P is a set and $<$ is a binary relation on P that is

irreflexive: $\neg(x < x)$

transitive: $x < y$ and $y < z \implies x < y$

Remark: The above definitions are related via: $x \le y \iff x < y$ or $x = y$ and $x < y \iff x \le y, x \ne y$.

For a partially ordered set $\mathbf{P}$, define the dual $\mathbf{P}^\partial = \langle P, \ge \rangle$ by $x \ge y \iff y \le x$. Then $\mathbf{P}^\partial$ is also a partially ordered set.

Formal Definition

$x \le x$

$x \le y$ and $y \le z \implies x \le z$

$x \le y$ and $y \le x \implies x = y$

Examples

Example 1: $\langle \mathbb{R}, \le \rangle$, the real numbers with the standard order.

Example 2: $\langle P(S), \subseteq \rangle$, the collection of subsets of a sets S, ordered by inclusion.

Example 3: Any poset is order-isomorphic to a poset of subsets of some set, ordered by inclusion.

Properties

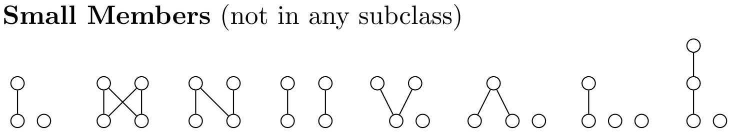
Classtype	Universal Horn class
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1$, $f_2 = 2$, $f_3 = 5$, $f_4 = 16$, $f_5 = 63$, $f_6 = 318$, $f_7 = 2045$, $f_8 = 16999$, $f_9 = 183231$, $f_{10} = 2567284$, $f_{11} = 46749427$, $f_{12} = 1104891746$, $f_{13} = 33823827452$, $f_{14} = 1338193159771$, $f_{15} = 68275077901156$, $f_{16} = 4483130665195087$

oeis.org/A000112

Small Members (not in any subclass)



$P_{3,1}$ $P_{4,1}$ $P_{4,2}$ $P_{4,3}$ $P_{4,4}$ $P_{4,5}$ $P_{4,6}$ $P_{4,7}$

Subclasses

Jslat: Join-semilattices

Mslat: Meet-semilattices

PoImpA: Partially ordered implication algebras

PoMag: Partially ordered magmas

PoNUn: Partially ordered negated unars

PoUn: Partially ordered unars

Set: The category of sets

pPos: Pointed posets

Superclasses

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pPos: Pointed posets

Definition

A *pointed poset* is a po-algebra $\mathbf{P} = \langle P, \le, c \rangle$ such that P is a partially ordered set and c is a constant operation on P.

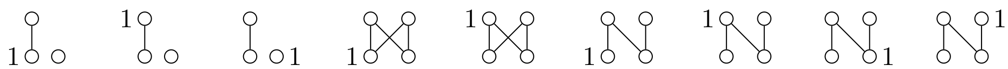
Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 11,\ f_4 = 47,\ f_5 = 243$

Small Members (not in any subclass)



$pP_{3,1}$ $pP_{3,2}$ $pP_{3,3}$ $pP_{4,1}$ $pP_{4,2}$ $pP_{4,3}$ $pP_{4,4}$ $pP_{4,5}$ $P_{4,6}$

Subclasses

PoMon: Partially ordered monoids

pJslat: Pointed join-semilattices

pMslat: Pointed meet-semilattices

pSet: The category of pointed sets

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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3. PoUn: Partially ordered unars

Definition

A *partially ordered unar* (also called a *po-unar* for short) is a po-algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a partially ordered set and f is a unary operation on P that is

order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$x \leq y \implies f(x) \leq f(y)$

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 43,\ f_4 = 452$

Subclasses

GalPos: Galois posets

JUn: Join-semilattice-ordered unars

MUn: Meet-semilattice-ordered unars

RPoUn: Residuated partially ordered unars

Unar: Unary Algebras

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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4. PoNUn: Partially ordered negated unars

Definition

A *partially ordered negated unar* (also called a *po-nunar* for short) is a po-algebra $\mathbf{P} = \langle P, \leq, \sim\rangle$ such that P is a partially ordered set and $\sim$ is a unary operation on P that is

order-reversing: $x \leq y \implies {\sim}y \leq {\sim}x$

Formal Definition

$x \leq y \implies {\sim}y \leq {\sim}x$

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 39,\ f_4 = 386,\ f_5 = 5203$

Subclasses

GalPos: Galois posets

JNUn: Join-semilattice-ordered negated unars

MNUn: Meet-semilattice-ordered negated unars

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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5. PoMag: Partially ordered magmas

Definition

A *partially ordered magma* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot \rangle$ such that

$\langle A, \cdot \rangle$ is a magma

$\langle A, \leq \rangle$ is a partially ordered set

$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 16,\ f_3 = 4051$

Subclasses

JMag: Join-semilattice-ordered magmas

LrPoMag: Left-residuated partially ordered magmas

MMag: Meet-semilattice-ordered magmas

PoSgrp: Partially ordered semigroups

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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6. PoSgrp: Partially ordered semigroups

Definition

A *partially ordered semigroup* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot \rangle$ such that

$\langle A, \cdot \rangle$ is a semigroup

$\langle A, \leq \rangle$ is a partially ordered set

$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Examples

Example 1: The natural numbers larger than 1, with addition, or with multiplication.

Properties

Classtype	Quasivariety

Finite Members

$f_1 = 1,\ f_2 = 11,\ f_3 = 173,\ f_4 = 4753,\ f_5 = 198838,\ f_6 = 13457454,\ f_7 = 4207546916$

Subclasses

CPoSgrp: Commutative partially ordered semigroups

IdPoSgrp: Idempotent partially ordered semigroups

JSgrp: Join-semilattice-ordered semigroups

LrPoSgrp: Left-residuated partially ordered semigroups

MSgrp: Meet-semilattice-ordered semigroups

PoMon: Partially ordered monoids

Superclasses

PoMag: Partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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7. PoMon: Partially ordered monoids

Definition

A *partially ordered monoid* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a monoid

$\langle A, \leq\rangle$ is a partially ordered set

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Basic Results

Every monoid with the discrete partial order is a po-monoid.

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 37,\ f_4 = 549$

Subclasses

CPoMon: Commutative partially ordered monoids

IPoMon: Integral partially ordered monoids

IdPoMon: Idempotent partially ordered monoids

JMon: Join-semilattice-ordered monoids

LrPoMon: Left-residuated partially ordered monoids

MMon: Meet-semilattice-ordered monoids

Superclasses

PoSgrp: Partially ordered semigroups

pPos: Pointed posets Cont|Po|J|M|L|D|To|B|U|Ind

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8. IPoMon: Integral partially ordered monoids

Definition

An *integral partially ordered monoid* is a partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that $x \leq 1$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 11,\ f_5 = 102,\ f_6 = 1609$

Subclasses

CIPoMon: Commutative integral partially ordered monoids
IJMon: Integral join-semilattice-ordered monoids
IMMon: Integral meet-semilattice-ordered monoids
Polrim: Partially ordered left-residuated integral monoids

Superclasses

PoMon: Partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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9. IdPoSgrp: Idempotent partially ordered semigroups

Definition

An *idempotent partially ordered semigroup* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot\rangle$ such that $\langle A, \leq, \cdot\rangle$ is a partially ordered semigroup and $\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 7,\ f_3 = 69,\ f_4 = 1035$

Subclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups
IdJSgrp: Idempotent join-semilattice-ordered semigroups
IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups
IdMSgrp: Idempotent meet-semilattice-ordered semigroups

IdPoMon: Idempotent partially ordered monoids

Superclasses

PoSgrp: Partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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10. IdPoMon: Idempotent partially ordered monoids

Definition

An *idempotent partially ordered monoid* is a partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 23,\ f_4 = 238,\ f_5 = 3356$

Subclasses

CIdPoMon: Commutative idempotent partially ordered monoids

IdJMon: Idempotent join-semilattice-ordered monoids

IdLrPoMon: Idempotent left-residuated partially ordered monoids

IdMMon: Idempotent meet-semilattice-ordered monoids

Superclasses

IdPoSgrp: Idempotent partially ordered semigroups

PoMon: Partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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11. PoImpA: Partially ordered implication algebras

Formal Definition

$x \leq y \implies y \to z \leq x \to z$

$x \leq y \implies z \to x \leq z \to y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 16,\ f_3 = 3981$

Subclasses

DivPos: Division posets

JImpA: Join-semilattice-ordered implication algebras

LrPoMag: Left-residuated partially ordered magmas

MImpA: Meet-semilattice-ordered implication algebras

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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12. LrPoMag: Left-residuated partially ordered magmas

Definition

A *left-residuated partially ordered magma* (or *lrpo-magma*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash \rangle$ such that
$\langle A, \leq \rangle$ is a partially ordered set,
$\langle A, \cdot \rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 110$

Subclasses

LrJMag: Left-residuated join-semilattice-ordered magmas
LrMMag: Left-residuated meet-semilattice-ordered magmas
LrPoSgrp: Left-residuated partially ordered semigroups
RPoMag: Residuated partially ordered magmas

Superclasses

PoImpA: Partially ordered implication algebras
PoMag: Partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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13. LrPoSgrp: Left-residuated partially ordered semigroups

Definition

A *left-residuated partially ordered semigroup* (or *lrpo-semigroup*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash \rangle$ such that
$\langle A, \leq \rangle$ is a partially ordered set,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 5,\ f_3 = 28,\ f_4 = 273,\ f_5 = 3788$

Subclasses

IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups
LrJSgrp: Left-residuated join-semilattice-ordered semigroups
LrMSgrp: Left-residuated meet-semilattice-ordered semigroups
LrPoMon: Left-residuated partially ordered monoids
RPoMon: Residuated partially ordered monoids
RPoSgrp: Residuated partially ordered semigroups

Superclasses

LrPoMag: Left-residuated partially ordered magmas

PoSgrp: Partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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14. LrPoMon: Left-residuated partially ordered monoids

Definition

A *left-residuated partially ordered monoid* (or *lrpo-monoid*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash\rangle$ such that
$\langle A, \leq\rangle$ is a partially ordered set,
$\langle A, \cdot, 1\rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 32,\ f_5 = 234,\ f_6 = 2493$

Subclasses

IdLrPoMon: Idempotent left-residuated partially ordered monoids

LrJMon: Left-residuated join-semilattice-ordered monoids

LrMMon: Left-residuated meet-semilattice-ordered monoids

Polrim: Partially ordered left-residuated integral monoids

RPoMon: Residuated partially ordered monoids

Superclasses

LrPoSgrp: Left-residuated partially ordered semigroups

PoMon: Partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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15. Polrim: Partially ordered left-residuated integral monoids

Definition

A *partially ordered left-residuated integral monoid* (or *polrim* for short) is a left-residuated partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash\rangle$ for which
$x \leq 1$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 51,\ f_6 = 409$

Subclasses

ILrJMon: Integral left-residuated join-semilattice-ordered monoids
ILrMMon: Integral left-residuated meet-semilattice-ordered monoids
Porim: Partially ordered residuated integral monoids

Superclasses

IPoMon: Integral partially ordered monoids
LrPoMon: Left-residuated partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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16. IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups

Definition

An *idempotent left-residuated partially ordered semigroup* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash \rangle$ such that $\langle A, \leq, \cdot, \backslash \rangle$ is a left-residuated partially ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 12,\ f_4 = 71,\ f_5 = 524$

Subclasses

IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups
IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups
IdLrPoMon: Idempotent left-residuated partially ordered monoids
IdRPoSgrp: Idempotent residuated partially ordered semigroups

Superclasses

IdPoSgrp: Idempotent partially ordered semigroups
LrPoSgrp: Left-residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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17. IdLrPoMon: Idempotent left-residuated partially ordered monoids

Definition

An *idempotent left-residuated partially ordered monoid* is a left-residuated partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 12,\ f_5 = 59,\ f_6 = 350$

Subclasses

IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids
IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids
IdRPoMon: Idempotent residuated partially ordered monoids

Superclasses

IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups
IdPoMon: Idempotent partially ordered monoids
LrPoMon: Left-residuated partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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18. RPoUn: Residuated partially ordered unars

Formal Definition

A *residuated partially ordered unar* (also called a *rpo-unar* for short) is a po-algebra $\mathbf{P} = \langle P, \leq, f, g\rangle$ such that $\langle P, \leq\rangle$ is a partially ordered set and f, g are unary operations on P that g is the upper residual of f, or equivalently, g is the right adjoint of f:

$f(x) \leq y \iff x \leq g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 16,\ f_4 = 87,\ f_5 = 562$

Subclasses

InPoMon: Involutive partially ordered monoids
RJUn: Residuated join-semilattice-ordered unars
RMUn: Residuated meet-semilattice-ordered unars

Superclasses

PoUn: Partially ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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19. DivPos: Division posets

Formal Definition

A *division poset* is a po-algebra $\mathbf{P} = \langle P, \leq, \backslash, /\rangle$ such that $\langle P, \leq\rangle$ is a partially ordered set,

$x \leq y \implies z\backslash x \leq z\backslash y$,
$x \leq y \implies x/z \leq y/z$ and

$x \leq z/y \iff y \leq x\backslash z.$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 123$

Subclasses

CDivPos: Commutative division posets
DivJslat: Division join-semilattices
DivMslat: Division meet-semilattices
RPoMag: Residuated partially ordered magmas

Superclasses

PoImpA: Partially ordered implication algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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20. RPoMag: Residuated partially ordered magmas

Definition

A *residuated partially ordered magma* (or *rpo-magma*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, /\rangle$ such that
$\langle A, \leq\rangle$ is a partially ordered set,
$\langle A, \cdot\rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 28,\ f_4 = 1200$

Subclasses

CRPoMag: Commutative residuated partially ordered magmas
InPoMag: Involutive partially ordered magmas
RJMag: Residuated join-semilattice-ordered magmas
RMMag: Residuated meet-semilattice-ordered magmas
RPoSgrp: Residuated partially ordered semigroups

Superclasses

DivPos: Division posets
LrPoMag: Left-residuated partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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21. RPoSgrp: Residuated partially ordered semigroups

Definition

A *residuated partially ordered semigroup* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, /\rangle$ such that
$\langle A, \leq\rangle$ is a partially ordered set,
$\langle A, \cdot\rangle$ is a semigroup and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 16,\ f_4 = 154,\ f_5 = 2100$

Subclasses

CRPoSgrp: Commutative residuated partially ordered semigroups
IdRPoSgrp: Idempotent residuated partially ordered semigroups
InPoSgrp: Involutive partially ordered semigroups
RJSgrp: Residuated join-semilattice-ordered semigroups
RMSgrp: Residuated meet-semilattice-ordered semigroups
RPoMon: Residuated partially ordered monoids

Superclasses

LrPoSgrp: Left-residuated partially ordered semigroups
RPoMag: Residuated partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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22. RPoMon: Residuated partially ordered monoids

Definition

A *residuated partially ordered monoid* (or *rpo-monoid*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that
$\langle A, \leq \rangle$ is a partially ordered set,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 28,\ f_5 = 186$

Subclasses

CRPoMon: Commutative residuated partially ordered monoids
IdRPoMon: Idempotent residuated partially ordered monoids

InPoMon: Involutive partially ordered monoids
Porim: Partially ordered residuated integral monoids
RJMon: Residuated join-semilattice-ordered monoids
RMMon: Residuated meet-semilattice-ordered monoids

Superclasses

LrPoMon: Left-residuated partially ordered monoids
LrPoSgrp: Left-residuated partially ordered semigroups
RPoSgrp: Residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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23. Porim: Partially ordered residuated integral monoids

Definition

A *partially ordered residuated integral monoid* is an rpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that
x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 365$

Subclasses

IRJMon: Integral residuated join-semilattice-ordered monoids
IRMMon: Meet-semilattice-ordered residuated integral monoids
InPorim: Involutive partially ordered integral monoids
Pocrim: Partially ordered commutative residuated integral monoids

Superclasses

Polrim: Partially ordered left-residuated integral monoids
RPoMon: Residuated partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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24. IdRPoSgrp: Idempotent residuated partially ordered semigroups

Definition

An *idempotent residuated partially ordered semigroup* is a residuated partially ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 169$

Subclasses

CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups

IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups

IdRPoMon: Idempotent residuated partially ordered monoids

Superclasses

IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups

RPoSgrp: Residuated partially ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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25. IdRPoMon: Idempotent residuated partially ordered monoids

Definition

An *idempotent residuated partially ordered monoid* is a residuated partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, /\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 32,\ f_6 = 148$

Subclasses

CIdRPoMon: Commutative idempotent residuated partially ordered monoids

IdRJMon: Idempotent residuated join-semilattice-ordered monoids

IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Superclasses

IdLrPoMon: Idempotent left-residuated partially ordered monoids

IdRPoSgrp: Idempotent residuated partially ordered semigroups

RPoMon: Residuated partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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26. GalPos: Galois posets

Definition

A *Galois poset* is a po-algebra $\mathbf{P} = \langle P, \leq, \sim, - \rangle$ such that P is a partially ordered set and $\sim, -$ are a pair of unary operations on P that form a
Galois connection: $x \leq {\sim}y \iff y \leq -x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

Basic Results

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 15,\ f_4 = 83,\ f_5 = 539$

Subclasses

GalJslat: Galois join-semilattices

GalMslat: Galois meet-semilattices

InPos: Involutive posets

Superclasses

PoNUn: Partially ordered negated unars

PoUn: Partially ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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27. InPos: Involutive posets

Definition

An *involutive poset* is a Galois poset $\mathbf{P} = \langle P, \leq, \sim, - \rangle$ such that $\sim, -$ are inverses of each other:

${\sim}{-}x = x$

$-{\sim}x = x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

${\sim}{-}x = x$

$-{\sim}x = x$

Basic Results

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 5,\ f_4 = 16,\ f_5 = 30,\ f_6 = 108$

Subclasses

InLat: Involutive lattices

InPoMag: Involutive partially ordered magmas

Superclasses

GalPos: Galois posets

Cont|Po|J|M|L|D|To|B|U|Ind

28. InPoMag: Involutive partially ordered magmas

Definition

An *involutive partially ordered magma* (or *inpo-magma*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\langle A, \leq, \cdot \rangle$ is a partially ordered magma,
$\sim, -$ is an involutive pair: $\sim - x = x = - \sim x$,
$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$ and
$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$.

Formal Definition

$\sim - x = x$

$- \sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 12,\ f_4 = 77,\ f_5 = 498$

Subclasses

CyInPoMag: Cyclic involutive partially ordered magmas
InLMag: Involutive lattice-ordered magmas
InPoSgrp: Involutive partially ordered semigroups

Superclasses

InPos: Involutive posets
RPoMag: Residuated partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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29. InPoSgrp: Involutive partially ordered semigroups

Definition

An *involutive partially ordered semigroup* (or *inpo-semigroup*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\langle A, \leq, \cdot \rangle$ is an involutive partially ordered magma and
$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$\sim - x = x$

$- \sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 10,\ f_4 = 50,\ f_5 = 210,\ f_6 = 1721$

Subclasses

CyInPoSgrp: Cyclic involutive partially ordered semigroups
InLSgrp: Involutive lattice-ordered semigroups
InPoMon: Involutive partially ordered monoids

Superclasses

InPoMag: Involutive partially ordered magmas

RPoSgrp: Residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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30. InPoMon: Involutive partially ordered monoids

Definition

An *involutive partially ordered monoid* (or *inpo-monoid*) is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that $\langle A, \leq, \cdot\rangle$ is an involutive partially ordered semigroup that has an
identity: $x \cdot 1 = x = 1 \cdot x$

Formal Definition

$\sim{-x} = x$

$-{\sim x} = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 5,\ f_4 = 20,\ f_5 = 39,\ f_6 = 179,\ f_7 = 500$

Subclasses

CyInPoMon: Cyclic involutive partially ordered monoids

InPorim: Involutive partially ordered integral monoids

PoGrp: Partially ordered groups

Superclasses

InPoSgrp: Involutive partially ordered semigroups

RPoMon: Residuated partially ordered monoids

RPoUn: Residuated partially ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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31. InPorim: Involutive partially ordered integral monoids

Definition

An *involutive partially ordered integral monoid* (or *in-porim*) is an involutive partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ that is
integral: $x \leq 1$

Formal Definition

$\sim{-x} = x$

$-{\sim x} = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 13,\ f_7 = 17,\ f_8 = 84$

Subclasses

CyInPorim: Cyclic involutive partially ordered integral monoids

Superclasses

InPoMon: Involutive partially ordered monoids

Porim: Partially ordered residuated integral monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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32. CyInPoMag: Cyclic involutive partially ordered magmas

Definition

A *cyclic involutive partially ordered magma* (or *cyinpo-magma*) is an inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 12,\ f_4 = 76,\ f_5 = 481$

Subclasses

CInPoMag: Commutative involutive partially ordered magmas

CyInLMag: Cyclic involutive lattice-ordered magmas

CyInPoSgrp: Cyclic involutive partially ordered semigroups

Superclasses

InPoMag: Involutive partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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33. CyInPoSgrp: Cyclic involutive partially ordered semigroups

Definition

A *cyclic involutive partially ordered semigroup* (or *cyinpo-semigroup*) is a cyinpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 10,\ f_4 = 50,\ f_5 = 196,\ f_6 = 1397$

Subclasses

CInPoSgrp: Commutative involutive partially ordered semigroups

CyInLSgrp: Cyclic involutive lattice-ordered semigroups
CyInPoMon: Cyclic involutive partially ordered monoids

Superclasses

CyInPoMag: Cyclic involutive partially ordered magmas
InPoSgrp: Involutive partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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34. CyInPoMon: Cyclic involutive partially ordered monoids

Definition

A *cyclic involutive partially ordered monoid* (or *cyinpo-monoid*) is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 5,\ f_4 = 20,\ f_5 = 39,\ f_6 = 176,\ f_7 = 493$

Subclasses

CInPoMon: Commutative involutive partially ordered monoids
CyInPorim: Cyclic involutive partially ordered integral monoids
PoGrp: Partially ordered groups

Superclasses

CyInPoSgrp: Cyclic involutive partially ordered semigroups
InPoMon: Involutive partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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35. CyInPorim: Cyclic involutive partially ordered integral monoids

Definition

A *cyclic involutive partially ordered integral monoid* (or *cyclic involutive porim*) is an involutive porim $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 15,\ f_8 = 79$

Subclasses

InPocrim: Involutive partially ordered commutative integral monoids

Superclasses

CyInPoMon: Cyclic involutive partially ordered monoids

InPorim: Involutive partially ordered integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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36. PoGrp: Partially ordered groups

Definition

A *partially ordered group* is a po-algebra $\mathbf{G} = \langle G, \cdot, ^{-1}, 1, \leq \rangle$ such that

$\langle G, \cdot, ^{-1}, 1 \rangle$ is a group

$\langle G, \leq \rangle$ is a partially ordered set

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x^{-1} \cdot x = 1$

$x \cdot x^{-1} = 1$

Examples

Example 1: The integers, the rationals and the reals with the usual order.

Basic Results

Any group is a partially ordered group with equality as partial order.

Any finite partially ordered group has only the equality relation as partial order.

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 5,\ f_9 = 2,\ f_{10} = 2$

Subclasses

AbPoGrp: Abelian partially ordered groups

LGrp: Lattice-ordered groups

Superclasses

CyInPoMon: Cyclic involutive partially ordered monoids

InPoMon: Involutive partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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37. CPoSgrp: Commutative partially ordered semigroups

Definition

A *commutative partially ordered semigroup* is a partially ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 7,\ f_3 = 83,\ f_4 = 1468,\ f_5 = 37248,\ f_6 = 1337698,\ f_7 = 71748346$

Subclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups
CJSgrp: Commutative join-semilattice-ordered semigroups
CMSgrp: Commutative meet-semilattice-ordered semigroups
CPoMon: Commutative partially ordered monoids
CRPoSgrp: Commutative residuated partially ordered semigroups

Superclasses

PoSgrp: Partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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38. CPoMon: Commutative partially ordered monoids

Definition

A *commutative partially ordered monoid* is a partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 27,\ f_4 = 301,\ f_5 = 4887$

Subclasses

CIPoMon: Commutative integral partially ordered monoids
CIdPoMon: Commutative idempotent partially ordered monoids
CJMon: Commutative join-semilattice-ordered monoids
CMMon: Commutative meet-semilattice-ordered monoids
CRPoMon: Commutative residuated partially ordered monoids

Superclasses

CPoSgrp: Commutative partially ordered semigroups
PoMon: Partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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39. CIPoMon: Commutative integral partially ordered monoids

Definition

A *commutative integral partially ordered monoid* is a integral partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 60,\ f_6 = 590$

Subclasses

CIJMon: Commutative Integral join-semilattice-ordered monoids
CIMMon: Commutative Integral meet-semilattice-ordered monoids
Pocrim: Partially ordered commutative residuated integral monoids

Superclasses

CPoMon: Commutative partially ordered monoids
IPoMon: Integral partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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40. CIdPoSgrp: Commutative idempotent partially ordered semigroups

Definition

A *commutative idempotent partially ordered semigroup* is a po-algebra $\mathbf{A} = \langle A, \leq, \cdot\rangle$ such that
$\langle A, \leq, \cdot\rangle$ is an idempotent partially ordered semigroup and
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 19,\ f_4 = 171,\ f_5 = 2069$

Subclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups
CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups
CIdPoMon: Commutative idempotent partially ordered monoids
CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

Superclasses

CPoSgrp: Commutative partially ordered semigroups

IdPoSgrp: Idempotent partially ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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41. CIdPoMon: Commutative idempotent partially ordered monoids

Definition

A *commutative idempotent partially ordered monoid* is an idempotent partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Basic Results

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 13,\ f_4 = 86,\ f_5 = 759$

Subclasses

CIdJMon: Commutative idempotent join-semilattice-ordered monoids

CIdMMon: Commutative idempotent meet-semilattice-ordered monoids

CIdRPoMon: Commutative idempotent residuated partially ordered monoids

Superclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups

CPoMon: Commutative partially ordered monoids

IdPoMon: Idempotent partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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42. CDivPos: Commutative division posets

Definition

A *commutative division partially ordered set* is a division poset $\mathbf{P} = \langle P, \leq, \backslash, /\rangle$ such that

$x \leq y \implies x/z \leq y/z$ and

$\backslash, /$ are commutative: $x/y = y\backslash x$.

Formal Definition

$x \leq y \implies x/z \leq y/z$

$x \leq z/y \iff y \leq x\backslash z$

$x/y = y\backslash x$

Basic Results

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 55,\ f_4 = 1434$

Subclasses

BCK: BCK-algebras
CDivJslat: Commutative division join-semilattices
CDivMslat: Commutative division meet-semilattices
CRPoMag: Commutative residuated partially ordered magmas

Superclasses

DivPos: Division posets Cont|Po|J|M|L|D|To|B|U|Ind

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43. BCK: BCK-algebras

Formal Definition

A *BCK-algebra* is an algebra $\langle A, \dot{-}, 0\rangle$ such that
(1) $((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) = 0$
(2) $x \dot{-} 0 = x$
(3) $x \dot{-} x = 0$
(4) $x \dot{-} y = y \dot{-} x = 0 \implies x = y$
The operation $\dot{-}$ satisfies the axioms of truncated subtraction or set-difference.

Definition

A *BCK-algebra* is an algebra $\langle A, \to, 1\rangle$ such that
(B) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$
(C) $x \to (y \to z) = y \to (x \to z)$
(K) $x \to (y \to x) = 1$
(4op) $x \to y = y \to x = 1 \implies x = y$
The name BCK-algebra comes from these equations. They are based on the λ-calculus combinators known as B, C, K.

Properties

Classtype	Quasivariety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No

Finite Members

Subclasses

BCKJslat: BCK-join-semilattices
BCKMslat: BCK-meet-semilattices
HilA: Hilbert algebras
Pocrim: Partially ordered commutative residuated integral monoids

Superclasses

CDivPos: Commutative division posets Cont|Po|J|M|L|D|To|B|U|Ind

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44. CRPoMag: Commutative residuated partially ordered magmas

Definition

A *commutative residuated partially ordered magma* is a residuated partially ordered magma $\mathbf{A} = \langle A, \leq, \cdot, \backslash, /\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 16,\ f_4 = 180,\ f_5 = 4761$

Subclasses

CInPoMag: Commutative involutive partially ordered magmas
CRJMag: Commutative residuated join-semilattice-ordered magmas
CRMMag: Commutative residuated meet-semilattice-ordered magmas
CRPoSgrp: Commutative residuated partially ordered semigroups

Superclasses

CDivPos: Commutative division posets
RPoMag: Residuated partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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45. HilA: Hilbert algebras

Definition

A *Hilbert algebra* is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 1\rangle$ such that
$x \to (y \to x) = 1$
$(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$
$x \to y = 1$ and $y \to x = 1 \implies x = y$

Definition

A *Hilbert algebra* is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 1\rangle$ such that
$x \to x = 1$
$1 \to x = x$
$x \to (y \to z) = (x \to y) \to (x \to z)$
$(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$

Formal Definition

$x \leq y \iff x \to y = 1$
$x \to x = 1$
$1 \to x = x$
$x \to (y \to z) = (x \to y) \to (x \to z)$
$(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$

Examples

Example 1: Given any poset with top element 1, $\langle A, \leq, 1\rangle$, define $a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise.} \end{cases}$ Then $\langle A, \to, 1\rangle$ is a Hilbert algebra.

Basic Results

Hilbert algebras are algebraic models of the implicational fragment of intuitionistic logic, i. e., they are $(\to, 1)$-subreducts of Heyting algebras.

The variety of Hilbert algebras is not generated as a quasivariety by any of its finite members Celani and Cabrer [2005].

Properties

Classtype	variety Diego [1966]
Locally finite	yes
Congruence distributive	yes
Congruence 1-regular	yes
Congruence extension property	yes
Equationally def. pr. cong.	yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 21,\ f_6 = 95$

Subclasses

TarA: Tarski algebras

Superclasses

BCK: BCK-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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46. TarA: Tarski algebras

Definition

A *Tarski algebra* is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 1\rangle$ such that $x \to x = 1$

$x \to (y \to x) = x$

$(x \to y) \to y = (y \to x) \to x$

$x \to (y \to z) = y \to (x \to z)$

Formal Definition

$x \le y \iff x \to y = 1$

$1 \to x = x$

$x \to x = 1$

$x \to (y \to z) = (x \to y) \to (x \to z)$

$(x \to y) \to y = (y \to x) \to x$

Basic Results

Tarski algebras are algebraic models of the implicational fragment of classical logic, i. e., they are $(\to, 1)$-subreducts of Boolean algebras.

Properties

Classtype	Variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 2,\ f_6 = 3,\ f_7 = 5,\ f_8 = 8,\ f_9 = 11,\ f_{10} = 18$

Subclasses

Superclasses

HilA: Hilbert algebras Cont|Po|J|M|L|D|To|B|U|Ind

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47. CRPoSgrp: Commutative residuated partially ordered semigroups

Definition

A *commutative residuated partially ordered semigroup* is a residuated partially ordered semigroup $\mathbf{A} = \langle A, \le, \cdot, \backslash, /\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 12,\ f_4 = 76,\ f_5 = 670$

Subclasses

CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

CInPoSgrp: Commutative involutive partially ordered semigroups

CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

CRPoMon: Commutative residuated partially ordered monoids

Superclasses

CPoSgrp: Commutative partially ordered semigroups

CRPoMag: Commutative residuated partially ordered magmas

RPoSgrp: Residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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48. CRPoMon: Commutative residuated partially ordered monoids

Definition

A *commutative residuated partially ordered monoid* is a residuated partially ordered monoid $\mathbf{A} = \langle A, \le, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Remark: These algebras are also known as *lineales*.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 24,\ f_5 = 131,\ f_6 = 1001$

Subclasses

CIdRPoMon: Commutative idempotent residuated partially ordered monoids

CInPoMon: Commutative involutive partially ordered monoids

CRJMon: Commutative residuated join-semilattice-ordered monoids

CRMMon: Commutative residuated meet-semilattice-ordered monoids

Pocrim: Partially ordered commutative residuated integral monoids

Superclasses

CPoMon: Commutative partially ordered monoids

CRPoSgrp: Commutative residuated partially ordered semigroups

RPoMon: Residuated partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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49. Pocrim: Partially ordered commutative residuated integral monoids

Definition

A *partially ordered residuated integral monoid* is a porim $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety
Congruence distributive	Yes (relatively) Blok and Raftery [1997]
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129$

Subclasses

CIRJMon: Commutative integral residuated join-semilattice-ordered monoids

CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

InPocrim: Involutive partially ordered commutative integral monoids

Superclasses

BCK: BCK-algebras

CIPoMon: Commutative integral partially ordered monoids

CRPoMon: Commutative residuated partially ordered monoids

Porim: Partially ordered residuated integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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50. CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

Definition

A *commutative idempotent residuated partially ordered semigroup* is an idempotent residuated partially ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 36,\ f_6 = 203$

Subclasses

CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

CIdRPoMon: Commutative idempotent residuated partially ordered monoids

Superclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups

CRPoSgrp: Commutative residuated partially ordered semigroups

IdRPoSgrp: Idempotent residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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51. CIdRPoMon: Commutative idempotent residuated partially ordered monoids

Definition

A *commutative idempotent residuated partially ordered monoid* is an idmpotent residuated partially ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 20,\ f_6 = 78$

Subclasses

CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Superclasses

CIdPoMon: Commutative idempotent partially ordered monoids

CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

CRPoMon: Commutative residuated partially ordered monoids

IdRPoMon: Idempotent residuated partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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52. CInPoMag: Commutative involutive partially ordered magmas

Definition

A *commutative involutive partially ordered magma* (or *cinpo-magma*) is a inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 12,\ f_4 = 69,\ f_5 = 354,\ f_6 = 3632$

Subclasses

CInLMag: Commutative involutive lattice-ordered magmas

CInPoSgrp: Commutative involutive partially ordered semigroups

Superclasses

CRPoMag: Commutative residuated partially ordered magmas

CyInPoMag: Cyclic involutive partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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53. CInPoSgrp: Commutative involutive partially ordered semigroups

Definition

A *commutative involutive partially ordered semigroup* (or *cinpo-semigroup*) is a inpo-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 10,\ f_4 = 50,\ f_5 = 194,\ f_6 = 1356$

Subclasses

CInLSgrp: Commutative involutive lattice-ordered semigroups

CInPoMon: Commutative involutive partially ordered monoids

Superclasses

CInPoMag: Commutative involutive partially ordered magmas

CRPoSgrp: Commutative residuated partially ordered semigroups

CyInPoSgrp: Cyclic involutive partially ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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54. CInPoMon: Commutative involutive partially ordered monoids

Definition

A *commutative involutive partially ordered monoid* (or *cinpo-monoid*) is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 5,\ f_4 = 20,\ f_5 = 39,\ f_6 = 174,\ f_7 = 488$

Subclasses

AbPoGrp: Abelian partially ordered groups

InPocrim: Involutive partially ordered commutative integral monoids

Superclasses

CInPoSgrp: Commutative involutive partially ordered semigroups

CRPoMon: Commutative residuated partially ordered monoids

CyInPoMon: Cyclic involutive partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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55. InPocrim: Involutive partially ordered commutative integral monoids

Definition

An *involutive partially ordered commutative integral monoid* (or *in-pocrim*) is an in-porim $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

$x \cdot 1 = x$

$x \leq 1$

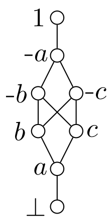
Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 15,\ f_8 = 73,\ f_9 = 116$

Small Members (not in any subclass)



$\mathbf{Pos}_{8,1}$

$\cdot$	-a	-c	-b	c	b	a
-a	a	a	a	a	a	$\bot$
-c	a	a	a	$\bot$	a	$\bot$
-b	a	a	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	a	a	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,1}$

$\cdot$	-a	-c	-b	c	b	a
-a	b	b	a	a	a	$\bot$
-c	b	b	a	$\bot$	a	$\bot$
-b	a	a	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	a	a	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,2}$

$\cdot$	-a	-c	-b	c	b	a
-a	-c	-c	b	a	b	$\bot$
-c	-c	**-c**	b	$\bot$	b	$\bot$
-b	b	b	a	a	$\bot$	$\bot$
c	a	$\bot$	a	$\bot$	$\bot$	$\bot$
b	b	b	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\mathbf{InPocrim}_{8,3}$

Subclasses

CIInFL: Commutative integral involutive FL-algebras

Superclasses

CInPoMon: Commutative involutive partially ordered monoids

CyInPorim: Cyclic involutive partially ordered integral monoids

Pocrim: Partially ordered commutative residuated integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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56. AbPoGrp: Abelian partially ordered groups

Definition

An *abelian partially ordered group* is a partially ordered group $\mathbf{A} = \langle A, \cdot, ^{-1}, 1, \leq \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x^{-1} \cdot x = 1$

$x \cdot x^{-1} = 1$

$x \cdot y = y \cdot x$

Properties

Classtype	po-variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1,\ f_8 = 3,\ f_9 = 2,\ f_{10} = 1$

Subclasses

AbLGrp: Abelian lattice-ordered groups

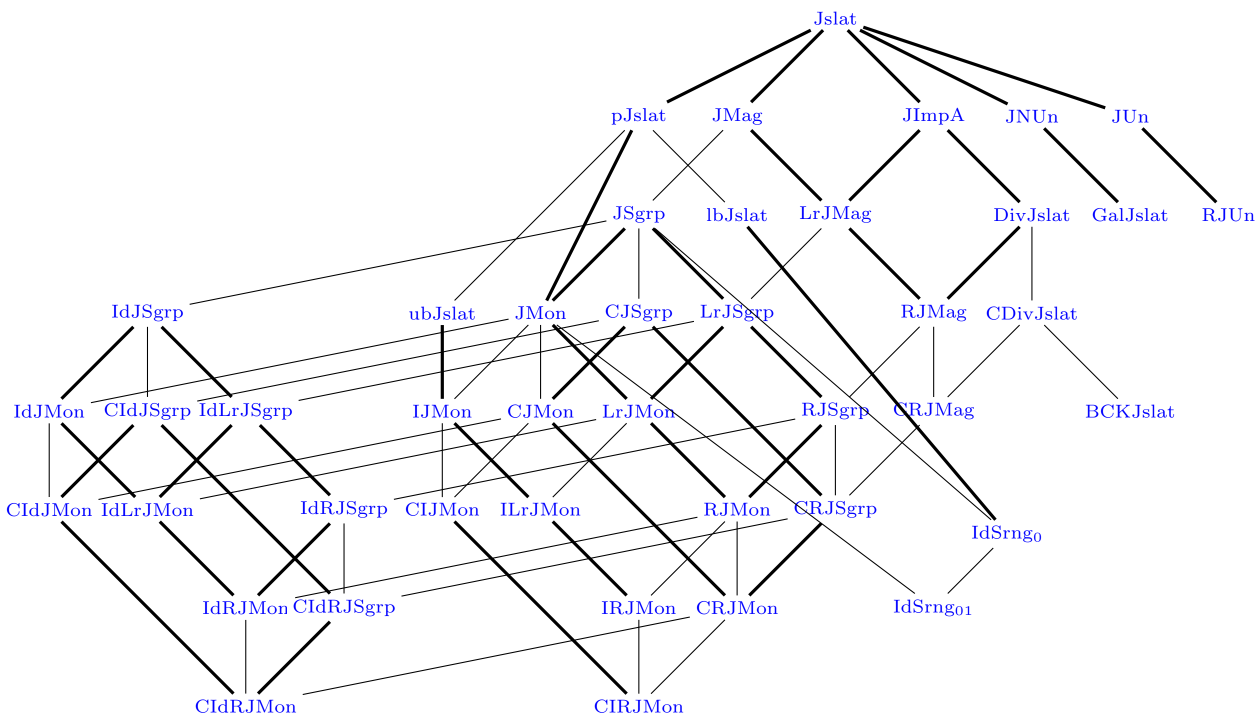
Superclasses

CInPoMon: Commutative involutive partially ordered monoids

PoGrp: Partially ordered groups Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 3

Join-semilattice-ordered algebras



In this chapter (and Chapters 5-9) the binary operation $\cdot$ (if present) is assumed to distribute over the join operation in both arguments. For the algebras where $\cdot$ has both residuals, this is a consequence of residuation, but for the other classes we add this property as two equational axioms (one suffices for left-residuated algebras). As for partially ordered algebras, we view these axioms as part of the ordertype (signature) of the algebra. They ensure that in a suitable complete extension $\cdot$ will have both residuals, and hence a convenient display calculus.

1. Jslat: Join-semilattices

Definition

A *join-semilattice* is an algebra $\langle S, \vee\rangle$ such that S is a set and $\vee$ is a binary operation on S that is

- associative: $(x \vee y) \vee z = x \vee (y \vee z)$
- commutative: $x \vee y = y \vee x$
- idempotent: $x \vee x = x$ and
- partially ordered: $x \le y \iff x \vee y = y$

Definition

A *join-semilattice* is an algebra $\mathbf{S} = \langle S, \leq, \vee \rangle$, where $\vee$ is an infix binary operation, called the *join*, such that
$\leq$ is a partial order,
$x \leq y \implies x \vee z \leq y \vee z$ and $z \vee x \leq z \vee y$,
$x \leq x \vee y$ and $y \leq x \vee y$,
$x \vee x \leq x$.
This definition shows that semilattices form a partially-ordered variety.

Definition

A *join-semilattice* is an algebra $\mathbf{S} = \langle S, \vee \rangle$, where $\vee$ is an infix binary operation, called the *join*, such that
$\leq$ is a partial order, where $x \leq y \iff x \vee y = y$
$x \vee y$ is the least upper bound of $\{x, y\}$.

Definition

A *meet-semilattice* is an algebra $\mathbf{S} = \langle S, \wedge \rangle$, where $\wedge$ is an infix binary operation, called the *meet*, such that
$\leq$ is a partial order, where $x \leq y \iff x \wedge y = x$
$x \wedge y$ is the greatest lower bound of $\{x, y\}$.

Formal Definition

associative: $(x \vee y) \vee z = x \vee (y \vee z)$
commutative: $x \vee y = y \vee x$
idempotent: $x \vee x = x$ and
partially ordered: $x \leq y \iff x \vee y = y$

Examples

Example 1: $\langle \mathcal{P}_\omega(X) - \{\emptyset\}, \cup \rangle$, the set of finite nonempty subsets of a set X, with union, is the free join-semilattice with singleton subsets of X as generators.

Properties

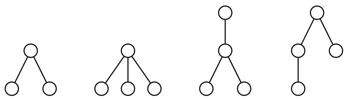
Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	Yes
Residual size	2
Congruence distributive	No
Congruence modular	No
Congruence meet-semidistributive	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 5$, $f_5 = 15$, $f_6 = 53$, $f_7 = 222$, $f_8 = 1078$, $f_9 = 5994$, $f_{10} = 37622$, $f_{11} = 262776$, $f_{12} = 2018305$, $f_{13} = 16873364$, $f_{14} = 152233518$, $f_{15} = 1471613387$, $f_{16} = 15150569446$, $f_{17} = 165269824761$

These results follow from Heitzig and Reinhold [2002] and the observation that semilattices with n elements are in 1-1 correspondence to lattices with $n+1$ elements.

Small Members (not in any subclass)



$\mathbf{S}_{3,1}$ $\mathbf{S}_{4,1}$ $\mathbf{S}_{4,2}$ $\mathbf{S}_{4,3}$

Subclasses

JImpA: Join-semilattice-ordered implication algebras

JMag: Join-semilattice-ordered magmas

JNUn: Join-semilattice-ordered negated unars

JUn: Join-semilattice-ordered unars

Lat: Lattices

pJslat: Pointed join-semilattices

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pJslat: Pointed join-semilattices

Definition

A *pointed join-semilattice* is an algebra $\langle S, \vee, c\rangle$ such that S is a join-semilattice and c is a constant operation on S.

Formal Definition

$c = c$

Basic Results

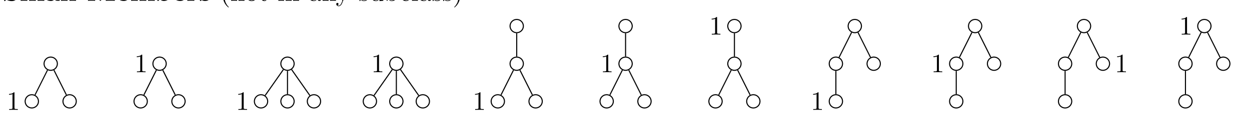
Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 16,\ f_5 = 60,\ f_6 = 262,\ f_7 = 1315$

Small Members (not in any subclass)



$\mathbf{pS}_{3,1}$ $\mathbf{pS}_{3,2}$ $\mathbf{pS}_{4,1}$ $\mathbf{pS}_{4,2}$ $\mathbf{pS}_{4,3}$ $\mathbf{pS}_{4,4}$ $\mathbf{pS}_{4,5}$ $\mathbf{pS}_{4,6}$ $\mathbf{pS}_{4,7}$ $\mathbf{pS}_{4,8}$ $\mathbf{pS}_{4,9}$

Subclasses

JMon: Join-semilattice-ordered monoids

lbJslat: Lower-bounded join-semilattices

pLat: Pointed lattices

ubJslat: Upper-bounded join-semilattices

Superclasses

Jslat: Join-semilattices

pPos: Pointed posets

Cont|Po|J|M|L|D|To|B|U|Ind

3. lbJslat: Lower-bounded join-semilattices

Definition

A *lower bounded join-semilattice* is an algebra $\mathbf{S} = \langle S, \vee, \bot\rangle$ such that

$\langle S, \cdot\rangle$ is a join-semilattice and

$\bot$ is an indentity for $\vee$: $x \vee \bot = x$

Formal Definition

$x \vee \bot = x$

Properties

Classtype	Variety
Equational theory	Decidable in PTIME
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No

Finite Members

Same as for lattices (since every complete semilattice is a lattice).

Subclasses

IdSrng_0: Idempotent semirings with zero

bLat: Bounded lattices

Superclasses

pJslat: Pointed join-semilattices Cont|Po|J|M|L|D|To|B|U|Ind

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4. ubJslat: Upper-bounded join-semilattices

Definition

An *upper-bounded join-semilattice* is an algebra $\mathbf{S} = \langle S, \vee, \top\rangle$ such that

$\langle S, \vee\rangle$ is a join-semilattice and

$\top$ is absorbing for $\vee$: $x \vee \top = \top$

Formal Definition

$x \vee \top = \top$

Properties

Classtype	Variety
Equational theory	Decidable in PTIME
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	Yes
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No

Finite Members

Same as for join-semilattices (since every complete join-semilattice has a top element).

Subclasses

IJMon: Integral join-semilattice-ordered monoids

bLat: Bounded lattices

Superclasses

pJslat: Pointed join-semilattices Cont|Po|J|M|L|D|To|B|U|Ind

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5. JUn: Join-semilattice-ordered unars

Definition

A *join-semilattice-ordered unar* (also called a *j-unar* for short) is an algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a join-semilattice and f is a unary operation on P that is

order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$f(x \vee y) = f(x) \vee f(y)$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 16,\ f_4 = 104,\ f_5 = 822$

Subclasses

GalJslat: Galois join-semilattices

LUn: Lattice-ordered unars

RJUn: Residuated join-semilattice-ordered unars

Superclasses

Jslat: Join-semilattices

PoUn: Partially ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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6. JNUn: Join-semilattice-ordered negated unars

Definition

A *join-semilattice-ordered negated unar* (also called a *jn-nunar* for short) is an algebra $\mathbf{P} = \langle P, \leq, \sim\rangle$ such that P is a join-semilattice and $\sim$ is a unary operation on P that is

order-reversing: $x \leq y \implies {\sim}y \leq {\sim}x$

Formal Definition

$x \leq y \implies {\sim}y \leq {\sim}x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 15,\ f_4 = 113,\ f_5 = 1167$

Subclasses

GalJslat: Galois join-semilattices

LNUn: Lattice-ordered negated unars

Superclasses

Jslat: Join-semilattices

PoNUn: Partially ordered negated unars Cont|Po|J|M|L|D|To|B|U|Ind

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7. GalJslat: Galois join-semilattices

Definition

A *Galois join-semilattice* is an algebra $\mathbf{P} = \langle P, \leq, \sim, -\rangle$ such that P is a join-semilattice and $\sim, -$ are a pair of unary operations on P that form a

Galois connection: $x \leq \sim y \iff y \leq -x$

Formal Definition

$x \leq \sim y \iff y \leq -x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 9,\ f_4 = 52,\ f_5 = 361,\ f_6 = 2947$

Subclasses

GalLat: Galois lattices

Superclasses

GalPos: Galois posets

JNUn: Join-semilattice-ordered negated unars

JUn: Join-semilattice-ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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8. JMag: Join-semilattice-ordered magmas

Definition

A *join-semilattice-ordered magma* or *multiplicative semilattice* (or *m-semilattice*, [Birkhoff, 1979, p. 323]) is an algebra $\mathbf{A} = \langle A, \vee, \cdot\rangle$ of type $\langle 2, 2\rangle$ such that

$\langle A, \vee\rangle$ is a semilattice

$\cdot$ distributes over $\vee$: $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 220$

Subclasses

JSgrp: Join-semilattice-ordered semigroups

LMag: Lattice-ordered magmas

LrJMag: Left-residuated join-semilattice-ordered magmas

Superclasses

Jslat: Join-semilattices

PoMag: Partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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9. JSgrp: Join-semilattice-ordered semigroups

Definition

A *join-semilattice-ordered semigroup* or *(additively) idempotent semiring* is an algebra $\mathbf{A} = \langle A, \vee, \cdot\rangle$ such that

$\langle A, \cdot\rangle$ is a semigroup

$\langle A, \vee\rangle$ is a join-semilattice

$\cdot$ is *joinpreserving*: $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$ and $(x \vee y) \cdot z = x \cdot z \vee y \cdot z$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 61,\ f_4 = 866$

Subclasses

CJSgrp: Commutative join-semilattice-ordered semigroups

IdJSgrp: Idempotent join-semilattice-ordered semigroups

JMon: Join-semilattice-ordered monoids

LSgrp: Lattice-ordered semigroups

LrJSgrp: Left-residuated join-semilattice-ordered semigroups

Superclasses

JMag: Join-semilattice-ordered magmas

PoSgrp: Partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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10. JMon: Join-semilattice-ordered monoids

Definition

A *join-semilattice-ordered monoid* or *(additively) idempotent unital semiring* is an algebra $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a monoid,

$\langle A, \vee\rangle$ is a join-semilattice and

$\cdot$ is *join-preserving*: $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$ and $(x \vee y) \cdot z = x \cdot z \vee y \cdot z$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Basic Results

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 11,\ f_4 = 73,\ f_5 = 703$

Subclasses

CJMon: Commutative join-semilattice-ordered monoids

IJMon: Integral join-semilattice-ordered monoids

IdJMon: Idempotent join-semilattice-ordered monoids

IdSrng_{01}: Idempotent semirings with identity and zero

LMon: Lattice-ordered monoids

LrJMon: Left-residuated join-semilattice-ordered monoids

Superclasses

JSgrp: Join-semilattice-ordered semigroups

PoMon: Partially ordered monoids

pJslat: Pointed join-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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11. $\mathbf{IdSrng_0}$: Idempotent semirings with zero

Definition

An *idempotent semiring with zero* is a semiring with zero $\mathbf{S} = \langle S, \vee, 0, \cdot\rangle$ such that $\vee$ is idempotent: $x \vee x = x$

Definition

An *idempotent semiring with and zero* is an algebra $\mathbf{S} = \langle S, \vee, 0, \cdot\rangle$ such that $\mathbf{S} = \langle S, \vee, \cdot\rangle$ is a join-semilattice-ordered semigroup,

0 is the bottom element: $x \vee 0 = x$ and

0 is absorbing: $x \cdot 0 = 0 = 0 \cdot x$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \vee 0 = x$

$x \cdot 0 = 0,\ 0 \cdot x = 0$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 129,\ f_5 = 1852$

Subclasses

IdSrng_{01}: Idempotent semirings with identity and zero

Superclasses

Srng_0: Semirings with zero

lbJslat: Lower-bounded join-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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12. IdSrng_{01}: Idempotent semirings with identity and zero

Definition

An *idempotent semiring with identity and zero* is a semiring with identity and zero $\mathbf{S} = \langle S, \vee, 0, \cdot, 1\rangle$ such that

$\vee$ is idempotent: $x \vee x = x$ ($1 \vee 1 = 1$ is sufficient).

Definition

An *idempotent semiring with identity and zero* is an algebra $\mathbf{S} = \langle S, \vee, 0, \cdot, 1\rangle$ such that $\mathbf{S} = \langle S, \vee, \cdot, 1\rangle$ is a join-semilattice-ordered monoid,

0 is the bottom element: $x \vee 0 = x$ and

0 is absorbing: $x \cdot 0 = 0 = 0 \cdot x$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \vee 0 = x$

$x \cdot 0 = 0,\ 0 \cdot x = 0$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence meet-semidistributive	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488,\ f_7 = 18554$

Subclasses

KA: Kleene algebras

Superclasses

IdSrng_0: Idempotent semirings with zero

JMon: Join-semilattice-ordered monoids

Srng_{01}: Semirings with identity and zero

Cont|Po|J|M|L|D|To|B|U|Ind

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13. KA: Kleene algebras

Definition

A *Kleene algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \cdot, 1, ^*\rangle$ of type $\langle 2, 0, 2, 0, 1\rangle$ such that $\langle A, \vee, 0, \cdot, 1\rangle$ is an idempotent semiring with identity and zero

$e \vee x \vee x^* x^* = x^*$

$x \cdot y \leq y \implies x^* y = y$

$y \cdot x \leq y \implies yx^* = y$

Properties

Classtype	Quasivariety
Equational theory	Decidable, PSPACE complete Stockmeyer and Meyer [1973]
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence meet-semidistributive	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488$

Subclasses

KLat: Kleene lattices

Superclasses

IdSrng_{01}: Idempotent semirings with identity and zero

Cont|Po|J|M|L|D|To|B|U|Ind

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14. IJMon: Integral join-semilattice-ordered monoids

Definition

An *integral join-semilattice-ordered monoid* is a join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that x is integral: $x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

CIJMon: Commutative Integral join-semilattice-ordered monoids

ILMon: Integral lattice-ordered monoids

ILrJMon: Integral left-residuated join-semilattice-ordered monoids

Superclasses

IPoMon: Integral partially ordered monoids

JMon: Join-semilattice-ordered monoids

ubJslat: Upper-bounded join-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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15. IdJSgrp: Idempotent join-semilattice-ordered semigroups

Definition

An *idempotent join-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \vee, \cdot\rangle$ such that
$\langle A, \vee, \cdot\rangle$ is a join-semilattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 23,\ f_4 = 166,\ f_5 = 1379$

Subclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups
IdJMon: Idempotent join-semilattice-ordered monoids
IdLSgrp: Idempotent lattice-ordered semigroups
IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups

Superclasses

IdPoSgrp: Idempotent partially ordered semigroups
JSgrp: Join-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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16. IdJMon: Idempotent join-semilattice-ordered monoids

Definition

An *idempotent join-semilattice-ordered monoid* is a join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 29,\ f_5 = 136$

Subclasses

CIdJMon: Commutative idempotent join-semilattice-ordered monoids
IdLMon: Idempotent lattice-ordered monoids
IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids

Superclasses

IdJSgrp: Idempotent join-semilattice-ordered semigroups

IdPoMon: Idempotent partially ordered monoids
JMon: Join-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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17. JImpA: Join-semilattice-ordered implication algebras

Formal Definition

$x \leq y \implies y \to z \leq x \to z$
$x \leq y \implies z \to x \leq z \to y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 245$

Subclasses

DivJslat: Division join-semilattices
LImpA: Lattice-ordered implication algebras
LrJMag: Left-residuated join-semilattice-ordered magmas

Superclasses

Jslat: Join-semilattices
PoImpA: Partially ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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18. LrJMag: Left-residuated join-semilattice-ordered magmas

Definition

A *left-residuated join-semilattice-ordered magma* (or *lrj-magma*) is an algebra $\mathbf{A} = \langle A, \vee, \cdot, \backslash \rangle$ such that
$\langle A, \vee \rangle$ is a join-semilattice,
$\langle A, \cdot \rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 52,\ f_4 = 4827$

Subclasses

LrJSgrp: Left-residuated join-semilattice-ordered semigroups
LrLMag: Left-residuated lattice-ordered magmas
RJMag: Residuated join-semilattice-ordered magmas

Superclasses

JImpA: Join-semilattice-ordered implication algebras
JMag: Join-semilattice-ordered magmas
LrPoMag: Left-residuated partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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19. LrJSgrp: Left-residuated join-semilattice-ordered semigroups

Definition

A *left-residuated join-semilattice-ordered semigroup* (or *lrj-semigroup*) is an algebra $\mathbf{A} = \langle A, \vee, \cdot, \backslash \rangle$ such that
$\langle A, \vee \rangle$ is a join-semilattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 19,\ f_4 = 192$

Subclasses

IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups
LrJMon: Left-residuated join-semilattice-ordered monoids
LrLSgrp: Left-residuated lattice-ordered semigroups
RJMon: Residuated join-semilattice-ordered monoids
RJSgrp: Residuated join-semilattice-ordered semigroups

Superclasses

JSgrp: Join-semilattice-ordered semigroups
LrJMag: Left-residuated join-semilattice-ordered magmas
LrPoSgrp: Left-residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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20. LrJMon: Left-residuated join-semilattice-ordered monoids

Definition

A *left-residuated join-semilattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash \rangle$ such that
$\langle A, \vee \rangle$ is a join-semilattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 23,\ f_5 = 169,\ f_6 = 1635$

Subclasses

ILrJMon: Integral left-residuated join-semilattice-ordered monoids

IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids
LrLMon: Left-residuated lattice-ordered monoids
RJMon: Residuated join-semilattice-ordered monoids

Superclasses

JMon: Join-semilattice-ordered monoids
LrJSgrp: Left-residuated join-semilattice-ordered semigroups
LrPoMon: Left-residuated partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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21. ILrJMon: Integral left-residuated join-semilattice-ordered monoids

Definition

A *join-semilattice-ordered left-residuated integral monoid* (or *ILrJMon* for short) is a left-residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash \rangle$ for which
$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

ILrLMon: Integral left-residuated lattice-ordered monoids
IRJMon: Integral residuated join-semilattice-ordered monoids

Superclasses

IJMon: Integral join-semilattice-ordered monoids
LrJMon: Left-residuated join-semilattice-ordered monoids
Polrim: Partially ordered left-residuated integral monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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22. IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups

Definition

An *idempotent left-residuated join-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \vee, \cdot \rangle$ such that
$\langle A, \vee, \cdot \rangle$ is a left-residuated join-semilattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 45,\ f_5 = 304$

Subclasses

IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids

IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups

IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups

Superclasses

IdJSgrp: Idempotent join-semilattice-ordered semigroups

IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups

LrJSgrp: Left-residuated join-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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23. IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids

Definition

An *idempotent left-residuated join-semilattice-ordered monoid* is a left-residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 11,\ f_5 = 46,\ f_6 = 215,\ f_7 = 1114$

Subclasses

IdLrLMon: Idempotent left-residuated lattice-ordered monoids

IdRJMon: Idempotent residuated join-semilattice-ordered monoids

Superclasses

IdJMon: Idempotent join-semilattice-ordered monoids

IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups

IdLrPoMon: Idempotent left-residuated partially ordered monoids

LrJMon: Left-residuated join-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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24. RJUn: Residuated join-semilattice-ordered unars

Formal Definition

A *residuated join-semilattice-ordered unar* (also called a *jsl-unar* for short) is a po-algebra $\mathbf{S} = \langle S, \vee, f, g\rangle$ such that $\langle S, \vee\rangle$ is a join-semilatttice-ordered set and f, g are unary operations on S that g is the upper residual of f, or equivalently, g is the right adjoint of f:

$f(x) \leq y \iff x \leq g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all joins and g preserves all existing meets.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

Subclasses

RLUn: Residuated lattice-ordered unars

Superclasses

JUn: Join-semilattice-ordered unars

RPoUn: Residuated partially ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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25. DivJslat: Division join-semilattices

Definition

A *division join-semilattice* is an algebra $\mathbf{S} = \langle S, \vee, \backslash, /\rangle$ such that $\langle S, \vee\rangle$ is a join-semilattice,

$x \leq y \implies z\backslash x \leq z\backslash y$,

$x \leq y \implies x/z \leq y/z$ and

$x \leq z/y \iff y \leq x\backslash z$

Formal Definition

$x \leq y \implies z\backslash x \leq z\backslash y$,

$x \leq y \implies x/z \leq y/z$ and

$x \leq z/y \iff y \leq x\backslash z$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 281$

Subclasses

CDivJslat: Commutative division join-semilattices

DivLat: Division lattices

RJMag: Residuated join-semilattice-ordered magmas

Superclasses

DivPos: Division posets

JImpA: Join-semilattice-ordered implication algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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26. RJMag: Residuated join-semilattice-ordered magmas

Definition

A *residuated join-semilattice-ordered magma* (or *rpo-magma*) is an algebra $\mathbf{A} = \langle A, \vee, \cdot, \backslash, /\rangle$ such that

$\langle A, \vee\rangle$ is a join-semilattice,

$\langle A, \cdot\rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 20,\ f_4 = 1116$

Subclasses

CRJMag: Commutative residuated join-semilattice-ordered magmas

RJSgrp: Residuated join-semilattice-ordered semigroups

RLMag: Residuated lattice-ordered magmas

Superclasses

DivJslat: Division join-semilattices

LrJMag: Left-residuated join-semilattice-ordered magmas

RPoMag: Residuated partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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27. RJSgrp: Residuated join-semilattice-ordered semigroups

Definition

A *residuated join-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \vee, \cdot, \backslash, / \rangle$ such that

$\langle A, \vee \rangle$ is a join-semilattice,

$\langle A, \cdot \rangle$ is a semigroup and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x \backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \le z \iff x \le z/y$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 129,\ f_5 = 1852$

Subclasses

CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups

RJMon: Residuated join-semilattice-ordered monoids

RLSgrp: Residuated lattice-ordered semigroups

Superclasses

LrJSgrp: Left-residuated join-semilattice-ordered semigroups

RJMag: Residuated join-semilattice-ordered magmas

RPoSgrp: Residuated partially ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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28. RJMon: Residuated join-semilattice-ordered monoids

Definition

A *residuated join-semilattice-ordered monoid* (or *rpj-monoid*) is an algebra $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that
$\langle A, \vee \rangle$ is a join-semilattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488$

Subclasses

CRJMon: Commutative residuated join-semilattice-ordered monoids
IRJMon: Integral residuated join-semilattice-ordered monoids
IdRJMon: Idempotent residuated join-semilattice-ordered monoids

Superclasses

LrJMon: Left-residuated join-semilattice-ordered monoids
LrJSgrp: Left-residuated join-semilattice-ordered semigroups
RJSgrp: Residuated join-semilattice-ordered semigroups
RPoMon: Residuated partially ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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29. IRJMon: Integral residuated join-semilattice-ordered monoids

Definition

An *integral residuated join-semilattice-ordered monoid* is a residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that
x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364,\ f_7 = 3335$

Subclasses

CIRJMon: Commutative integral residuated join-semilattice-ordered monoids

Superclasses

ILrJMon: Integral left-residuated join-semilattice-ordered monoids

Porim: Partially ordered residuated integral monoids

RJMon: Residuated join-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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30. IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups

Definition

An *idempotent residuated join-semilattice-ordered semigroup* is a residuated join-semilattice-ordered semigroup $\mathbf{A} = \langle A, \vee, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 169,\ f_6 = 1404$

Subclasses

CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

IdRJMon: Idempotent residuated join-semilattice-ordered monoids

IdRLSgrp: Idempotent residuated lattice-ordered semigroups

Superclasses

IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups

IdRPoSgrp: Idempotent residuated partially ordered semigroups

RJSgrp: Residuated join-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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31. IdRJMon: Idempotent residuated join-semilattice-ordered monoids

Definition

An *idempotent residuated join-semilattice-ordered monoid* is a residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 32,\ f_6 = 147,\ f_7 = 759$

Subclasses

CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

Superclasses

IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids
IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups
IdRPoMon: Idempotent residuated partially ordered monoids
RJMon: Residuated join-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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32. CJSgrp: Commutative join-semilattice-ordered semigroups

Definition

A *commutative join-semilattice-ordered semigroup* is a join-semilattice-ordered semigroup $\mathbf{A} = \langle A, \vee, \cdot\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 29,\ f_4 = 289$

Subclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups
CJMon: Commutative join-semilattice-ordered monoids
CLSgrp: Commutative lattice-ordered semigroups
CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

Superclasses

CPoSgrp: Commutative partially ordered semigroups
JSgrp: Join-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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33. CJMon: Commutative join-semilattice-ordered monoids

Definition

A *commutative join-semilattice-ordered monoid* is a join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 9,\ f_4 = 55,\ f_5 = 437$

Subclasses

CIJMon: Commutative Integral join-semilattice-ordered monoids

CIdJMon: Commutative idempotent join-semilattice-ordered monoids

CLMon: Commutative lattice-ordered monoids

CRJMon: Commutative residuated join-semilattice-ordered monoids

Superclasses

CJSgrp: Commutative join-semilattice-ordered semigroups

CPoMon: Commutative partially ordered monoids

JMon: Join-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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34. CIJMon: Commutative Integral join-semilattice-ordered monoids

Definition

A *commutative integral join-semilattice-ordered monoid* is a integral join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129$

Subclasses

CILMon: Commutative Integral lattice-ordered monoids

CIRJMon: Commutative integral residuated join-semilattice-ordered monoids

Superclasses

CIPoMon: Commutative integral partially ordered monoids

CJMon: Commutative join-semilattice-ordered monoids

IJMon: Integral join-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

35. CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups

Definition

A *commutative idempotent join-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \vee, \cdot\rangle$ such that $\langle A, \vee, \cdot, \rangle$ is an idempotent join-semilattice-ordered semigroup and
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 33,\ f_5 = 185$

Subclasses

CIdJMon: Commutative idempotent join-semilattice-ordered monoids
CIdLSgrp: Commutative idempotent lattice-ordered semigroups
CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

Superclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups
CJSgrp: Commutative join-semilattice-ordered semigroups
IdJSgrp: Idempotent join-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

36. CIdJMon: Commutative idempotent join-semilattice-ordered monoids

Definition

A *commutative idempotent join-semilattice-ordered monoid* is an idempotent join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1\rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 17,\ f_5 = 66,\ f_6 = 288$

Subclasses

CIdLMon: Commutative idempotent lattice-ordered monoids
CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

Superclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups
CIdPoMon: Commutative idempotent partially ordered monoids
CJMon: Commutative join-semilattice-ordered monoids
IdJMon: Idempotent join-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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37. CDivJslat: Commutative division join-semilattices

Definition

A *commutative division join-semilattice* is a division join-semilattice $\mathbf{P} = \langle P, \vee, \backslash, / \rangle$ such that P is a join-semilattice and

$\backslash, /$ are commutative: $x/y = y\backslash x$.

Formal Definition

$x \leq z/y \iff y \leq x\backslash z$

$x/y = y\backslash x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 79,\ f_4 = 7545$

Subclasses

BCKJslat: BCK-join-semilattices
CDivLat: Commutative division lattices
CRJMag: Commutative residuated join-semilattice-ordered magmas

Superclasses

CDivPos: Commutative division posets
DivJslat: Division join-semilattices Cont|Po|J|M|L|D|To|B|U|Ind

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38. BCKJslat: BCK-join-semilattices

Definition

A *BCK-join-semilattice* is an algebra $\mathbf{A} = \langle A, \vee, \to, 1 \rangle$ such that

$\langle A, \vee \rangle$ is a join-semilattice and

(1): $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

(2): $1 \to x = x$

(3): $x \to 1 = 1$

(4): $x \to (x \vee y) = 1$

(5): $x \vee ((x \to y) \to y) = ((x \to y) \to y)$

Formal Definition

$(x \vee y) \to z \leq x \to z$

$x \to y \leq x \to (y \vee z)$

$(x \to y) \to ((y \to z) \to (x \to z)) = 1$

$1 \to x = x$

$x \to 1 = 1$

$x \to (x \vee y) = 1$

$x \leq ((x \to y) \to y)$

Properties

Classtype	Variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 14,\ f_5 = 87,\ f_6 = 745$

Subclasses

BCKLat: BCK-lattices

Superclasses

BCK: BCK-algebras

CDivJslat: Commutative division join-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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39. CRJMag: Commutative residuated join-semilattice-ordered magmas

Definition

A *commutative residuated join-semilattice-ordered magma* is a residuated join-semilattice-ordered magma such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 148,\ f_5 = 4398$

Subclasses

CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

CRLMag: Commutative residuated lattice-ordered magmas

Superclasses

CDivJslat: Commutative division join-semilattices

CRPoMag: Commutative residuated partially ordered magmas

RJMag: Residuated join-semilattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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40. CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

Definition

A *commutative residuated join-semilattice-ordered semigroup* is a residuated join-semilattice-ordered semigroup $\mathbf{A} = \langle A, \vee, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 57,\ f_5 = 550$

Subclasses

CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

CRJMon: Commutative residuated join-semilattice-ordered monoids

CRLSgrp: Commutative residuated lattice-ordered semigroups

Superclasses

CJSgrp: Commutative join-semilattice-ordered semigroups

CRJMag: Commutative residuated join-semilattice-ordered magmas

CRPoSgrp: Commutative residuated partially ordered semigroups

RJSgrp: Residuated join-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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41. CRJMon: Commutative residuated join-semilattice-ordered monoids

Definition

A *commutative residuated join-semilattice-ordered monoid* is a residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 100,\ f_6 = 794$

Subclasses

CIRJMon: Commutative integral residuated join-semilattice-ordered monoids

CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

Superclasses

CJMon: Commutative join-semilattice-ordered monoids

CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

CRPoMon: Commutative residuated partially ordered monoids

RJMon: Residuated join-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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42. CIRJMon: Commutative integral residuated join-semilattice-ordered monoids

Definition

A *commutative integral residuated join-semilattice-ordered monoid* is an integral residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129,\ f_7 = 723$

Subclasses

Superclasses

CIJMon: Commutative Integral join-semilattice-ordered monoids

CRJMon: Commutative residuated join-semilattice-ordered monoids

IRJMon: Integral residuated join-semilattice-ordered monoids

Pocrim: Partially ordered commutative residuated integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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43. CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

Definition

A *commutative idempotent residuated join-semilattice-ordered semigroup* is an idempotent residuated join-semilattice-ordered semigroup $\mathbf{A} = \langle A, \vee, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 36,\ f_6 = 202$

Subclasses

CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

Superclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups

CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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44. CIdRJMon: Commutative idempotent residuated join-semilattice-ordered monoids

Definition

A *commutative idempotent residuated join-semilattice-ordered monoid* is an idmpotent residuated join-semilattice-ordered monoid $\mathbf{A} = \langle A, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 20,\ f_6 = 77,\ f_7 = 333$

Subclasses

Superclasses

CIdJMon: Commutative idempotent join-semilattice-ordered monoids

CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

CIdRPoMon: Commutative idempotent residuated partially ordered monoids

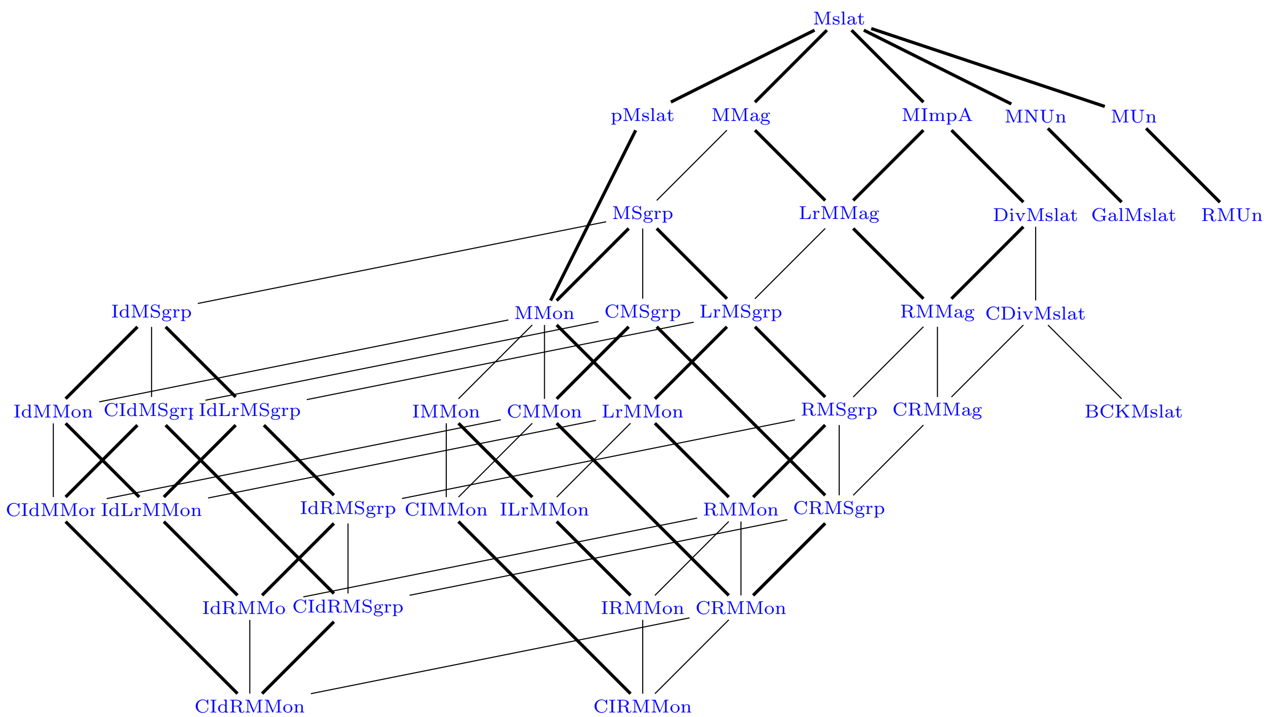
CRJMon: Commutative residuated join-semilattice-ordered monoids

IdRJMon: Idempotent residuated join-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 4

Meet-semilattice-ordered algebras



1. Mslat: Meet-semilattices

Definition

A *meet-semilattice* is a po-algebra $\mathbf{A} = \langle A, \leq, \wedge \rangle$ such that $\langle A, \leq \rangle$ is a poset in which all pairs of elements x, y have a meet $x \wedge y$, i. e.,

- $\wedge$ is order-preserving in each argument: $x \leq y \implies x \wedge z \leq y \wedge z$ and $z \wedge x \leq z \wedge y$
- $x \wedge y$ is a lower bound for x, y: $x \wedge y \leq x$ and $x \wedge y \leq y$
- $x \leq x \wedge x$

It follows that $x \wedge y$ is the greatest lower bound: $z \leq x$ and $z \leq y \implies z \leq z \wedge z \leq x \wedge z \leq x \wedge y$

Definition

A *meet-semilattice* is an algebra $\mathbf{A} = \langle A, \wedge \rangle$ where $\wedge$ is a binary operation that is

- associative: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
- commutative: $x \wedge y = y \wedge x$
- idempotent: $x \wedge x = x$ and
- $x \leq y \iff x \wedge y = x$

Formal Definition

$x \leq y \implies x \wedge z \leq y \wedge z$ and $z \wedge x \leq z \wedge y$

$x \wedge y \leq x$

$x \wedge y \leq y$

$x \leq x \wedge x$

Examples

Example 1: $\langle \mathbb{R}, \leq \rangle$, the real numbers with the standard order.

Example 2: $\langle P(S), \subseteq \rangle$, the collection of subsets of a sets S, ordered by inclusion.

Example 3: Any meet-semilattice is order-isomorphic to a meet-semilattice of subsets of some set, ordered by inclusion.

Basic Results

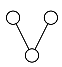
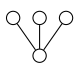
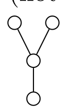
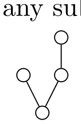
Properties

Classtype	Variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 5$, $f_5 = 15$, $f_6 = 53$, $f_7 = 222$, $f_8 = 1078$, $f_9 = 5994$, $f_{10} = 37622$, $f_{11} = 262776$, $f_{12} = 2018305$, $f_{13} = 16873364$, $f_{14} = 152233518$, $f_{15} = 1471613387$, $f_{16} = 15150569446$, $f_{17} = 165269824761$, $f_{18} = 1901910625578$

Small Members (not in any subclass)

$\mathbf{M}_{3,1}$ $\mathbf{M}_{4,1}$ $\mathbf{M}_{4,2}$ $\mathbf{M}_{4,3}$

Subclasses

Lat: Lattices

MImpA: Meet-semilattice-ordered implication algebras

MMag: Meet-semilattice-ordered magmas

MNUn: Meet-semilattice-ordered negated unars

MUn: Meet-semilattice-ordered unars

pMslat: Pointed meet-semilattices

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pMslat: Pointed meet-semilattices

Definition

A *pointed meet-semilattice* is an algebra $\mathbf{P} = \langle P, \wedge, c \rangle$ such that P is a meet-semilattice and c is a constant operation on P.

Formal Definition

$c = c$

Basic Results

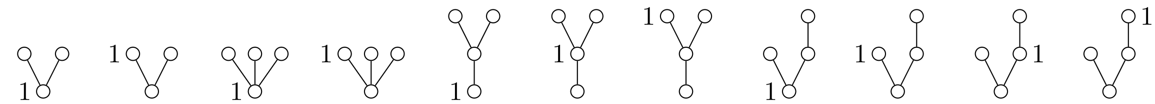
Properties

Classtype	variety

Finite Members

$f_1 = 1$, $f_2 = 2$, $f_3 = 5$, $f_4 = 16$, $f_5 = 60$, $f_6 = 262$, $f_7 = 1315$

Small Members (not in any subclass)



$\mathbf{pM}_{3,1}$ $\mathbf{pM}_{3,2}$ $\mathbf{pM}_{4,1}$ $\mathbf{pM}_{4,2}$ $\mathbf{pM}_{4,3}$ $\mathbf{pM}_{4,4}$ $\mathbf{pM}_{4,5}$ $\mathbf{pM}_{4,6}$ $\mathbf{pM}_{4,7}$ $\mathbf{pM}_{4,8}$ $\mathbf{pM}_{4,9}$

Subclasses

MMon: Meet-semilattice-ordered monoids

pLat: Pointed lattices

Superclasses

Mslat: Meet-semilattices

pPos: Pointed posets

Cont|Po|J|M|L|D|To|B|U|Ind

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3. MUn: Meet-semilattice-ordered unars

Definition

A *meet-semilattice-ordered unar* (also called a *po-unar* for short) is an algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a meet-semilattice and f is a unary operation on P that is

order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$x \leq y \implies f(x) \leq f(y)$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 17,\ f_4 = 138,\ f_5 = 1555$

Subclasses

GalMslat: Galois meet-semilattices

LUn: Lattice-ordered unars

RMUn: Residuated meet-semilattice-ordered unars

Superclasses

Mslat: Meet-semilattices

PoUn: Partially ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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4. MNUn: Meet-semilattice-ordered negated unars

Definition

A *meet-semilattice-ordered negated unar* is an algebra $\mathbf{P} = \langle P, \leq, \sim\rangle$ such that P is a meet-semilattice and $\sim$ is a unary operation on P that is

order-reversing: $x \leq y \implies {\sim}y \leq {\sim}x$

Formal Definition

$x \leq y \implies {\sim}y \leq {\sim}x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 15,\ f_4 = 113,\ f_5 = 1167$

Subclasses

GalMslat: Galois meet-semilattices

LNUn: Lattice-ordered negated unars

Superclasses

Mslat: Meet-semilattices

PoNUn: Partially ordered negated unars

Cont|Po|J|M|L|D|To|B|U|Ind

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5. GalMslat: Galois meet-semilattices

Definition

A *Galois meet-semilattice* is an algebra $\mathbf{P} = \langle P, \leq, \sim, -\rangle$ such that P is a meet-semilattice and $\sim, -$ are a pair of unary operations on P that form a

Galois connection: $x \leq \sim y \iff y \leq -x$

Formal Definition

$x \leq \sim y \iff y \leq -x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 30,\ f_5 = 184,\ f_6 = 1373$

Subclasses

GalLat: Galois lattices

Superclasses

GalPos: Galois posets

MNUn: Meet-semilattice-ordered negated unars

MUn: Meet-semilattice-ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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6. MMag: Meet-semilattice-ordered magmas

Definition

A *meet-semilattice-ordered magma* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot\rangle$ such that

$\langle A, \cdot\rangle$ is a magma

$\langle A, \wedge\rangle$ is a meet-semilattice.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 280$

Subclasses

LMag: Lattice-ordered magmas

LrMMag: Left-residuated meet-semilattice-ordered magmas

MSgrp: Meet-semilattice-ordered semigroups

Superclasses

Mslat: Meet-semilattices

PoMag: Partially ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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7. MSgrp: Meet-semilattice-ordered semigroups

Definition

A *meet-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot \rangle$ such that

$\langle A, \cdot \rangle$ is a semigroup

$\langle A, \wedge \rangle$ is a meet-semilattice

$\cdot$ is *orderpreserving*: $x \le y \implies x \cdot z \le y \cdot z$ and $z \cdot x \le z \cdot y$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 70,\ f_4 = 1437$

Subclasses

CMSgrp: Commutative meet-semilattice-ordered semigroups

IdMSgrp: Idempotent meet-semilattice-ordered semigroups

LSgrp: Lattice-ordered semigroups

LrMSgrp: Left-residuated meet-semilattice-ordered semigroups

MMon: Meet-semilattice-ordered monoids

Superclasses

MMag: Meet-semilattice-ordered magmas

PoSgrp: Partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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8. MMon: Meet-semilattice-ordered monoids

Definition

A *meet-semilattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, 1 \rangle$ such that

$\langle A, \cdot, 1 \rangle$ is a monoid

$\langle A, \wedge \rangle$ is a meet-semilattice

$\cdot$ is *orderpreserving*: $x \le y \implies wxz \le wyz$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 14,\ f_4 = 168,\ f_5 = 3488$

Subclasses

CMMon: Commutative meet-semilattice-ordered monoids
IMMon: Integral meet-semilattice-ordered monoids
IdMMon: Idempotent meet-semilattice-ordered monoids
LMon: Lattice-ordered monoids
LrMMon: Left-residuated meet-semilattice-ordered monoids

Superclasses

MSgrp: Meet-semilattice-ordered semigroups
PoMon: Partially ordered monoids
pMslat: Pointed meet-semilattices Cont|Po|J|M|L|D|To|B|U|Ind

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9. IMMon: Integral meet-semilattice-ordered monoids

Definition

An *integral meet-semilattice-ordered monoid* is a meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1\rangle$ such that $x \le 1$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \le 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 11,\ f_5 = 102,\ f_6 = 1569$

Subclasses

CIMMon: Commutative Integral meet-semilattice-ordered monoids
ILMon: Integral lattice-ordered monoids
ILrMMon: Integral left-residuated meet-semilattice-ordered monoids

Superclasses

IPoMon: Integral partially ordered monoids
MMon: Meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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10. IdMSgrp: Idempotent meet-semilattice-ordered semigroups

Definition

An *idempotent meet-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot\rangle$ such that
$\langle A, \wedge, \cdot\rangle$ is a meet-semilattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 28,\ f_4 = 308,\ f_5 = 4694$

Subclasses

CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups
IdLSgrp: Idempotent lattice-ordered semigroups
IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups
IdMMon: Idempotent meet-semilattice-ordered monoids

Superclasses

IdPoSgrp: Idempotent partially ordered semigroups
MSgrp: Meet-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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11. IdMMon: Idempotent meet-semilattice-ordered monoids

Definition

An *idempotent meet-semilattice-ordered monoid* is a meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1\rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 81,\ f_5 = 950$

Subclasses

CIdMMon: Commutative idempotent meet-semilattice-ordered monoids
IdLMon: Idempotent lattice-ordered monoids
IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids

Superclasses

IdMSgrp: Idempotent meet-semilattice-ordered semigroups
IdPoMon: Idempotent partially ordered monoids
MMon: Meet-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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12. MImpA: Meet-semilattice-ordered implication algebras

Formal Definition

$x \le y \implies y \to z \le x \to z$

$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 220$

Subclasses

DivMslat: Division meet-semilattices

LImpA: Lattice-ordered implication algebras

LrMMag: Left-residuated meet-semilattice-ordered magmas

Superclasses

Mslat: Meet-semilattices

PoImpA: Partially ordered implication algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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13. LrMMag: Left-residuated meet-semilattice-ordered magmas

Definition

A *left-residuated meet-semilattice-ordered magma* (or *lrm-magma*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, \rangle$ such that
$\langle A, \wedge \rangle$ is a meet-semilattice,
$\langle A, \cdot \rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x \backslash z$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$x \cdot y \le z \iff y \le x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 52,\ f_4 = 4827$

Subclasses

LrLMag: Left-residuated lattice-ordered magmas

LrMSgrp: Left-residuated meet-semilattice-ordered semigroups

RMMag: Residuated meet-semilattice-ordered magmas

Superclasses

LrPoMag: Left-residuated partially ordered magmas

MImpA: Meet-semilattice-ordered implication algebras

MMag: Meet-semilattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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14. LrMSgrp: Left-residuated meet-semilattice-ordered semigroups

Definition

A *left-residuated meet-semilattice-ordered semigroup* (or *lrm-semigroup*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \backslash \rangle$ such that
$\langle A, \wedge \rangle$ is a meet-semilattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x \backslash z$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 19,\ f_4 = 199,\ f_5 = 2946$

Subclasses

IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups
LrLSgrp: Left-residuated lattice-ordered semigroups
LrMMon: Left-residuated meet-semilattice-ordered monoids
RMSgrp: Residuated meet-semilattice-ordered semigroups

Superclasses

LrMMag: Left-residuated meet-semilattice-ordered magmas
LrPoSgrp: Left-residuated partially ordered semigroups
MSgrp: Meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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15. LrMMon: Left-residuated meet-semilattice-ordered monoids

Definition

A *left-residuated meet-semilattice-ordered monoid* (or *lrm-monoid*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash \rangle$ such that
$\langle A, \wedge \rangle$ is a meet-semilattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 195,\ f_6 = 2146$

Subclasses

ILrMMon: Integral left-residuated meet-semilattice-ordered monoids
IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids
LrLMon: Left-residuated lattice-ordered monoids
RMMon: Residuated meet-semilattice-ordered monoids

Superclasses

LrMSgrp: Left-residuated meet-semilattice-ordered semigroups
LrPoMon: Left-residuated partially ordered monoids
MMon: Meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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16. ILrMMon: Integral left-residuated meet-semilattice-ordered monoids

Definition

An *integral left-residuated meet-semilattice-ordered monoid* is a left-residuated meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash\rangle$ for which
$x \leq 1$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 51,\ f_6 = 408$

Subclasses

ILrLMon: Integral left-residuated lattice-ordered monoids
IRMMon: Meet-semilattice-ordered residuated integral monoids
RtHp: Right hoops

Superclasses

IMMon: Integral meet-semilattice-ordered monoids
LrMMon: Left-residuated meet-semilattice-ordered monoids
Polrim: Partially ordered left-residuated integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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17. RtHp: Right hoops

Definition

A *right hoop* is an algebra $\mathbf{A} = \langle A, \cdot, /, 1\rangle$ such that
$\langle A, \cdot, 1\rangle$ is a monoid
$x/(y \cdot z) = (x/z)/y$
$x/x = 1$
$(x/y) \cdot y = (y/x) \cdot x$
Remark: This definition shows that right hoops form a variety.
Right hoops are partially ordered by the relation $x \leq y \iff y/x = 1$.
The operation $x \wedge y = (x/y) \cdot y$ is a meet with respect to this order.

Definition

A *right hoop* is an algebra $\mathbf{A} = \langle A, \cdot, /, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that
$\langle A, \cdot, 1\rangle$ is a commutative monoid
and if $x \leq y$ is defined by $y/x = 1$ then
$\leq$ is a partial order,
$/$ is the right residual of $\cdot$, i.e., $x \cdot y \leq z \iff x \leq z/y$, and
$(x/y) \cdot y = (y/x) \cdot x$.

Formal Definition

$x \leq y \iff y/x = 1$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x/(y \cdot z) = (x/z)/y$
$x/x = 1$
$(x/y) \cdot y = (y/x) \cdot x$

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 24,\ f_6 = 91$

Subclasses

Hp: Hoops

Superclasses

ILrMMon: Integral left-residuated meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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18. IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups

Definition

An *idempotent left-residuated meet-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot \rangle$ such that $\langle A, \wedge, \cdot \rangle$ is a left-residuated meet-semilattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 46,\ f_5 = 345,\ f_6 = 3180$

Subclasses

IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups
IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids
IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups

Superclasses

IdLrPoSgrp: Idempotent left-residuated partially ordered semigroups
IdMSgrp: Idempotent meet-semilattice-ordered semigroups
LrMSgrp: Left-residuated meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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19. IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids

Definition

An *idempotent left-residuated meet-semilattice-ordered monoid* is a left-residuated meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1 \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 12,\ f_5 = 59,\ f_6 = 348,\ f_7 = 2372$

Subclasses

IdLrLMon: Idempotent left-residuated lattice-ordered monoids

IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Superclasses

IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups

IdLrPoMon: Idempotent left-residuated partially ordered monoids

IdMMon: Idempotent meet-semilattice-ordered monoids

LrMMon: Left-residuated meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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20. RMUn: Residuated meet-semilattice-ordered unars

Formal Definition

A *residuated meet-semilattice-ordered unar* (also called a *msl-unar* for short) is a po-algebra $\mathbf{S} = \langle S, \wedge, f, g \rangle$ such that $\langle S, \wedge \rangle$ is a meet-semilattice-ordered set and f, g are unary operations on S that g is the upper residual of f, or equivalently, g is the right adjoint of f:

$f(x) \leq y \iff x \leq g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets. In particular, $g(x \wedge y) = g(x) \wedge g(y)$.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

Subclasses

RLUn: Residuated lattice-ordered unars

Superclasses

MUn: Meet-semilattice-ordered unars

RPoUn: Residuated partially ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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21. DivMslat: Division meet-semilattices

Definition

A *division meet-semilattice* is an algebra $\mathbf{P} = \langle P, \wedge, \backslash, /\rangle$ such that P is a meet-semilattice,
$x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z$,
$(x \wedge y)/z = x/z \wedge y/z$ and
$x \leq z/y \iff y \leq x\backslash z$

Formal Definition

$x \leq z/y \iff y \leq x\backslash z$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 216$

Subclasses

CDivMslat: Commutative division meet-semilattices
DivLat: Division lattices
RMMag: Residuated meet-semilattice-ordered magmas

Superclasses

DivPos: Division posets
MImpA: Meet-semilattice-ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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22. RMMag: Residuated meet-semilattice-ordered magmas

Definition

A *residuated meet-semilattice-ordered magma* (or *rpo-magma*) is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, /\rangle$ such that
$\langle A, \wedge\rangle$ is a meet-semilattice,
$\langle A, \cdot\rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 20,\ f_4 = 1116$

Subclasses

CRMMag: Commutative residuated meet-semilattice-ordered magmas
RLMag: Residuated lattice-ordered magmas
RMSgrp: Residuated meet-semilattice-ordered semigroups

Superclasses

DivMslat: Division meet-semilattices
LrMMag: Left-residuated meet-semilattice-ordered magmas
RPoMag: Residuated partially ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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23. RMSgrp: Residuated meet-semilattice-ordered semigroups

Definition

A *residuated meet-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, / \rangle$ such that
$\langle A, \wedge \rangle$ is a meet-semilattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 129,\ f_5 = 1852$

Subclasses

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups
IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups
RLSgrp: Residuated lattice-ordered semigroups
RMMon: Residuated meet-semilattice-ordered monoids

Superclasses

LrMSgrp: Left-residuated meet-semilattice-ordered semigroups
RMMag: Residuated meet-semilattice-ordered magmas
RPoSgrp: Residuated partially ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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24. RMMon: Residuated meet-semilattice-ordered monoids

Definition

A *residuated meet-semilattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, / \rangle$ such that
$\langle A, \wedge \rangle$ is a meet-semilattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488$

Subclasses

CRMMon: Commutative residuated meet-semilattice-ordered monoids
IRMMon: Meet-semilattice-ordered residuated integral monoids
IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Superclasses

LrMMon: Left-residuated meet-semilattice-ordered monoids
RMSgrp: Residuated meet-semilattice-ordered semigroups
RPoMon: Residuated partially ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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25. IRMMon: Meet-semilattice-ordered residuated integral monoids

Definition

A *meet-semilattice-ordered residuated integral monoid* is an rm-monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, / \rangle$ such that
x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

Superclasses

ILrMMon: Integral left-residuated meet-semilattice-ordered monoids
Porim: Partially ordered residuated integral monoids
RMMon: Residuated meet-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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26. IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups

Definition

An *idempotent residuated meet-semilattice-ordered semigroup* is a residuated meet-semilattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, / \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 169,\ f_6 = 1404$

Subclasses

CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

IdRLSgrp: Idempotent residuated lattice-ordered semigroups

IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Superclasses

IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups

IdRPoSgrp: Idempotent residuated partially ordered semigroups

RMSgrp: Residuated meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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27. IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Definition

An *idempotent residuated meet-semilattice-ordered monoid* is a residuated meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 32,\ f_6 = 147$

Subclasses

CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Superclasses

IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids

IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups

IdRPoMon: Idempotent residuated partially ordered monoids

RMMon: Residuated meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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28. CMSgrp: Commutative meet-semilattice-ordered semigroups

Definition

A *commutative meet-semilattice-ordered semigroup* is a meet-semilattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \cdot \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 32,\ f_4 = 432$

Subclasses

CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups

CLSgrp: Commutative lattice-ordered semigroups

CMMon: Commutative meet-semilattice-ordered monoids

CRLSgrp: Commutative residuated lattice-ordered semigroups

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

Superclasses

CPoSgrp: Commutative partially ordered semigroups

MSgrp: Meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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29. CMMon: Commutative meet-semilattice-ordered monoids

Definition

A *commutative meet-semilattice-ordered monoid* is a meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 92,\ f_5 = 1322$

Subclasses

CIMMon: Commutative Integral meet-semilattice-ordered monoids

CIdMMon: Commutative idempotent meet-semilattice-ordered monoids

CLMon: Commutative lattice-ordered monoids

CRMMon: Commutative residuated meet-semilattice-ordered monoids

Superclasses

CMSgrp: Commutative meet-semilattice-ordered semigroups

CPoMon: Commutative partially ordered monoids

MMon: Meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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30. CIMMon: Commutative Integral meet-semilattice-ordered monoids

Definition

A *commutative integral meet-semilattice-ordered monoid* is a integral meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \le 1$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 60,\ f_6 = 572$

Subclasses

CILMon: Commutative Integral lattice-ordered monoids

CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

Superclasses

CIPoMon: Commutative integral partially ordered monoids

CMMon: Commutative meet-semilattice-ordered monoids

IMMon: Integral meet-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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31. CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups

Definition

A *commutative idempotent meet-semilattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \cdot\rangle$ such that $\langle A, \wedge, \cdot\rangle$ is an idempotent meet-semilattice-ordered semigroup and

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 53,\ f_5 = 498$

Subclasses

CIdLSgrp: Commutative idempotent lattice-ordered semigroups

CIdMMon: Commutative idempotent meet-semilattice-ordered monoids

CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

Superclasses

CIdPoSgrp: Commutative idempotent partially ordered semigroups
CMSgrp: Commutative meet-semilattice-ordered semigroups
IdMSgrp: Idempotent meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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32. CIdMMon: Commutative idempotent meet-semilattice-ordered monoids

Definition

A *commutative idempotent meet-semilattice-ordered monoid* is an idempotent meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 31,\ f_5 = 228,\ f_6 = 2205$

Subclasses

CIdLMon: Commutative idempotent lattice-ordered monoids
CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Superclasses

CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups
CIdPoMon: Commutative idempotent partially ordered monoids
CMMon: Commutative meet-semilattice-ordered monoids
IdMMon: Idempotent meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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33. CDivMslat: Commutative division meet-semilattices

Definition

A *commutative division meet-semilattice* is a division meet-semilattice $\mathbf{P} = \langle P, \wedge\rangle$ such that P is a meet-semilattice and

$\backslash, /$ are commutative: $x/y = y\backslash x$.

Formal Definition

$x \le z/y \iff y \le x\backslash z$
$x/y = y\backslash x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 64,\ f_4 = 6208$

Subclasses

BCKMslat: BCK-meet-semilattices

CDivLat: Commutative division lattices

CRMMag: Commutative residuated meet-semilattice-ordered magmas

Superclasses

CDivPos: Commutative division posets

DivMslat: Division meet-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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34. BCKMslat: BCK-meet-semilattices

Definition

A *BCK-meet-semilattice* is an algebra $\mathbf{A} = \langle A, \wedge, \to, 1\rangle$ such that

$\mathbf{A} = \langle A, \wedge\rangle$ is a meet-semilattice and

(1): $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

(2): $1 \to x = x$

(3): $x \to 1 = 1$

(4): $(x \wedge y) \to y = 1$

(5): $x \wedge ((x \to y) \to y) = x$

Remark: $x \leq y \iff x \to y = 1$ is a partial order, with 1 as greatest element, and $\wedge$ is a meet in this partial order. Idziak [1984]

Formal Definition

$x \leq y \iff x \to y = 1$

$(x \to y) \to ((y \to z) \to (x \to z)) = 1$

$1 \to x = x$

$x \to 1 = 1$

$(x \wedge y) \to y = 1$

$x \wedge ((x \to y) \to y) = x$

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 38,\ f_6 = 265$

Subclasses

BCKLat: BCK-lattices

Superclasses

BCK: BCK-algebras

CDivMslat: Commutative division meet-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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35. CRMMag: Commutative residuated meet-semilattice-ordered magmas

Definition

A *commutative residuated meet-semilattice-ordered magma* is a residuated meet-semilattice-ordered magma such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 148,\ f_5 = 4398$

Subclasses

CRLMag: Commutative residuated lattice-ordered magmas

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

Superclasses

CDivMslat: Commutative division meet-semilattices

CRPoMag: Commutative residuated partially ordered magmas

RMMag: Residuated meet-semilattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

36. CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

Definition

A *commutative residuated meet-semilattice-ordered semigroup* is a residuated meet-semilattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 57,\ f_5 = 550$

Subclasses

CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

CRLSgrp: Commutative residuated lattice-ordered semigroups

CRMMon: Commutative residuated meet-semilattice-ordered monoids

Superclasses

CMSgrp: Commutative meet-semilattice-ordered semigroups

CRMMag: Commutative residuated meet-semilattice-ordered magmas

CRPoSgrp: Commutative residuated partially ordered semigroups

RMSgrp: Residuated meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

37. CRMMon: Commutative residuated meet-semilattice-ordered monoids

Definition

A *commutative residuated meet-semilattice-ordered monoid* is a residuated meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, /\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Remark: These algebras are also known as *lineales*.[(dePaiva2005)]

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 100,\ f_6 = 794$

Subclasses

CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Superclasses

CMMon: Commutative meet-semilattice-ordered monoids

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

CRPoMon: Commutative residuated partially ordered monoids

RMMon: Residuated meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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38. CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

Definition

A *commutative integral residuated Mmetsemilattice-ordered monoid* is an integral residuated Mmetsemilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, /\rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety
Congruence distributive	Yes (relatively) Blok and Raftery [1997]
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129,\ f_7 = 723$

Subclasses

CIRL: Commutative integral residuated lattices

Superclasses

CIMMon: Commutative Integral meet-semilattice-ordered monoids

CRMMon: Commutative residuated meet-semilattice-ordered monoids

IRMMon: Meet-semilattice-ordered residuated integral monoids

Pocrim: Partially ordered commutative residuated integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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39. Hp: Hoops

Definition

A *hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a commutative monoid

$x \to (y \to z) = (x \cdot y) \to z$

$x \to x = 1$

$(x \to y) \cdot x = (y \to x) \cdot y$

Remark: This definition shows that hoops form a variety.

Hoops are partially ordered by the relation $x \le y \iff x \to y = 1$.

The operation $x \wedge y = (x \to y) \cdot x$ is a meet with respect to this order.

Definition

A *hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a commutative monoid

and if $x \le y$ is defined by $x \to y = 1$ then

$\le$ is a partial order,

$\to$ is the residual of $\cdot$, i.e., $x \cdot y \le z \iff y \le x \to z$, and

$(x \to y) \cdot x = (y \to x) \cdot y$.

Formal Definition

$x \wedge y = (x \to y) \cdot x$

$x \cdot y = y \cdot x$

$x \cdot 1 = x$

$x \to (y \to z) = (x \cdot y) \to z$

$x \to x = 1$

$(x \to y) \cdot x = (y \to x) \cdot y$

Basic Results

Finite hoops are the same as generalized BL-algebras (= divisible residuated lattices) since the join always exists in a finite meet-semilattice with top, and since all finite GBL-algebras are commutative and integral.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 10,\ f_6 = 23,\ f_7 = 49$

Subclasses

BrSlat: Brouwerian semilattices

WaHp: Wajsberg hoops

Superclasses

RtHp: Right hoops

Cont|Po|J|M|L|D|To|B|U|Ind

40. CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

Definition

A *commutative idempotent residuated meet-semilattice-ordered semigroup* is an idempotent residuated meet-semilattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 36,\ f_6 = 202$

Subclasses

CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Superclasses

CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups

CIdRPoSgrp: Commutative idempotent residuated partially ordered semigroups

CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups

IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

41. CIdRMMon: Commutative idempotent residuated meet-semilattice-ordered monoids

Definition

A *commutative idempotent residuated meet-semilattice-ordered monoid* is an idmpotent residuated meet-semilattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \le z \iff y \le x \backslash z$
$x \cdot y \le z \iff x \le z / y$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 20,\ f_6 = 77$

Subclasses

Superclasses

CIdMMon: Commutative idempotent meet-semilattice-ordered monoids
CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups
CIdRPoMon: Commutative idempotent residuated partially ordered monoids
CRMMon: Commutative residuated meet-semilattice-ordered monoids
IdRMMon: Idempotent residuated meet-semilattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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42. BrSlat: Brouwerian semilattices

Abbreviation: **BrSlat**

Definition

A *Brouwerian semilattice* is an algebra $\mathbf{A} = \langle A, \wedge, 1, \to\rangle$ such that
$\langle A, \wedge, 1\rangle$ is a semilattice with identity
$\to$ gives the residual of $\wedge$: $x \wedge y \le z \iff y \le x \to z$

Definition

A *Brouwerian semilattice* is a hoop $\mathbf{A} = \langle A, \cdot, 1, \to\rangle$ such that
$\cdot$ is idempotent: $x \cdot x = x$

Formal Definition

$x \wedge y \le z \iff y \le x \to z$
$x \le \top$

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	Yes
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8,\ f_8 = 15,\ f_9 = 26,\ f_{10} = 47,\ f_{11} = 82,\ f_{12} = 151,$
$f_{13} = 269,\ f_{14} = 494,\ f_{15} = 891,\ f_{16} = 1639,\ f_{17} = 2978,\ f_{18} = 5483,\ f_{19} = 10006,\ f_{20} = 18428$

Values known up to size 49 Erné et al. [2002]

Subclasses

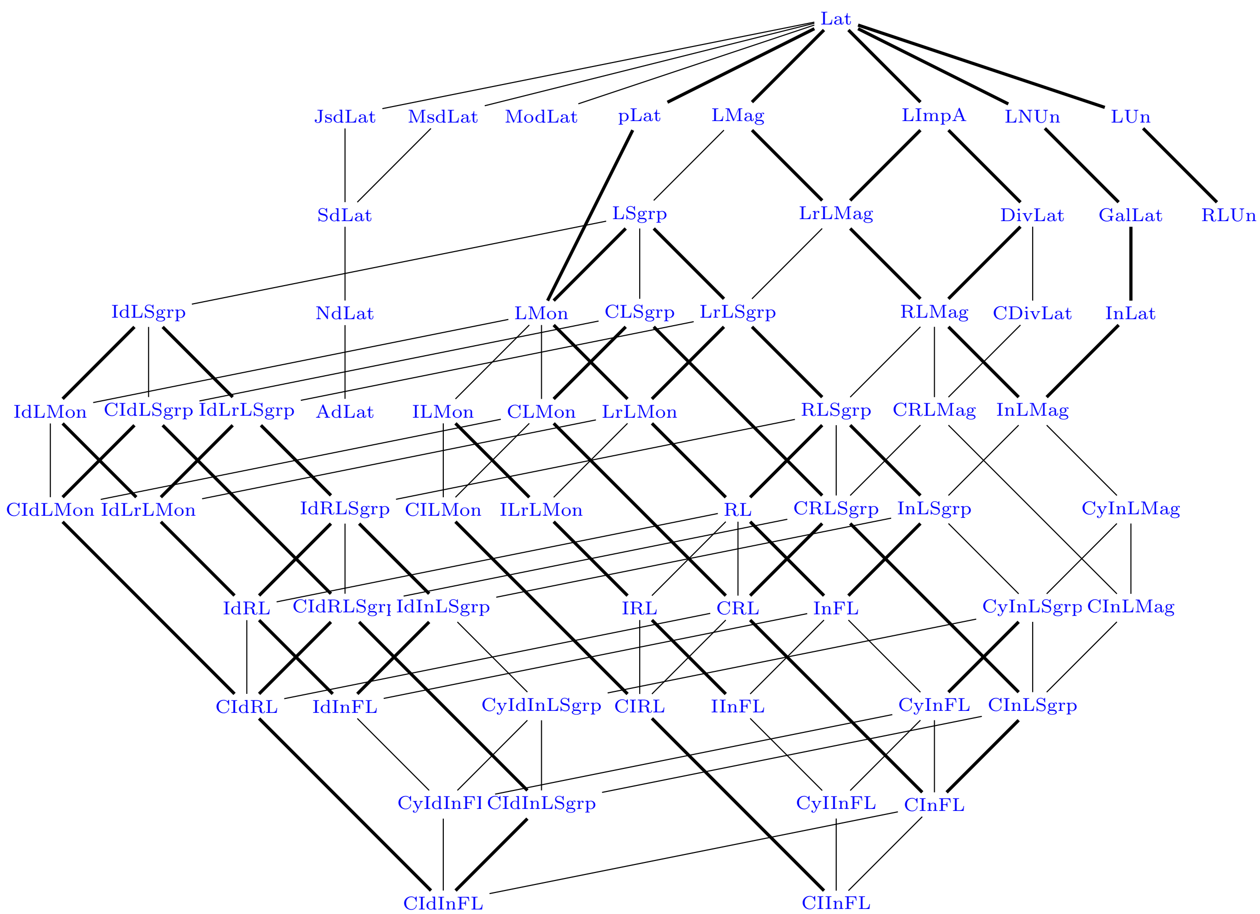
BrA: Brouwerian algebras

Superclasses

Hp: Hoops

CHAPTER 5

Lattice-ordered algebras



1. Lat: Lattices

Definition

A *lattice* is an algebra $\mathbf{L} = \langle L, \vee, \wedge\rangle$, where $\vee$ and $\wedge$ are infix binary operations called the *join* and *meet*, such that

$\vee, \wedge$ are associative: $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

$\vee, \wedge$ are commutative: $x \vee y = y \vee x$, $x \wedge y = y \wedge x$

$\vee, \wedge$ are absorbtive: $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$.

Remark: It follows that $\vee$ and $\wedge$ are idempotent: $x \vee x = x$, $x \wedge x = x$.

This definition shows that lattices form a variety.

A partial order $\leq$ is definable in any lattice by $x \leq y \iff x \wedge y = x$, or equivalently by $x \leq y \iff x \vee y = y$.

Definition

A *lattice* is an algebra $\mathbf{L} = \langle L, \vee, \wedge\rangle$ of type $\langle 2, 2\rangle$ such that

$\langle L, \vee\rangle$ and $\langle L, \wedge\rangle$ are semilattices, and
$\vee, \wedge$ are absorbtive: $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$

Definition

A *lattice* is an algebra $\mathbf{L} = \langle L, \leq\rangle$ that is a partially ordered set in which all elements $x, y \in L$ have a least upper bound: $z = x \vee y \iff x \leq z,\ y \leq z$ and $\forall w\ (x \leq w$ and $y \leq w \implies z \leq w)$ and a greatest lower bound: $z = x \wedge y \iff z \leq x,\ z \leq y$ and $\forall w\ (w \leq x$ and $w \leq y \implies w \leq z)$

Definition

A *lattice* is an algebra $\mathbf{L} = \langle L, \vee, \wedge, \leq\rangle$ such that $\langle L, \leq\rangle$ is a partially ordered set and the following quasiequations hold:

$\vee$-left: $x \leq z$ and $y \leq z \implies x \vee y \leq z$
$\vee$-right: $z \leq x \implies z \leq x \vee y$, $\quad z \leq y \implies z \leq x \vee y$
$\wedge$-right: $z \leq x$ and $z \leq y \implies z \leq x \wedge y$
$\wedge$-left: $x \leq z \implies x \wedge y \leq z$, $\quad y \leq z \implies x \wedge y \leq z$
Remark: These quasiequations give a cut-free Gentzen system to decide the equational theory of lattices.

Examples

Example 1: $\langle P(S), \cup, \cap, \subseteq\rangle$, the collection of subsets of a sets S, ordered by inclusion.

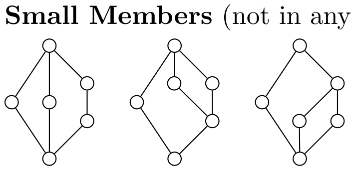
Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	yes Funayama and Nakayama [1942]
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	yes Jonsson [1956]
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 5$, $f_6 = 15$, $f_7 = 53$, $f_8 = 222$, $f_9 = 1078$, $f_{10} = 5994$, $f_{11} = 37622$, $f_{12} = 262776$, $f_{13} = 2\,018\,305$, $f_{14} = 16\,873\,364$, $f_{15} = 152\,233\,518$, $f_{16} = 1\,471\,613\,387$, $f_{17} = 15\,150\,569\,446$, $f_{18} = 165\,269\,824\,761$ Heitzig and Reinhold [2002], $f_{19} = 1\,901\,910\,625\,578$ Jipsen and Lawless [2015], $f_{20} = 23\,003\,059\,864\,006$ Gebhardt and Tawn [2020]

Small Members (not in any subclass)



$\mathbf{L}_{6,1}$ $\quad$ $\mathbf{L}_{6,2}$ $\quad$ $\mathbf{L}_{6,3}$

Subclasses

JsdLat: Join-semidistributive lattices
LImpA: Lattice-ordered implication algebras
LMag: Lattice-ordered magmas

LNUn: Lattice-ordered negated unars
LUn: Lattice-ordered unars
ModLat: Modular lattices
MsdLat: Meet-semidistributive lattices
OLat: Ortholattices
pLat: Pointed lattices

Superclasses

Jslat: Join-semilattices
Mslat: Meet-semilattices
SkLat: Skew lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pLat: Pointed lattices

Definition

A *pointed lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, c\rangle$ such that $\mathbf{A} = \langle A, \wedge, \vee\rangle$ is a lattice and c is a constant operation on A.

Formal Definition

$c = c$

Basic Results

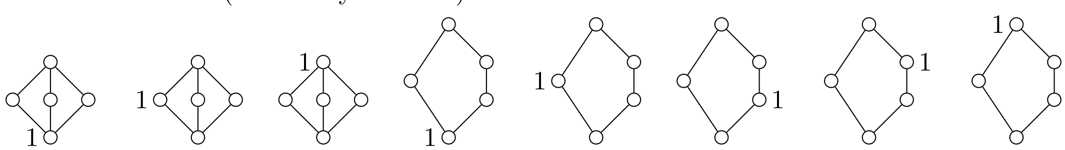
Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 7,\ f_5 = 21,\ f_6 = 75,\ f_7 = 315$

Small Members (not in any subclass)



$\mathbf{pL}_{5,1}$ $\mathbf{pL}_{5,2}$ $\mathbf{pL}_{5,3}$ $\mathbf{pL}_{5,4}$ $\mathbf{pL}_{5,5}$ $\mathbf{pL}_{5,6}$ $\mathbf{pL}_{5,7}$ $\mathbf{pL}_{5,8}$

Subclasses

LMon: Lattice-ordered monoids
pDLat: Pointed distributive lattices

Superclasses

Lat: Lattices
pJslat: Pointed join-semilattices
pMslat: Pointed meet-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

3. bLat: Bounded lattices

Definition

A *bounded lattice* is an algebra $\mathbf{L} = \langle L, \vee, \bot, \wedge, \top\rangle$ such that

$\langle L, \wedge, \vee\rangle$ is a lattice

$\bot$ is the least element: $\bot \leq x$

$\top$ is the greatest element: $x \leq \top$

Formal Definition

$\bot \leq x$

$x \leq \top$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 5,\ f_6 = 15,\ f_7 = 53$, Same as for finite lattices since every complete lattice is bounded.

Subclasses

CplmLat: Complemented lattices

bDLat: Bounded distributive lattices

Superclasses

lbJslat: Lower-bounded join-semilattices

ubJslat: Upper-bounded join-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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4. LUn: Lattice-ordered unars

Definition

A *lattice-ordered unar* is an algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a lattice and f is a unary operation on P that is

order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$f(x \vee y) = f(x) \vee f(y)$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 50,\ f_5 = 313$

Subclasses

DLUn: Distributive lattice-ordered unars

RLUn: Residuated lattice-ordered unars

Superclasses

JUn: Join-semilattice-ordered unars

Lat: Lattices

MUn: Meet-semilattice-ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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5. LNUn: Lattice-ordered negated unars

Definition

A *lattice-ordered negated unar* (also called a *po-nunar* for short) is an algebra $\mathbf{P} = \langle P, \leq, \sim\rangle$ such that P is a lattice and $\sim$ is a unary operation on P that is

order-reversing: $x \leq y \implies {\sim}y \leq {\sim}x$

Formal Definition

$x \leq y \implies {\sim}y \leq {\sim}x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 56,\ f_5 = 457$

Subclasses

DLNUn: Distributive lattice-ordered negated unars

GalLat: Galois lattices

Superclasses

JNUn: Join-semilattice-ordered negated unars

Lat: Lattices

MNUn: Meet-semilattice-ordered negated unars Cont|Po|J|M|L|D|To|B|U|Ind

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6. LMag: Lattice-ordered magmas

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

DLMag: Distributive lattice-ordered magmas

LSgrp: Lattice-ordered semigroups

LrLMag: Left-residuated lattice-ordered magmas

MultLat: Multiplicative lattices

Superclasses

JMag: Join-semilattice-ordered magmas

Lat: Lattices

MMag: Meet-semilattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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7. LSgrp: Lattice-ordered semigroups

Definition

A *lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that
$\langle A, \cdot \rangle$ is a semigroup
$\langle A, \wedge, \vee \rangle$ is a lattice
$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 44,\ f_4 = 479$

Subclasses

CLSgrp: Commutative lattice-ordered semigroups
DLSgrp: Distributive lattice-ordered semigroups
IdLSgrp: Idempotent lattice-ordered semigroups
LMon: Lattice-ordered monoids
LrLSgrp: Left-residuated lattice-ordered semigroups

Superclasses

DLImpA: Distributive lattice-ordered implication algebras
JSgrp: Join-semilattice-ordered semigroups
LMag: Lattice-ordered magmas
MSgrp: Meet-semilattice-ordered semigroups
MultLat: Multiplicative lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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8. LMon: Lattice-ordered monoids

Definition

A *lattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1 \rangle$ such that
$\langle A, \cdot, 1 \rangle$ is a monoid
$\langle A, \wedge, \vee \rangle$ is a lattice
$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$

Basic Results

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 45,\ f_5 = 347$

Subclasses

CLMon: Commutative lattice-ordered monoids
DLMon: Distributive lattice-ordered monoids
ILMon: Integral lattice-ordered monoids
IdLMon: Idempotent lattice-ordered monoids
LrLMon: Left-residuated lattice-ordered monoids

Superclasses

JMon: Join-semilattice-ordered monoids
LSgrp: Lattice-ordered semigroups
MMon: Meet-semilattice-ordered monoids
pLat: Pointed lattices Cont|Po|J|M|L|D|To|B|U|Ind

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9. KLat: Kleene lattices

Definition

A *Kleene lattice* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, 0, \cdot, 1, ^* \rangle$ of type $\langle 2, 2, 0, 2, 0, 1\rangle$ such that
$\langle A, \vee, 0, \cdot, 1, ^* \rangle$ is a Kleene algebra
$\langle A, \vee, \wedge\rangle$ is a lattice

Properties

Classtype	Quasivariety
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 149,\ f_6 = 1488$

Subclasses

ActLat: Action lattices

Superclasses

KA: Kleene algebras Cont|Po|J|M|L|D|To|B|U|Ind

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10. ActLat: Action lattices

Definition

An *action lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, 0, \cdot, 1, ^*, \backslash, /\rangle$ of type $\langle 2, 2, 0, 2, 0, 1, 2, 2\rangle$ such that
$\langle A, \vee, 0, \cdot, 1, ^* \rangle$ is a Kleene algebra
$\langle A, \wedge, \vee\rangle$ is a lattice
$\backslash$ is the left residual of $\cdot$: $y \le x\backslash z \iff xy \le z$
$/$ is the right residual of $\cdot$: $x \le z/y \iff xy \le z$

Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \le z \iff y \le x\backslash z$
$x \cdot y \le z \iff x \le z/y$

$x \cdot 0 = 0$

$0 \cdot x = 0$

$1 \vee x \vee x^* \cdot x^* = x^*$

$x \cdot y \leq y \implies x^* \cdot y = y$

$y \cdot x \leq y \implies y \cdot x^* = y$

Properties

Classtype	variety
Equational theory	?
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 149,\ f_6 = 1488$

Subclasses

TrivA: Trivial algebras

Superclasses

KLat: Kleene lattices

RL: Residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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11. ModLat: Modular lattices

Definition

A *modular lattice* is a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ that satisfies the *modular identity*: $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

Definition

A *modular lattice* is a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ that satisfies the *modular law*: $x \leq z \implies (x \vee y) \wedge z \leq x \vee (y \wedge z)$

Definition

A *modular lattice* is a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ such that $\mathbf{L}$ has no sublattice isomorphic to the pentagon $\mathbf{N}_5$

Examples

Example 1: M_3 is the smallest nondistributive modular lattice. By a result of Dedekind [1900] this lattice occurs as a sublattice of every nondistributive modular lattice.

Properties

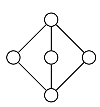
Classtype	Variety
Equational theory	Undecidable Freese [1980], Herrmann [1983]
Quasiequational theory	Undecidable Lipshitz [1974]
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

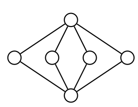
$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 4,\ f_6 = 8,\ f_7 = 16,\ f_8 = 34,\ f_9 = 72,\ f_{10} = 157,\ f_{11} = 343,\ f_{12} = 766,\ f_{13} = 1718,\ f_{14} = 3899,\ f_{15} = 8898,\ f_{16} = 20475,\ f_{17} = 47321,\ f_{18} = 110024,\ f_{19} = 256791,\ f_{20} = 601991,\ f_{21} = 1415768,\ f_{22} = 3340847,\ f_{23} = 7904700,\ f_{24} = 18752942$

Jipsen and Lawless [2015], A006981

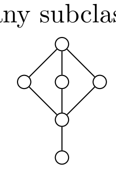
Small Members (not in any subclass)



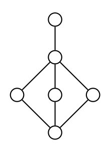
$\mathbf{ModL}_{5,1}$



$\mathbf{ModL}_{6,1}$



$\mathbf{ModL}_{6,2}$



$\mathbf{ModL}_{6,3}$

Subclasses

DLat: Distributive lattices

Superclasses

Lat: Lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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12. MultLat: Multiplicative lattices

Definition

A *multiplicative lattice* (or *m-lattice*) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\langle A, \wedge, \vee\rangle$ is a lattice

$\cdot$ distributes over $\vee$: $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$ and

$\cdot$ distributes over $\wedge$: $x(y \wedge z) = xy \wedge xz$, $(x \wedge y)z = xz \wedge yz$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z,\ (x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$x \cdot (y \wedge z) = x \cdot y \wedge x \cdot z,\ (x \wedge y) \cdot z = x \cdot z \wedge y \cdot z$

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

LSgrp: Lattice-ordered semigroups

Superclasses

LMag: Lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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13. ILMon: Integral lattice-ordered monoids

Definition

An *integral lattice-ordered monoid* is a lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that
$x \leq 1$.

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

CILMon: Commutative Integral lattice-ordered monoids
DILMon: Distributive integral lattice-ordered monoids
ILrLMon: Integral left-residuated lattice-ordered monoids

Superclasses

IJMon: Integral join-semilattice-ordered monoids
IMMon: Integral meet-semilattice-ordered monoids
LMon: Lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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14. IdLSgrp: Idempotent lattice-ordered semigroups

Definition

An *idempotent lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that
$\langle A, \wedge, \vee, \cdot\rangle$ is a lattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 17,\ f_4 = 100,\ f_5 = 674$

Subclasses

CIdLSgrp: Commutative idempotent lattice-ordered semigroups
DIdLSgrp: Distributive idempotent lattice-ordered semigroups
IdLMon: Idempotent lattice-ordered monoids
IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups

Superclasses

IdJSgrp: Idempotent join-semilattice-ordered semigroups
IdMSgrp: Idempotent meet-semilattice-ordered semigroups
LSgrp: Lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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15. IdLMon: Idempotent lattice-ordered monoids

Definition

An *idempotent lattice-ordered monoid* is a lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 22,\ f_5 = 93,\ f_6 = 439$

Subclasses

CIdLMon: Commutative idempotent lattice-ordered monoids
DIdLMon: Distributive idempotent lattice-ordered monoids
IdLrLMon: Idempotent left-residuated lattice-ordered monoids

Superclasses

IdJMon: Idempotent join-semilattice-ordered monoids
IdLSgrp: Idempotent lattice-ordered semigroups
IdMMon: Idempotent meet-semilattice-ordered monoids
LMon: Lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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16. LImpA: Lattice-ordered implication algebras

Formal Definition

$(x \vee y) \to z = (x \to z) \wedge (y \to z)$
$z \to (x \wedge y) = (z \to x) \wedge (z \to y)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

DLImpA: Distributive lattice-ordered implication algebras
DivLat: Division lattices
LrLMag: Left-residuated lattice-ordered magmas

Superclasses

JImpA: Join-semilattice-ordered implication algebras
Lat: Lattices
MImpA: Meet-semilattice-ordered implication algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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17. LrLMag: Left-residuated lattice-ordered magmas

Definition

A *left-residuated lattice-ordered magma* (or *lrpo-magma*) is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq\rangle$ is a lattice,
$\langle A, \cdot\rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 50,\ f_4 = 4441$

Subclasses

DLrLMag: Distributive left-residuated lattice-ordered magmas
LrLSgrp: Left-residuated lattice-ordered semigroups
RLMag: Residuated lattice-ordered magmas

Superclasses

LImpA: Lattice-ordered implication algebras
LMag: Lattice-ordered magmas
LrJMag: Left-residuated join-semilattice-ordered magmas
LrMMag: Left-residuated meet-semilattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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18. LrLSgrp: Left-residuated lattice-ordered semigroups

Definition

A *left-residuated lattice-ordered semigroup* (or *lrpo-semigroup*) is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq\rangle$ is a lattice,
$\langle A, \cdot\rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 18,\ f_4 = 183,\ f_5 = 2500$

Subclasses

DLrLSgrp: Distributive left-residuated lattice-ordered semigroups
IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups
LrLMon: Left-residuated lattice-ordered monoids
RLSgrp: Residuated lattice-ordered semigroups

Superclasses

LSgrp: Lattice-ordered semigroups
LrJSgrp: Left-residuated join-semilattice-ordered semigroups
LrLMag: Left-residuated lattice-ordered magmas
LrMSgrp: Left-residuated meet-semilattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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19. LrLMon: Left-residuated lattice-ordered monoids

Definition

A *left-residuated lattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a lattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 23,\ f_5 = 169,\ f_6 = 1635$

Subclasses

DLrLMon: Distributive left-residuated lattice-ordered monoids
ILrLMon: Integral left-residuated lattice-ordered monoids
IdLrLMon: Idempotent left-residuated lattice-ordered monoids
RL: Residuated lattices

Superclasses

LMon: Lattice-ordered monoids
LrJMon: Left-residuated join-semilattice-ordered monoids
LrLSgrp: Left-residuated lattice-ordered semigroups
LrMMon: Left-residuated meet-semilattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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20. ILrLMon: Integral left-residuated lattice-ordered monoids

Definition

A *integral left-residuated lattice-ordered monoid* (or *ilrℓ-monoid* for short) is a left-residuated lattice-ordered monoid $\langle A, \leq, \cdot, 1, \backslash, \rangle$ that satisfies $x \leq 1$.

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

DILrLMon: Distributive integral left-residuated lattice-ordered monoids

IRL: Integral residuated lattices

Superclasses

ILMon: Integral lattice-ordered monoids

ILrJMon: Integral left-residuated join-semilattice-ordered monoids

ILrMMon: Integral left-residuated meet-semilattice-ordered monoids

LrLMon: Left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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21. IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups

Definition

An *idempotent left-residuated lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that
$\langle A, \wedge, \vee, \cdot \rangle$ is a left-residuated lattice-ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 40,\ f_5 = 273$

Subclasses

DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups

IdLrLMon: Idempotent left-residuated lattice-ordered monoids

IdRLSgrp: Idempotent residuated lattice-ordered semigroups

Superclasses

IdLSgrp: Idempotent lattice-ordered semigroups

IdLrJSgrp: Idempotent left-residuated join-semilattice-ordered semigroups

IdLrMSgrp: Idempotent left-residuated meet-semilattice-ordered semigroups
LrLSgrp: Left-residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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22. IdLrLMon: Idempotent left-residuated lattice-ordered monoids

Definition

An *idempotent left-residuated lattice-ordered monoid* is a left-residuated lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 11,\ f_5 = 46,\ f_6 = 215$

Subclasses

DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids
IdRL: Idempotent residuated lattices

Superclasses

IdLMon: Idempotent lattice-ordered monoids
IdLrJMon: Idempotent left-residuated join-semilattice-ordered monoids
IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups
IdLrMMon: Idempotent left-residuated meet-semilattice-ordered monoids
LrLMon: Left-residuated lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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23. RLUn: Residuated lattice-ordered unars

Formal Definition

A *residuated lattice-ordered unar* (also called an *ℓ-unar* for short) is a po-algebra $\langle L, \wedge, \vee, f, g\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice and f, g are unary operations on L such that g is the upper residual of f, or equivalently, g is the right adjoint of f:

$f(x) \leq y \iff x \leq g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets. In particular, $f(x \vee y) = f(x) \vee f(y)$ and $g(x \wedge y) = g(x) \wedge g(y)$.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

Subclasses

DRLUn: Distributive residuated lattice-ordered unars

Superclasses

LUn: Lattice-ordered unars

RJUn: Residuated join-semilattice-ordered unars

RMUn: Residuated meet-semilattice-ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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24. DivLat: Division lattices

Definition

A *division lattice* is an algebra $\mathbf{P} = \langle P, \leq, \backslash, /\rangle$ such that P is a lattice and

$x \leq z/y \iff y \leq x\backslash z$

Formal Definition

$x \leq z/y \iff y \leq x\backslash z$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 216$

Subclasses

CDivLat: Commutative division lattices

DDivLat: Distributive division lattices

RLMag: Residuated lattice-ordered magmas

Superclasses

DivJslat: Division join-semilattices

DivMslat: Division meet-semilattices

LImpA: Lattice-ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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25. RLMag: Residuated lattice-ordered magmas

Definition

A *residuated lattice-ordered magma* (or *rpo-magma*) is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, /\rangle$ such that

$\langle A, \leq\rangle$ is a lattice,

$\langle A, \cdot\rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 20,\ f_4 = 1116$

Subclasses

CRLMag: Commutative residuated lattice-ordered magmas
DRLMag: Distributive residuated lattice-ordered magmas
InLMag: Involutive lattice-ordered magmas
RLSgrp: Residuated lattice-ordered semigroups

Superclasses

DivLat: Division lattices
LrLMag: Left-residuated lattice-ordered magmas
RJMag: Residuated join-semilattice-ordered magmas
RMMag: Residuated meet-semilattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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26. RLSgrp: Residuated lattice-ordered semigroups

Definition

A *residuated lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\langle A, \leq \rangle$ is a lattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 129,\ f_5 = 1852$

Subclasses

CRLSgrp: Commutative residuated lattice-ordered semigroups
DRLSgrp: Distributive residuated lattice-ordered semigroups
IdRLSgrp: Idempotent residuated lattice-ordered semigroups
InLSgrp: Involutive lattice-ordered semigroups
RL: Residuated lattices

Superclasses

LrLSgrp: Left-residuated lattice-ordered semigroups
RJSgrp: Residuated join-semilattice-ordered semigroups
RLMag: Residuated lattice-ordered magmas
RMSgrp: Residuated meet-semilattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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27. RL: Residuated lattices

Definition

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that
$\langle A, \wedge, \vee \rangle$ is a lattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	Variety
Equational theory	Decidable [(OK1985)] ((implementation))
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488,\ f_7 = 18554,\ f_8 = 295292$

Subclasses

ActLat: Action lattices

CRL: Commutative residuated lattices

CanRL: Cancellative residuated lattices

DRL: Distributive residuated lattices

FL: Full Lambek algebras

IRL: Integral residuated lattices

IdRL: Idempotent residuated lattices

bRL: Bounded residuated lattices

Superclasses

LrLMon: Left-residuated lattice-ordered monoids

RLSgrp: Residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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28. bRL: Bounded residuated lattices

Definition

A *bounded residuated lattice* is an algebra $\langle A, \wedge, \vee, \bot, \top, \cdot, 1, \backslash, /\rangle$ such that $\langle A, \wedge, \vee, \cdot, 1, \backslash, /\rangle$ is a residuated lattice,

$\bot$ is the least element: $\bot \vee x = x$ and

$\top$ is the greatest element: $\top \vee x = \top$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$\bot \vee x = x$

$\top \vee x = \top$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	no
Residual size	Unbounded
Congruence distributive	yes
Congruence modular	yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	yes
Congruence uniform	no
Congruence extension property	yes
Definable principal congruences	no
Equationally def. pr. cong.	no

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 149,\ f_6 = 1488$ Same as for finite residuated lattices.

Subclasses

ILLA: Intuitionistic linear logic algebras

MALLA: Multiplicative additive linear logic algebras

Superclasses

FL: Full Lambek algebras

RL: Residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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29. IRL: Integral residuated lattices

Definition

An *integral residuated lattice* is an residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 364$

Subclasses

CIRL: Commutative integral residuated lattices

DIRL: Distributive integral residuated lattices
IInFL: Integral involutive FL-algebras

Superclasses

ILrLMon: Integral left-residuated lattice-ordered monoids
RL: Residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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30. IdRLSgrp: Idempotent residuated lattice-ordered semigroups

Definition

An *idempotent residuated lattice-ordered semigroup* is a residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 169$

Subclasses

CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups
DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups
IdRL: Idempotent residuated lattices

Superclasses

IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups
IdRJSgrp: Idempotent residuated join-semilattice-ordered semigroups
IdRMSgrp: Idempotent residuated meet-semilattice-ordered semigroups
RLSgrp: Residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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31. IdRL: Idempotent residuated lattices

Definition

An *idempotent residuated lattice* is a residuated lattice-ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 32,\ f_6 = 147$

Subclasses

CIdRL: Commutative idempotent residuated lattices

DIdRL: Distributive idempotent residuated lattices

Superclasses

IdLrLMon: Idempotent left-residuated lattice-ordered monoids

IdRLSgrp: Idempotent residuated lattice-ordered semigroups

RL: Residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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32. FL: Full Lambek algebras

Definition

A *full Lambek algebra*, or *FL-algebra*, is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, /, 0\rangle$ of type $\langle 2, 2, 2, 0, 2, 2, 0\rangle$ such that

$\langle A, \wedge, \vee, \cdot, 1, \backslash, /\rangle$ is a residuated lattice and

0 is an additional constant (can denote any element).

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z/y$

$d = d$

Properties

Classtype	Variety
Equational theory	Decidable Ono and Komori [1985]
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, n=2
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 9,\ f_4 = 79,\ f_5 = 737$

Subclasses

FL_c: Full Lambek algebras with contraction

FL_e: Full Lambek algebras with exchange

FL_w: Full Lambek algebras with weakening
InFL: Involutive FL-algebras
bRL: Bounded residuated lattices

Superclasses

RL: Residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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33. FL_c: Full Lambek algebras with contraction

Definition

A FL_c*-algebra* is an FL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, /, 0\rangle$ such that

$\cdot$ is *contractive*: $x \leq x \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$d = d$

$x \leq x \cdot x$

Properties

Equational theory	undecidable[(CH2016)]
Quasiequational theory	undecidable
First-order theory	undecidable
Locally finite	no
Residual size	infinite
Congruence distributive	yes
Congruence modular	yes
Congruence n-permutable	yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 39,\ f_5 = 279$

Subclasses

FL_{ec}: Full Lambek algebras with exchange and contraction

Superclasses

FL: Full Lambek algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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34. FL_e: Full Lambek algebras with exchange

Definition

A *full Lambek algebra with exchange*, or FL_e*-algebra*, is an FL-algebra $\langle A, \wedge, \vee, \cdot, 1, \backslash, /, 0\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$d = d$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 9,\ f_4 = 63,\ f_5 = 492$

Subclasses

FL_{ec}: Full Lambek algebras with exchange and contraction

FL_{ew}: Full Lambek algebras with exchange and weakening

ILLA: Intuitionistic linear logic algebras

Superclasses

FL: Full Lambek algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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35. FL_w: Full Lambek algebras with weakening

Definition

A *FL_w-algebra* is an FL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, /, 0\rangle$ that is *integral* (i.e. satisfies the weakening rules): $0 \le x \le 1$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \le z \iff y \le x\backslash z$

$x \cdot y \le z \iff x \le z/y$

$x \cdot y = y \cdot x$

$0 \le x$

$x \le 1$

Properties

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129,\ f_7 = 723$

Subclasses

Superclasses

FL: Full Lambek algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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36. $\mathbf{FL}_{ec}$: Full Lambek algebras with exchange and contraction

Definition

A *full Lambek algebra with exchange and contraction*, or FL_{ec}*-algebra*, is an FL_e-algebra $\langle A, \vee, 0, \wedge, T, \cdot, 1, \backslash, /\rangle$ such that

$\cdot$ is contractive or square-increasing: $x \leq x \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$d = d$

$x \leq x \cdot x$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 31,\ f_5 = 199$

Subclasses

Superclasses

FL_c: Full Lambek algebras with contraction

FL_e: Full Lambek algebras with exchange

Cont|Po|J|M|L|D|To|B|U|Ind

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37. $\mathbf{FL}_{ew}$: Full Lambek algebras with exchange and weakening

Definition

A FL_{ew}*-algebra* is an FL_e-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, /, 0\rangle$ that is *integral* and bounded (i.e. satisfies the weakening rules): $0 \leq x \leq 1$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$
$0 \leq x$
$x \leq 1$

Properties

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129,\ f_7 = 723$

Subclasses

Superclasses

FL_e: Full Lambek algebras with exchange Cont|Po|J|M|L|D|To|B|U|Ind

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38. GalLat: Galois lattices

Definition

A *Galois lattice* is an algebra $\mathbf{P} = \langle P, \leq, \sim, - \rangle$ such that P is a lattice and $\sim, -$ are a pair of unary operations on P that form a
Galois connection: $x \leq {\sim}y \iff y \leq -x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

Basic Results

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 30,\ f_5 = 184$

Subclasses

DGalLat: Distributive Galois lattices
InLat: Involutive lattices

Superclasses

GalJslat: Galois join-semilattices
GalMslat: Galois meet-semilattices
LNUn: Lattice-ordered negated unars Cont|Po|J|M|L|D|To|B|U|Ind

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39. InLat: Involutive lattices

Definition

An *involutive lattice* is a Galois lattice $\mathbf{P} = \langle P, \leq, \sim, - \rangle$ such that $\sim, -$ are inverses of each other:
${\sim}{-}x = x$
$-{\sim}x = x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$
${\sim}{-}x = x$
$-{\sim}x = x$

Basic Results

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 5,\ f_6 = 14,\ f_7 = 27$

Subclasses

Bilat: Bilattices

DInLat: Distributive involutive lattices

InLMag: Involutive lattice-ordered magmas

Superclasses

GalLat: Galois lattices

InPos: Involutive posets

Cont|Po|J|M|L|D|To|B|U|Ind

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40. InLMag: Involutive lattice-ordered magmas

Definition

An *involutive lattice-ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\langle A, \leq, \cdot \rangle$ is a lattice-ordered magma,

$\sim, -$ is an involutive pair: $\sim{-x} = x = -{\sim x}$,

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$ and

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$.

Formal Definition

$\sim{-x} = x$

$-{\sim x} = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 42,\ f_5 = 342$

Subclasses

CyInLMag: Cyclic involutive lattice-ordered magmas

DInLMag: Distributive involutive lattice-ordered magmas

InLSgrp: Involutive lattice-ordered semigroups

Superclasses

InLat: Involutive lattices

InPoMag: Involutive partially ordered magmas

RLMag: Residuated lattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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41. InLSgrp: Involutive lattice-ordered semigroups

Definition

An *involutive lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\langle A, \leq, \cdot \rangle$ is an involutive lattice-ordered magma and

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 146,\ f_6 = 1308$

Subclasses

CyInLSgrp: Cyclic involutive lattice-ordered semigroups

DInLSgrp: Distributive involutive lattice-ordered semigroups

InFL: Involutive FL-algebras

Superclasses

InLMag: Involutive lattice-ordered magmas

InPoSgrp: Involutive partially ordered semigroups

RLSgrp: Residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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42. InFL: Involutive FL-algebras

Definition

An *involutive FL-algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim, -\rangle$ such that $\langle A, \wedge, \vee, \cdot, \sim, -\rangle$ is an involutive lattice-ordered semigroup that has an identity: $x \cdot 1 = x = 1 \cdot x$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety
Equational theory	Decidable Galatos and Jipsen [2013]
Locally finite	No
Residual size	∞
Congruence distributive	Yes
Congruence modular	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 21,\ f_6 = 101,\ f_7 = 284,\ f_8 = 1464$

Subclasses

CyInFL: Cyclic involutive FL-algebras

DInFL: Distributive involutive FL-algebras

IInFL: Integral involutive FL-algebras

Superclasses

FL: Full Lambek algebras

InLSgrp: Involutive lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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43. IInFL: Integral involutive FL-algebras

Definition

An *integral involutive FL-algebra* is an involutive FL-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ that is integral: $x \leq 1$

Formal Definition

$\sim - x = x$

$- \sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 17,\ f_8 = 78$

Subclasses

CyIInFL: Cyclic involutive lattice-ordered integral monoids

DIInFL: Distributive integral involutive FL-algebras

Superclasses

IRL: Integral residuated lattices

InFL: Involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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44. CyInLMag: Cyclic involutive lattice-ordered magmas

Definition

A *cyclic involutive lattice-ordered magma* (or *cyinpo-magma*) is an inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that $\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$- - x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 42,\ f_5 = 328$

Subclasses

CInLMag: Commutative involutive lattice-ordered magmas

CyDInLMag: Cyclic distributive involutive lattice-ordered magmas

CyInLSgrp: Cyclic involutive lattice-ordered semigroups

Superclasses

CyInPoMag: Cyclic involutive partially ordered magmas

InLMag: Involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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45. CyInLSgrp: Cyclic involutive lattice-ordered semigroups

Definition

A *cyclic involutive lattice-ordered semigroup* (or *cyinpo-semigroup*) is a cyinpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 132,\ f_6 = 1018$

Subclasses

CInLSgrp: Commutative involutive lattice-ordered semigroups

CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups

CyInFL: Cyclic involutive FL-algebras

Superclasses

CyInLMag: Cyclic involutive lattice-ordered magmas

CyInPoSgrp: Cyclic involutive partially ordered semigroups

InLSgrp: Involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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46. CyInFL: Cyclic involutive FL-algebras

Definition

A *cyclic involutive FL-algebra* is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	Variety
Equational theory	Decidable Galatos and Jipsen [2013]
Locally finite	No
Residual size	∞
Congruence distributive	Yes
Congruence modular	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 21,\ f_6 = 101,\ f_7 = 279,\ f_8 = 1433$

Subclasses

CInFL: Commutative involutive FL-algebras

CyDInFL: Cyclic distributive involutive FL-algebras

CyIInFL: Cyclic involutive lattice-ordered integral monoids

Superclasses

CyInLSgrp: Cyclic involutive lattice-ordered semigroups

InFL: Involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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47. CyIInFL: Cyclic involutive lattice-ordered integral monoids

Definition

A *cyclic integral involutive FL-algebra* is an inporim $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 15,\ f_8 = 75$

Subclasses

CIInFL: Commutative integral involutive FL-algebras

CyDIInFL: Cyclic distributive involutive lattice-ordered integral monoids

Superclasses

CyInFL: Cyclic involutive FL-algebras

IInFL: Integral involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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48. CLSgrp: Commutative lattice-ordered semigroups

Definition

A *commutative lattice-ordered semigroup* is a lattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety
Congruence distributive	yes
Congruence modular	yes

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 20,\ f_4 = 149,\ f_5 = 1427$

Subclasses

CDLSgrp: Commutative distributive lattice-ordered semigroups

CIdLSgrp: Commutative idempotent lattice-ordered semigroups

CLMon: Commutative lattice-ordered monoids

CRLSgrp: Commutative residuated lattice-ordered semigroups

Superclasses

CJSgrp: Commutative join-semilattice-ordered semigroups

CMSgrp: Commutative meet-semilattice-ordered semigroups

LSgrp: Lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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49. CLMon: Commutative lattice-ordered monoids

Definition

A *commutative lattice-ordered monoid* is a lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Congruence distributive	yes
Congruence modular	yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 31,\ f_5 = 199$

Subclasses

CDLMon: Commutative distributive lattice-ordered monoids

CILMon: Commutative Integral lattice-ordered monoids

CIdLMon: Commutative idempotent lattice-ordered monoids

CRL: Commutative residuated lattices

Superclasses

CJMon: Commutative join-semilattice-ordered monoids

CLSgrp: Commutative lattice-ordered semigroups

CMMon: Commutative meet-semilattice-ordered monoids

LMon: Lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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50. CILMon: Commutative Integral lattice-ordered monoids

Definition

A *commutative integral lattice-ordered monoid* is a integral lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129$

Subclasses

CDILMon: Commutative distributive integral lattice-ordered monoids

CIRL: Commutative integral residuated lattices

Superclasses

CIJMon: Commutative Integral join-semilattice-ordered monoids

CIMMon: Commutative Integral meet-semilattice-ordered monoids

CLMon: Commutative lattice-ordered monoids

ILMon: Integral lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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51. CIdLSgrp: Commutative idempotent lattice-ordered semigroups

Definition

A *commutative idempotent lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\langle A, \wedge, \vee, \cdot\rangle$ is an idempotent lattice-ordered semigroup and

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 19,\ f_5 = 86,\ f_6 = 462$

Subclasses

CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups
CIdLMon: Commutative idempotent lattice-ordered monoids
CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

Superclasses

CIdJSgrp: Commutative idempotent join-semilattice-ordered semigroups
CIdMSgrp: Commutative idempotent meet-semilattice-ordered semigroups
CLSgrp: Commutative lattice-ordered semigroups
IdLSgrp: Idempotent lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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52. CIdLMon: Commutative idempotent lattice-ordered monoids

Definition

A *commutative idempotent lattice-ordered monoid* is an idempotent lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$z \cdot (x \vee y) = z \cdot x \vee z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 4,\ f_4 = 12,\ f_5 = 41,\ f_6 = 159$

Subclasses

CDIdLMon: Commutative distributive idempotent lattice-ordered monoids
CIdRL: Commutative idempotent residuated lattices

Superclasses

CIdJMon: Commutative idempotent join-semilattice-ordered monoids
CIdLSgrp: Commutative idempotent lattice-ordered semigroups
CIdMMon: Commutative idempotent meet-semilattice-ordered monoids
CLMon: Commutative lattice-ordered monoids
IdLMon: Idempotent lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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53. CDivLat: Commutative division lattices

Definition

A *commutative division lattice* is a division lattice $\mathbf{P} = \langle P, \leq\rangle$ such that P is a lattice and

Formal Definition

$x \leq z/y \iff y \leq x\backslash z$
$x/y = y\backslash x$

Basic Results

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 64,\ f_4 = 6208$

Subclasses

CDDivLat: Commutative distributive division lattices

CRLMag: Commutative residuated lattice-ordered magmas

Superclasses

CDivJslat: Commutative division join-semilattices

CDivMslat: Commutative division meet-semilattices

DivLat: Division lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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54. BCKLat: BCK-lattices

Definition

A *BCK-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \to, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that

$\langle A, \vee, \to, 1\rangle$ is a BCK-join-semilattice

$\langle A, \wedge, \to, 1\rangle$ is a BCK-meet-semilattice

Remark: $x \le y \iff x \to y = 1$ is a partial order, with 1 as greatest element, and $\vee$, $\wedge$ are a join and meet for this order. Idziak [1984]

Formal Definition

$(x \vee y) \to z = (x \to z) \wedge (y \to z)$

$z \to (x \wedge y) = (z \to x) \wedge (z \to y)$

$(x \to y) \to ((y \to z) \to (x \to z)) = 1$

$1 \to x = x$

$x \to 1 = 1$

$x \to (x \vee y) = 1$

$x \vee ((x \to y) \to y) = ((x \to y) \to y)$

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	yes $n = 2$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129$

Subclasses

HA: Heyting algebras

Superclasses

BCKJslat: BCK-join-semilattices

BCKMslat: BCK-meet-semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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55. CRLMag: Commutative residuated lattice-ordered magmas

Definition

A *commutative residuated lattice-ordered magma* is a residuated lattice-ordered magma such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 148,\ f_5 = 4398$

Subclasses

CDRLMag: Commutative distributive residuated lattice-ordered magmas
CInLMag: Commutative involutive lattice-ordered magmas
CRLSgrp: Commutative residuated lattice-ordered semigroups

Superclasses

CDivLat: Commutative division lattices
CRJMag: Commutative residuated join-semilattice-ordered magmas
CRMMag: Commutative residuated meet-semilattice-ordered magmas
RLMag: Residuated lattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

56. CRLSgrp: Commutative residuated lattice-ordered semigroups

Definition

A *commutative residuated lattice-ordered semigroup* is a residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$x \cdot y = y \cdot x$

Properties

Classtype	variety
Congruence distributive	yes
Congruence modular	yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 57,\ f_5 = 550$

Subclasses

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups
CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups
CInLSgrp: Commutative involutive lattice-ordered semigroups
CRL: Commutative residuated lattices

Superclasses

CLSgrp: Commutative lattice-ordered semigroups
CMSgrp: Commutative meet-semilattice-ordered semigroups
CRJSgrp: Commutative residuated join-semilattice-ordered semigroups

CRLMag: Commutative residuated lattice-ordered magmas
CRMSgrp: Commutative residuated meet-semilattice-ordered semigroups
RLSgrp: Residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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57. CRL: Commutative residuated lattices

Definition

A *commutative residuated lattice* is a residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, n=2
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 100,\ f_6 = 794,\ f_7 = 7493,\ f_8 = 84961$

Subclasses

CDRL: Commutative distributive residuated lattices
CIRL: Commutative integral residuated lattices
CIdRL: Commutative idempotent residuated lattices
CInFL: Commutative involutive FL-algebras

Superclasses

CLMon: Commutative lattice-ordered monoids
CRLSgrp: Commutative residuated lattice-ordered semigroups
RL: Residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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58. CIRL: Commutative integral residuated lattices

Definition

A *lattice-ordered residuated integral monoid* is a residuated lattice-ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety
Congruence distributive	Yes (relatively) Blok and Raftery [1997]
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 129,\ f_7 = 723$

Subclasses

CDIRL: Commutative distributive integral residuated lattices

CIInFL: Commutative integral involutive FL-algebras

Superclasses

CILMon: Commutative Integral lattice-ordered monoids

CIRMMon: Commutative integral residuated meet-semilattice-ordered monoids

CRL: Commutative residuated lattices

IRL: Integral residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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59. CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

Definition

A *commutative idempotent residuated lattice-ordered semigroup* is an idempotent residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 36,\ f_6 = 202$

Subclasses

CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups

CIdRL: Commutative idempotent residuated lattices

Superclasses

CIdLSgrp: Commutative idempotent lattice-ordered semigroups

CIdRJSgrp: Commutative idempotent residuated join-semilattice-ordered semigroups

CIdRMSgrp: Commutative idempotent residuated meet-semilattice-ordered semigroups

CRLSgrp: Commutative residuated lattice-ordered semigroups

IdRLSgrp: Idempotent residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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60. CIdRL: Commutative idempotent residuated lattices

Definition

A *commutative idempotent residuated lattice* is an idmpotent residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that $\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 20,\ f_6 = 77$

Subclasses

CDIdRL: Commutative distributive idempotent residuated lattices

CIdInFL: Commutative idempotent involutive FL-algebras

Superclasses

CIdLMon: Commutative idempotent lattice-ordered monoids

CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

CRL: Commutative residuated lattices

IdRL: Idempotent residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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61. CIdInFL: Commutative idempotent involutive FL-algebras

Definition

A *commutative idempotent involutive FL-algebra* or *commutative idempotent involutive residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim \rangle$ of type $\langle 2, 2, 2, 0, 1 \rangle$ such that

$\langle A, \wedge, \vee \rangle$ is a lattice

$\langle A, \cdot, 1\rangle$ is a semilattice with top

$\sim$ is an *involution*: $\sim\sim x = x$ and

$xy \leq z \iff x \leq \sim(y(\sim z))$

Definition

A *commutative involutive FL-algebra* or *commutative involutive residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim\rangle$ of type $\langle 2, 2, 2, 0, 1\rangle$ such that

$\langle A, \vee\rangle$ is a semilattice

$\langle A, \cdot\rangle$ is a semilattice and

$x \leq z \iff x \cdot \sim y \leq \sim 1$, where $x \leq y \iff x \vee y = y$.

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot x = x$

Properties

Classtype	Value
Equational theory	Decidable Galatos and Jipsen [2013]
Locally finite	No
Residual size	∞
Congruence distributive	Yes
Congruence modular	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 2,\ f_6 = 4,\ f_7 = 4,\ f_8 = 9,\ f_9 = 10,\ f_{10} = 21,\ f_{11} = 22,\ f_{12} = 49,\ f_{13} = 52,\ f_{14} = 114,\ f_{15} = 121,\ f_{16} = 270$

Subclasses

Superclasses

CIdRL: Commutative idempotent residuated lattices

CInFL: Commutative involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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62. CInLMag: Commutative involutive lattice-ordered magmas

Definition

A *commutative involutive lattice-ordered magma* (or *cinpo-magma*) is a inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 38,\ f_5 = 238,\ f_6 = 2722$

Subclasses

CDInLMag: Commutative distributive involutive lattice-ordered magmas
CInLSgrp: Commutative involutive lattice-ordered semigroups

Superclasses

CInPoMag: Commutative involutive partially ordered magmas
CRLMag: Commutative residuated lattice-ordered magmas
CyInLMag: Cyclic involutive lattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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63. CInLSgrp: Commutative involutive lattice-ordered semigroups

Definition

A *commutative involutive lattice-ordered semigroup* (or *cinpo-semigroup*) is a inpo-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 130,\ f_6 = 984$

Subclasses

CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups
CInFL: Commutative involutive FL-algebras

Superclasses

CInLMag: Commutative involutive lattice-ordered magmas
CInPoSgrp: Commutative involutive partially ordered semigroups
CRLSgrp: Commutative residuated lattice-ordered semigroups
CyInLSgrp: Cyclic involutive lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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64. CInFL: Commutative involutive FL-algebras

Definition

A *commutative involutive FL-algebra* is an involutive FL-algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety
Equational theory	Decidable Galatos and Jipsen [2013]
Locally finite	No
Residual size	∞
Congruence distributive	Yes
Congruence modular	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 21,\ f_6 = 100,\ f_7 = 276,\ f_8 = 1392$

Subclasses

CDInFL: Commutative distributive involutive FL-algebras

CIInFL: Commutative integral involutive FL-algebras

CIdInFL: Commutative idempotent involutive FL-algebras

MALLA: Multiplicative additive linear logic algebras

Superclasses

CInLSgrp: Commutative involutive lattice-ordered semigroups

CRL: Commutative residuated lattices

CyInFL: Cyclic involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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65. CIInFL: Commutative integral involutive FL-algebras

Definition

A *commutative integral involutive FL-algebra* is an in-porim $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that $\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

$x \cdot 1 = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 15,\ f_8 = 70,\ f_9 = 112$

Subclasses

CDIInFL: Commutative distributive integral involutive FL-algebras

Superclasses

CIRL: Commutative integral residuated lattices

CInFL: Commutative involutive FL-algebras

CyIInFL: Cyclic involutive lattice-ordered integral monoids

InPocrim: Involutive partially ordered commutative integral monoids Cont|Po|J|M|L|D|To|B|U|Ind

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66. JsdLat: Join-semidistributive lattices

Definition

A *join-semidistributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ that satisfies the join-semidistributive law $\mathrm{SD}_\vee$: $x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z)$

Examples

Example 1: $D[d] = \langle D \cup \{d'\}, \vee, \wedge\rangle$, where D is any distributive lattice and d is an element in it that is split into two elements d, d' using Alan Day's doubling construction.

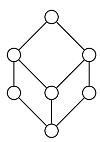
Properties

Classtype	Quasivariety
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Amalgamation property	No
Strong amalgamation property	No
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 4$, $f_6 = 9$, $f_7 = 23$, $f_8 = 65$, $f_9 = 197$, $f_{10} = 636$, $f_{11} = 2171$, $f_{12} = 7756$, $f_{13} = 28822$, $f_{14} = 110805$

Small Members (not in any subclass)



$\mathbf{JsdL}_{7,1}$

Subclasses

SdLat: Semidistributive lattices

Superclasses

Lat: Lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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67. MsdLat: Meet-semidistributive lattices

Definition

A *meet-semidistributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ that satisfies

the meet-semidistributive law $\mathrm{SD}_\wedge$: $x \wedge y = x \wedge z \implies x \wedge y = x \wedge (y \vee z)$

Examples

Example 1: $D[d] = \langle D \cup \{d'\}, \vee, \wedge\rangle$, where D is any distributive lattice and d is an element in it that is split into two elements d, d' using Alan Day's doubling construction.

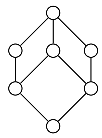
Properties

Classtype	Quasivariety
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Amalgamation property	No
Strong amalgamation property	No
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 4$, $f_6 = 9$, $f_7 = 23$, $f_8 = 65$, $f_9 = 197$, $f_{10} = 636$, $f_{11} = 2171$, $f_{12} = 7756$, $f_{13} = 28822$, $f_{14} = 110805$

Small Members (not in any subclass)



MsdL$_{7,1}$

Subclasses

SdLat: Semidistributive lattices

Superclasses

Lat: Lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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68. SdLat: Semidistributive lattices

Definition

A *semidistributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ such that

SD$_\wedge$: $x \wedge y = x \wedge z \implies x \wedge y = x \wedge (y \vee z)$

SD$_\vee$: $x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z)$

Examples

Example 1: $D[d] = \langle D \cup \{d'\}, \vee, \wedge \rangle$, where D is any distributive lattice and d is an element in it that is split into two elements d, d' using Alan Day's doubling construction.

Properties

Classtype	Quasivariety
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Amalgamation property	No
Strong amalgamation property	No
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 4$, $f_6 = 9$, $f_7 = 22$, $f_8 = 60$, $f_9 = 174$, $f_{10} = 534$, $f_{11} = 1720$, $f_{12} = 5767$, $f_{13} = 20013$, $f_{14} = 71546$

Subclasses

NdLat: Neardistributive lattices

Superclasses

JsdLat: Join-semidistributive lattices

MsdLat: Meet-semidistributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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69. NdLat: Neardistributive lattices

Definition

A *neardistributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ such that

$\mathrm{SD}^2_\wedge$: $x \wedge (y \vee z) = x \wedge [y \vee (x \wedge [z \vee (x \wedge y)])]$
$\mathrm{SD}^2_\vee$: $x \vee (y \wedge z) = x \vee [y \wedge (x \vee [z \wedge (x \vee y)])]$

Formal Definition

$x \wedge (y \vee z) = x \wedge [y \vee (x \wedge [z \vee (x \wedge y)])]$
$x \vee (y \wedge z) = x \vee [y \wedge (x \vee [z \wedge (x \vee y)])]$

Examples

Example 1: $D[d] = \langle D \cup \{d'\}, \vee, \wedge\rangle$, where D is any distributive lattice and d is an element in it that is split into two elements d, d' using Alan Day's doubling construction.

Properties

Classtype	Variety
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Amalgamation property	No
Strong amalgamation property	No
Locally finite	No
Residual size	Unbounded

Finite Members

Subclasses

AdLat: Almost distributive lattices

Superclasses

SdLat: Semidistributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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70. AdLat: Almost distributive lattices

Definition

An *almost distributive lattice* is a neardistributive lattice $\mathbf{L} = \langle L, \wedge, \vee\rangle$ such that

$\mathrm{AD}_\wedge$: $v \wedge [u \vee (x \wedge [y \vee (x \wedge z)])] \leq u \vee [(x \wedge [y \vee (x \wedge z)]) \wedge (v \vee (x \wedge y) \vee (x \wedge z))]$
$\mathrm{AD}_\vee$: $v \vee [u \wedge (x \vee [y \wedge (x \vee z)])] \geq u \wedge [(x \vee [y \wedge (x \vee z)]) \vee (v \wedge (x \vee y) \wedge (x \vee z))]$

Formal Definition

$v \wedge [u \vee (x \wedge [y \vee (x \wedge z)])] \leq u \vee [(x \wedge [y \vee (x \wedge z)]) \wedge (v \vee (x \wedge y) \vee (x \wedge z))]$
$v \vee [u \wedge (x \vee [y \wedge (x \vee z)])] \geq u \wedge [(x \vee [y \wedge (x \vee z)]) \vee (v \wedge (x \vee y) \wedge (x \vee z))]$

Examples

Example 1: $D[d] = \langle D \cup \{d'\}, \vee, \wedge\rangle$, where D is any distributive lattice and d is an element in it that is split into two elements d, d' using Alan Day's doubling construction.

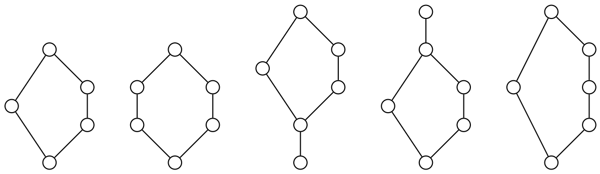
Properties

Classtype	Variety
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 4$

Small Members (not in any subclass)



$\mathbf{L}_{5,2}$ $\mathbf{L}_{6,2}$ $\mathbf{L}_{6,8}$ $\mathbf{L}_{6,9}$ $\mathbf{L}_{6,10}$

Subclasses

DLat: Distributive lattices

Superclasses

NdLat: Neardistributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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71. CplmLat: Complemented lattices

Definition

A *complemented lattice* is a bounded lattice $\mathbf{L} = \langle L, \vee, \bot, \wedge, \top \rangle$ such that every element has a complement: $\exists y(x \vee y = \top \text{ and } x \wedge y = \bot)$

Formal Definition

$\bot \vee x = x$

$\top \vee x = \top$

$\exists y(x \vee y = \top \text{ and } x \wedge y = \bot)$

Examples

Example 1: $\langle P(S), \cup, \emptyset, \cap, S \rangle$, the collection of subsets of a set S, with union, empty set, intersection, and the whole set S.

Properties

Classtype	first-order
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 2$

Subclasses

CplmModLat: Complemented modular lattices

Superclasses

bLat: Bounded lattices

Cont|Po|J|M|L|D|To|B|U|Ind

72. OLat: Ortholattices

Definition

An *ortholattice* is an algebra $\mathbf{L} = \langle L, \vee, \bot, \wedge, \top, ' \rangle$ such that

$\langle L, \vee, \bot, \wedge, \top \rangle$ is a bounded lattice

$'$ is complementation: $x \vee x' = \bot$, $x \wedge x' = \top$, $x'' = x$

$'$ satisfies De Morgan's laws: $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$

Examples

Example 1: $\langle P(S), \cup, \emptyset, \cap, S \rangle$, the collection of subsets of a set S, with union, empty set, intersection, and the whole set S.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes [(BrunsHarding1997)]

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 2,\ f_7 = 0,\ f_8 = 5,\ f_9 = 0,\ f_{10} = 15$

Subclasses

OModLat: Orthomodular lattices

Superclasses

Lat: Lattices

Cont|Po|J|M|L|D|To|B|U|Ind

73. OModLat: Orthomodular lattices

Definition

An *orthomodular lattice* is an ortholattice $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ' \rangle$ such that
the orthomodular law holds: $x \leq y \implies x \vee (x' \wedge y) = y$.
This law is equivalent to satisfying the identity $x \vee (x' \wedge (x \vee y)) = x \vee y$.

Examples

Example 1: The closed subspaces of (countably dimensional) Hilbert Space form an orthomodular lattice that is not modular (for finite dimensional vector spaces all subspaces are closed, hence the lattice of closed subspaces is modular).

Example 2: The smallest nonmodular orthomodular lattice has 10 elements and is isomorphic to a parallel sum of a 4-element Boolean algebra and an 8-element Boolean algebra. A failure of the modular law $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$ occurs when x, z are atoms of the 8-element algebra and y is an atom of the 4-element algebra.

Properties

Classtype	Variety
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 1,\ f_7 = 0,\ f_8 = 2$

Many Greechie diagrams of orthomodular lattices with blocks containing 3 atoms have been computed at http://cs.anu.edu.au/ Brendan.McKay/nauty/greechie.html

Subclasses

ModOLat: Modular ortholattices

Superclasses

OLat: Ortholattices

Cont|Po|J|M|L|D|To|B|U|Ind

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74. ModOLat: Modular ortholattices

Definition

A *modular ortholattice* is an ortholattice $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, ' \rangle$ such that
the *modular law* holds: $x \leq z \implies (x \vee y) \wedge z \leq x \vee (y \wedge z)$

Properties

Finite Members

Subclasses

BA: Boolean algebras

Superclasses

OModLat: Orthomodular lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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75. Bilat: Bilattices

Definition

A *bilattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \oplus, \otimes, \neg\rangle$ such that
$\langle L, \wedge, \vee\rangle$ is a lattice,
$\langle L, \oplus, \otimes\rangle$ is a lattice,
$\neg$ is a De Morgan operation for $\vee$, $\wedge$: $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg\neg x = x$ and
$\neg$ commutes with $\oplus$, $\otimes$: $\neg(x \oplus y) = \neg x \oplus \neg y$, $\neg(x \otimes y) = \neg x \otimes \neg y$.

Properties

Classtype	Variety
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_3 = 0,\ f_4 = 1,\ f_5 = 3,\ f_6 = 32,\ f_7 = 284$

Subclasses

TrivA: Trivial algebras

Superclasses

InLat: Involutive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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76. CanRL: Cancellative residuated lattices

Definition

A *cancellative residuated lattice* is a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, /\rangle$ such that
$\cdot$ is right-cancellative: $x \cdot z = y \cdot z \implies x = y$
$\cdot$ is left-cancellative: $z \cdot x = z \cdot y \implies x = y$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot z = y \cdot z \implies x = y$
$z \cdot x = z \cdot y \implies x = y$

Properties

Classtype	Variety
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_n = 0$ for $n > 1$

Subclasses

TrivA: Trivial algebras

Superclasses

RL: Residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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77. CplmModLat: Complemented modular lattices

Definition

A *complemented modular lattice* is a complemented lattice $\mathbf{L} = \langle L, \vee, 0, \wedge, 1\rangle$ that is a modular lattice: $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

Formal Definition

$((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

$\bot \vee x = x$

$\top \vee x = \top$

$\exists y(x \vee y = \top$ and $x \wedge y = \bot)$

Basic Results

This class generates the same variety as the class of its finite members plus the non-desargean planes.

Properties

Classtype	first-order
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence regular	No
Congruence uniform	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 1$

Subclasses

BA: Boolean algebras

Superclasses

CplmLat: Complemented lattices Cont|Po|J|M|L|D|To|B|U|Ind

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78. FRng: Function rings

Definition

A *function ring* (or *f-ring*) is a lattice-ordered ring $\mathbf{F} = \langle F, \wedge, \vee, +, -, 0, \cdot\rangle$ such that

$x \wedge y = 0$ and $z \geq 0 \implies x \cdot z \wedge y = 0$ and $z \cdot x \wedge y = 0$

Basic Results

The variety of f-rings is generated by the class of linearly ordered ℓ-rings. This means f-rings are subdirect products of linearly ordered ℓ-rings, i.e. f-rings are representable ℓ-rings (see e.g. [G. Birkhoff, Lattice Theory, 1967]).

Properties

Classtype	Variety
Congruence distributive	Yes, see lattices
Congruence n-permutable	Yes, $n = 2$, see groups
Congruence regular	Yes, see groups
Congruence uniform	Yes, see groups

Finite Members

Only the one-element f-ring.

Subclasses

TrivA: Trivial algebras

Superclasses

LRng: Lattice-ordered rings Cont|Po|J|M|L|D|To|B|U|Ind

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79. ILLA: Intuitionistic linear logic algebras

Definition

An *intuitionistic linear logic algebra* (or IL-algebra with storage Troelstra [1992]) is an algebra $\langle A, \vee, \bot, \wedge, \top, \cdot, 1, \backslash, /, 0, !\rangle$ such that $\langle A, \wedge, \vee, \cdot, 1, \to, 0\rangle$ is an FL_e-algebra

$\bot$ is the least element: $\bot \leq x$

$\top$ is the greatest element: $x \leq \top$

! is a storage operator: $!x \leq x$

$!x \leq y \implies !x \leq !y$

$!\top = 1$

$!(x \wedge y) = !x \cdot !y$

Properties

Classtype	variety

Finite Members

Subclasses

LLA: Linear logic algebras

Superclasses

FL_e: Full Lambek algebras with exchange

bRL: Bounded residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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80. LLA: Linear logic algebras

Definition

A *linear logic algebra* is an algebra $\mathbf{A} = \langle A, \vee, \bot, \wedge, \top, \cdot, 1, +, 0, \neg\rangle$ such that
$\langle A, \wedge, \vee, \cdot, 1, \neg\rangle$ is a commutative involutive FL-algebra
$\bot$ is the least element: $\bot \leq x$
$\top$ is the greatest element: $x \leq \top$
$+$ is the dual of $\cdot$: $x + y = \neg(\neg x \cdot \neg y)$
0 is the dual of 1: $0 = \neg 1$

Properties

Finite Members

Subclasses

TrivA: Trivial algebras

Superclasses

ILLA: Intuitionistic linear logic algebras
MALLA: Multiplicative additive linear logic algebras Cont|Po|J|M|L|D|To|B|U|Ind

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81. MALLA: Multiplicative additive linear logic algebras

Definition

A *multiplicative additive linear logic algebra* is an algebra $\mathbf{A} = \langle A, \vee, \bot, \wedge, \top, +, 0, \cdot, 1, ^{\perp}\rangle$ such that
$\langle A, \wedge, \vee, \cdot, 1, ^{\perp}\rangle$ is a commutative involutive FL-algebra,
$\bot$ is the least element: $\bot \leq x$
$\top$ is the greatest element: $x \leq \top$
$+$ is the dual of $\cdot$: $x + y = (x^{\perp} \cdot y^{\perp})^{\perp}$
0 is the dual of 1: $0 = 1^{\perp}$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence uniform	No

Finite Members

Subclasses

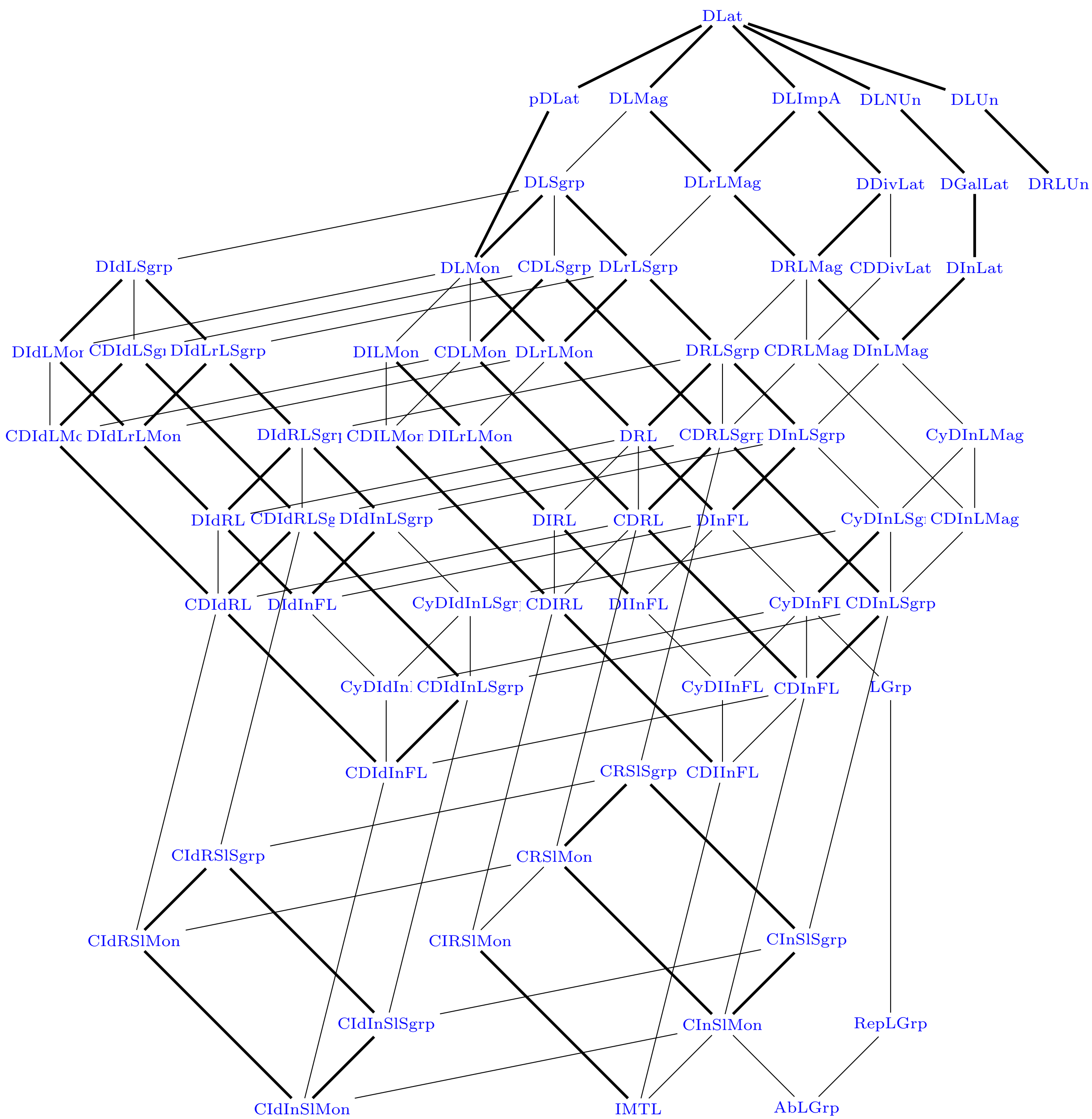
LLA: Linear logic algebras

Superclasses

CInFL: Commutative involutive FL-algebras
bRL: Bounded residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 6

Distributive lattice-ordered algebras



1. DLat: Distributive lattices

Formal Definition

A *distributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ such that
$\wedge$ distributes over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
and $\vee$ distributes over $\wedge$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Definition

A *distributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ such that
$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

Definition

A *distributive lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ such that $\mathbf{L}$ has no sublattice isomorphic to the diamond $\mathbf{M}_3$ or the pentagon $\mathbf{N}_5$

Definition

A *distributive lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ of type $\langle 2, 2 \rangle$ such that
$x \wedge (x \vee y) = x$ and
$x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$.[(Sholander1951)]

Examples

Example 1: $\langle P(S), \cup, \cap, \subseteq \rangle$, the collection of subsets of a sets S, ordered by inclusion.

Properties

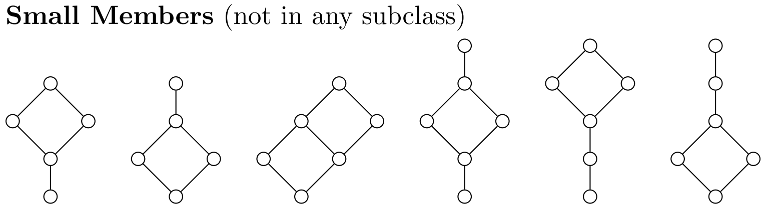
Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	No
Epimorphisms are surjective	No
Locally finite	Yes
Residual size	2

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8,\ f_8 = 15,\ f_9 = 26,\ f_{10} = 47,\ f_{11} = 82,\ f_{12} = 151,\ f_{13} = 269,\ f_{14} = 494,\ f_{15} = 891,\ f_{16} = 1639,\ f_{17} = 2978,\ f_{18} = 5483,\ f_{19} = 10006,\ f_{20} = 18428$

Values known up to size 49 Erné et al. [2002]

Small Members (not in any subclass)



$\mathbf{D}_{5,1}$ $\mathbf{D}_{5,2}$ $\mathbf{D}_{6,1}$ $\mathbf{D}_{6,2}$ $\mathbf{D}_{6,3}$ $\mathbf{D}_{6,4}$

Subclasses

BA: Boolean algebras

DLImpA: Distributive lattice-ordered implication algebras

DLMag: Distributive lattice-ordered magmas
DLNUn: Distributive lattice-ordered negated unars
DLUn: Distributive lattice-ordered unars
ToLat: Totally ordered lattices
bDLat: Bounded distributive lattices
pDLat: Pointed distributive lattices
pcDLat: Pseudocomplemented distributive lattices

Superclasses

AdLat: Almost distributive lattices
ModLat: Modular lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pDLat: Pointed distributive lattices

Definition

A *pointed distributive lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, c\rangle$ such that $\mathbf{A} = \langle A, \wedge, \vee\rangle$ is a distributive lattice and c is a constant operation on A.

Formal Definition

$c = c$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 7,\ f_5 = 13,\ f_6 = 27,\ f_7 = 50$

Subclasses

DLMon: Distributive lattice-ordered monoids
bDLat: Bounded distributive lattices
pBA: Pointed Boolean algebras
pToLat: Pointed totally ordered lattices

Superclasses

DLat: Distributive lattices
ToLat: Totally ordered lattices
pLat: Pointed lattices

Cont|Po|J|M|L|D|To|B|U|Ind

3. bDLat: Bounded distributive lattices

Definition

A *bounded distributive lattice* is an algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1\rangle$ such that

$\langle L, \vee, \wedge\rangle$ is a distributive lattice

0 is the least element: $0 \leq x$

1 is the greatest element: $x \leq 1$

Examples

Example 1: $\langle \mathcal{P}(S), \cup, \emptyset, \cap, S\rangle$, the collection of subsets of a set S, with union, empty set, intersection, and the whole set S.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	No
Epimorphisms are surjective	No
Locally finite	Yes
Residual size	2

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8,\ f_8 = 15,\ f_9 = 26,\ f_{10} = 47,\ f_{11} = 82,\ f_{12} = 151$
$f_{13} = 269,\ f_{14} = 494,\ f_{15} = 891,\ f_{16} = 1639,\ f_{17} = 2978,\ f_{18} = 5483,\ f_{19} = 10006,\ f_{20} = 18428$

Values known up to size 49 Erné et al. [2002].

Subclasses

BA: Boolean algebras

BoolLat: Boolean lattices

DdpAlg: Distributive dual p-algebras

DpAlg: Distributive p-algebras

OckA: Ockham algebras

Superclasses

DLat: Distributive lattices

bLat: Bounded lattices

pDLat: Pointed distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

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4. DLUn: Distributive lattice-ordered unars

Definition

A *distributive lattice-ordered unar* is an algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a distributive lattice and f is a unary operation on P that is

order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$f(x \vee y) = f(x) \vee f(y)$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 50,\ f_5 = 226$

Subclasses

BUn: Boolean unars

DGalLat: Distributive Galois lattices

DRLUn: Distributive residuated lattice-ordered unars
ToUn: Totally ordered unars

Superclasses

DLat: Distributive lattices
LUn: Lattice-ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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5. DLNUn: Distributive lattice-ordered negated unars

Definition

A *distributive lattice-ordered negated unar* is an algebra $\mathbf{P} = \langle P, \leq, \sim\rangle$ such that P is a distributive lattice and $\sim$ is a unary operation on P that is

order-reversing: $x \leq y \implies \sim y \leq \sim x$

Formal Definition

$x \leq y \implies \sim y \leq \sim x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 56,\ f_5 = 276$

Subclasses

BNUn: Boolean negated unars
DGalLat: Distributive Galois lattices
OckA: Ockham algebras
ToNUn: Totally ordered negated unars

Superclasses

DLat: Distributive lattices
LNUn: Lattice-ordered negated unars

Cont|Po|J|M|L|D|To|B|U|Ind

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6. pcDLat: Pseudocomplemented distributive lattices

Definition

A *pseudocomplemented distributive lattice* (also called a *Distributive p-algebra*) is an algebra $\mathbf{L} = \langle L, \vee, \bot, \wedge, ^* \rangle$ such that

$\langle L, \vee, \bot, \wedge\rangle$ is a distributive lattice with bottom element $\bot$

x^* is the *pseudo complement* of x: $y \leq x^* \iff x \wedge y = \bot$

Formal Definition

A *pseudocomplemented distributive lattice* is an algebra $\mathbf{L} = \langle L, \vee, \bot, \wedge, ^* \rangle$ such that

$\langle L, \wedge, \vee\rangle$ is a distributive lattice

$\bot$ is the bottom element: $\bot \leq x$

$x \wedge (x \wedge y)^* = x \wedge y^*$

$x \wedge \bot^* = x$

$(0^*)^* = 0$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Amalgamation property	Yes

Finite Members

Subclasses

DpAlg: Distributive p-algebras

Superclasses

DLat: Distributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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7. OckA: Ockham algebras

Definition

An *Ockham algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, ' \rangle$ such that

$\langle A, \vee, 0, \wedge, 1\rangle$ is a bounded distributive lattice

$'$ is a dual endomorphism: $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$, $0' = 1$, $1' = 0$

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

Subclasses

DmA: De Morgan algebras

Superclasses

DLNUn: Distributive lattice-ordered negated unars

bDLat: Bounded distributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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8. DmA: De Morgan algebras

Definition

A *De Morgan algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg\rangle$ such that

$\langle A, \vee, 0, \wedge, 1\rangle$ is a bounded distributive lattice

$\neg$ is a De Morgan involution: $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg\neg x = x$

Remark: It follows that $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg 1 = 0$ and $\neg 0 = 1$ (e.g. $\neg 1 = \neg 1 \vee 0 = \neg 1 \vee \neg\neg 0 = \neg(1 \wedge \neg 0) = \neg\neg 0 = 0$). Thus $\neg$ is a dual automorphism.

Examples

Example 1: Let $\{0 < a, b < 1\}$ be the 4-element lattice with a, b incomparable, and define $'$ by $0' = 1, a' = a, b' = b$.

Basic Results

The algebra in Example 1 generates the variety of De Morgan algebras, see e.g. `www.math.uic.edu/~kauffman/DeMorgan.pdf`

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes
Locally finite	Yes
Residual size	4

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 1,\ f_6 = 4,\ f_7 = 2,\ f_8 = 9,\ f_9 = 5,\ f_{10} = 14$

Subclasses

KLA: Kleene logic algebras

LA_n: Lukasiewicz algebras of order n

Superclasses

OckA: Ockham algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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9. DmMon: De Morgan monoids

Definition

A *De Morgan monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, ' , \rangle$ such that

$\langle A, \wedge, \vee \rangle$ is a distributive lattice,

$\langle A, \cdot, 1 \rangle$ is a commutative monoid,

$\cdot$ is involutive residuated: $x \cdot y \leq z \iff y \leq (z' \cdot x)'$ and

$\cdot$ is square-increasing: $x \leq x \cdot x$.

Remark: It follows that $x'' = x$ and that $(x \vee y)' = x' \wedge y'$.

Note that a De Morgan monoid is the same thing as a commutative distributive involutive square-increasing residuated lattice.

Properties

Classtype	Variety

Finite Members

Subclasses

Superclasses

CDInFL: Commutative distributive involutive FL-algebras

DunnMon: Dunn monoid

Cont|Po|J|M|L|D|To|B|U|Ind

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10. DpAlg: Distributive p-algebras

Definition

A *distributive p-algebra* is an algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ^* \rangle$ such that

$\langle L, \vee, 0, \wedge, 1 \rangle$ is a bounded distributive lattice

x^* is the *pseudo complement* of x: $y \leq x^* \iff x \wedge y = 0$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Amalgamation property	Yes

Finite Members

Subclasses

StAlg: Stone algebras

Superclasses

bDLat: Bounded distributive lattices

pcDLat: Pseudocomplemented distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

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11. DdpAlg: Distributive dual p-algebras

Definition

A *distributive dual p-algebra* is an algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ^+ \rangle$ such that

$\langle L, \vee, 0, \wedge, 1\rangle$ is a bounded distributive lattice

x^+ is the *dual pseudocomplement* of x: $x^+ \leq y \iff x \vee y = 1$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes
Amalgamation property	Yes

Finite Members

Subclasses

DDblpAlg: Distributive double p-algebras

Superclasses

bDLat: Bounded distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

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12. DDblpAlg: Distributive double p-algebras

Definition

A *distributive double p-algebra* is an algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ^*, ^+ \rangle$ such that

$\langle L, \vee, 0, \wedge, 1, ^* \rangle$ is a distributive p-algebra and

$\langle L, \vee, 0, \wedge, 1, ^+ \rangle$ is a distributive dual p-algebra

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes

Finite Members

Subclasses

DblStAlg: Double Stone algebras

Superclasses

DdpAlg: Distributive dual p-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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13. StAlg: Stone algebras

Definition

A *Stone algebra* is a distributive p-algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ^* \rangle$ such that

$(x^*)^* \vee x^* = 1$, $0^* = 1$

Properties

Equational theory	decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes
Amalgamation property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 2,\ f_6 = 4,\ f_7 = 5,\ f_8 = 10,\ f_9 = 16,\ f_{10} = 28$

Subclasses

DblStAlg: Double Stone algebras

Superclasses

DpAlg: Distributive p-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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14. DblStAlg: Double Stone algebras

Definition

A *double Stone algebra* is an algebra $\mathbf{L} = \langle L, \vee, 0, \wedge, 1, ^* \rangle$ such that

$\langle L, \vee, 0, \wedge, 1, ^* \rangle$ is a Stone algebra

$\langle L, \wedge, 1, \vee, 0, ^* \rangle$ is a Stone algebra

Properties

Classtype	Variety
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes

Finite Members

Subclasses

BA: Boolean algebras

Superclasses

DDblpAlg: Distributive double p-algebras

StAlg: Stone algebras Cont|Po|J|M|L|D|To|B|U|Ind

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15. DLMag: Distributive lattice-ordered magmas

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

BMag: Boolean magmas

DLSgrp: Distributive lattice-ordered semigroups

DLrLMag: Distributive left-residuated lattice-ordered magmas

ToMag: Totally ordered magmas

Superclasses

DLat: Distributive lattices

LMag: Lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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16. DLSgrp: Distributive lattice-ordered semigroups

Definition

A *distributive lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that

$\langle A, \cdot \rangle$ is a semigroup

$\langle G, \leq \rangle$ is a distributive lattice

$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 44,\ f_4 = 479$

Subclasses

BSgrp: Boolean semigroups

CDLSgrp: Commutative distributive lattice-ordered semigroups

DIdLSgrp: Distributive idempotent lattice-ordered semigroups

DLMon: Distributive lattice-ordered monoids

DLrLSgrp: Distributive left-residuated lattice-ordered semigroups

ToSgrp: Totally ordered semigroups

Superclasses

DLMag: Distributive lattice-ordered magmas

LSgrp: Lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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17. DLMon: Distributive lattice-ordered monoids

Definition

A *distributive lattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1 \rangle$ such that

$\langle A, \cdot, 1 \rangle$ is a monoid

$\langle G, \leq \rangle$ is a distributive lattice

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 45,\ f_5 = 279$

Subclasses

BMon: Boolean monoids
CDLMon: Commutative distributive lattice-ordered monoids
DILMon: Distributive integral lattice-ordered monoids
DIdLMon: Distributive idempotent lattice-ordered monoids
DLrLMon: Distributive left-residuated lattice-ordered monoids
ToMon: Totally ordered monoids

Superclasses

DLSgrp: Distributive lattice-ordered semigroups
LMon: Lattice-ordered monoids
pDLat: Pointed distributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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18. DILMon: Distributive integral lattice-ordered monoids

Definition

A *distributive integral lattice-ordered monoid* is a distributive lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that
$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 359$

Subclasses

BIMon: Boolean integral monoids
CDILMon: Commutative distributive integral lattice-ordered monoids
DILrLMon: Distributive integral left-residuated lattice-ordered monoids
IToMon: Integral totally ordered monoids

Superclasses

DLMon: Distributive lattice-ordered monoids
ILMon: Integral lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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19. DIdLSgrp: Distributive idempotent lattice-ordered semigroups

Definition

An *distributive idempotent lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that
$\langle A, \wedge, \vee, \cdot\rangle$ is a distributive lattice-ordered semigroup and
$\cdot$ is *distributive idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 17,\ f_4 = 100,\ f_5 = 576$

Subclasses

BIdSgrp: Boolean idempotent semigroups
CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups
DIdLMon: Distributive idempotent lattice-ordered monoids
DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups
IdToSgrp: Idempotent totally ordered semigroups

Superclasses

DLSgrp: Distributive lattice-ordered semigroups
IdLSgrp: Idempotent lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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20. DIdLMon: Distributive idempotent lattice-ordered monoids

Definition

An *distributive idempotent lattice-ordered monoid* is a distributive lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *distributive idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 22,\ f_5 = 75,\ f_6 = 274$

Subclasses

BIdMon: Boolean idempotent monoids
CDIdLMon: Commutative distributive idempotent lattice-ordered monoids
DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids
IdToMon: Idempotent totally ordered monoids

Superclasses

DIdLSgrp: Distributive idempotent lattice-ordered semigroups
DLMon: Distributive lattice-ordered monoids
IdLMon: Idempotent lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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21. DLImpA: Distributive lattice-ordered implication algebras

Formal Definition

$x \leq y \implies y \to z \leq x \to z$
$x \leq y \implies z \to x \leq z \to y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

BImpA: Boolean implication algebras
CDLSgrp: Commutative distributive lattice-ordered semigroups
DDivLat: Distributive division lattices
DLrLMag: Distributive left-residuated lattice-ordered magmas
ImpLat: Implicative lattices
LSgrp: Lattice-ordered semigroups
ToImpA: Totally ordered implication algebras

Superclasses

DLat: Distributive lattices
LImpA: Lattice-ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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22. DLrLMag: Distributive left-residuated lattice-ordered magmas

Definition

A *distributive left-residuated lattice-ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a distributive lattice,
$\langle A, \cdot \rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 50,\ f_4 = 4441$

Subclasses

BLrMag: Boolean left-residuated magmas
DLrLSgrp: Distributive left-residuated lattice-ordered semigroups
DRLMag: Distributive residuated lattice-ordered magmas
LrToMag: Left-residuated totally ordered magmas

Superclasses

DLImpA: Distributive lattice-ordered implication algebras
DLMag: Distributive lattice-ordered magmas
LrLMag: Left-residuated lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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23. DLrLSgrp: Distributive left-residuated lattice-ordered semigroups

Definition

A *distributive left-residuated lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq\rangle$ is a distributive lattice,
$\langle A, \cdot\rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 18,\ f_4 = 183,\ f_5 = 1968$

Subclasses

BLrSgrp: Boolean left-residuated semigroups
DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups
DLrLMon: Distributive left-residuated lattice-ordered monoids
DRLSgrp: Distributive residuated lattice-ordered semigroups
LrToSgrp: Left-residuated totally ordered semigroups

Superclasses

DLSgrp: Distributive lattice-ordered semigroups
DLrLMag: Distributive left-residuated lattice-ordered magmas
LrLSgrp: Left-residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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24. DLrLMon: Distributive left-residuated lattice-ordered monoids

Definition

A *distributive left-residuated lattice-ordered monoid* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ such that
$\langle A, \leq\rangle$ is a distributive lattice,
$\langle A, \cdot, 1\rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 23,\ f_5 = 130,\ f_6 = 976$

Subclasses

BILrMon: Boolean integral left-residuated monoids
DILrLMon: Distributive integral left-residuated lattice-ordered monoids
DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids

DRL: Distributive residuated lattices
LrToMon: Left-residuated totally ordered monoids

Superclasses

DLMon: Distributive lattice-ordered monoids
DLrLSgrp: Distributive left-residuated lattice-ordered semigroups
LrLMon: Left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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25. DILrLMon: Distributive integral left-residuated lattice-ordered monoids

Definition

A *distributive lattice-ordered left-residuated integral monoid* is a distributive left-residuated lattice-ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ for which
$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 359$

Subclasses

BIdLrSgrp: Boolean idempotent left-residuated semigroups
DIRL: Distributive integral residuated lattices
ILrToMon: Integral left-residuated totally ordered monoids

Superclasses

DILMon: Distributive integral lattice-ordered monoids
DLrLMon: Distributive left-residuated lattice-ordered monoids
ILrLMon: Integral left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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26. DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups

Definition

An *distributive idempotent left-residuated lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that $\langle A, \wedge, \vee, \cdot \rangle$ is a distributive left-residuated lattice-ordered semigroup and
$\cdot$ is *distributive idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 40,\ f_5 = 213$

Subclasses

BIdLrMon: Boolean idempotent left-residuated monoids

DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids

DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups

IdLrToSgrp: Idempotent left-residuated totally ordered semigroups

Superclasses

DIdLSgrp: Distributive idempotent lattice-ordered semigroups

DLrLSgrp: Distributive left-residuated lattice-ordered semigroups

IdLrLSgrp: Idempotent left-residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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27. DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids

Definition

An *distributive idempotent left-residuated lattice-ordered monoid* is a distributive left-residuated lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 11,\ f_5 = 37,\ f_6 = 134$

Subclasses

BRUn: Boolean residuated unars

DIdRL: Distributive idempotent residuated lattices

IdLrToMon: Idempotent left-residuated totally ordered monoids

Superclasses

DIdLMon: Distributive idempotent lattice-ordered monoids

DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups

DLrLMon: Distributive left-residuated lattice-ordered monoids

IdLrLMon: Idempotent left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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28. DRLUn: Distributive residuated lattice-ordered unars

Definition

A *distributive residuated lattice-ordered unar* (also called an *drℓ-unar* for short) is a residuated lattice-ordered unar $\langle D, \wedge, \vee, f, g\rangle$ such that $\langle D, \wedge, \vee\rangle$ is a distributive lattice.

Formal Definition

$f(x) \leq y \iff x \leq g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets. In particular, $f(x \vee y) = f(x) \vee f(y)$ and $g(x \wedge y) = g(x) \wedge g(y)$.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

Subclasses

BDivLat: Boolean division lattices

RToUn: Residuated totally-ordered unars

Superclasses

DLUn: Distributive lattice-ordered unars

RLUn: Residuated lattice-ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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29. DDivLat: Distributive division lattices

Definition

A *distributive division lattice* is a division lattice $\langle D, \wedge, \vee, \backslash, /\rangle$ such that $\langle D, \wedge, \vee\rangle$ is a distributive lattice.

Formal Definition

$x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z$,

$(x \wedge y)/z = x/z \wedge y/z$ and

$x \leq z/y \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 216$

Subclasses

BRMag: Boolean residuated magmas

CDDivLat: Commutative distributive division lattices

DRLMag: Distributive residuated lattice-ordered magmas

ToDivLat: Totally ordered division lattices

Superclasses

DLImpA: Distributive lattice-ordered implication algebras

DivLat: Division lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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30. DRLMag: Distributive residuated lattice-ordered magmas

Definition

A *distributive residuated lattice-ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, /\rangle$ such that

$\langle A, \leq\rangle$ is a distributive lattice,

$\langle A, \cdot\rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 20,\ f_4 = 1116$

Subclasses

BRSgrp: Boolean residuated semigroups
CDRLMag: Commutative distributive residuated lattice-ordered magmas
DInLMag: Distributive involutive lattice-ordered magmas
DRLSgrp: Distributive residuated lattice-ordered semigroups
RToMag: Residuated totally ordered magmas

Superclasses

DDivLat: Distributive division lattices
DLrLMag: Distributive left-residuated lattice-ordered magmas
RLMag: Residuated lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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31. DRLSgrp: Distributive residuated lattice-ordered semigroups

Definition

A *distributive residuated lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\langle A, \leq \rangle$ is a distributive lattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 129,\ f_5 = 1437$

Subclasses

BRL: Boolean residuated lattices
CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups
DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups
DInLSgrp: Distributive involutive lattice-ordered semigroups
DRL: Distributive residuated lattices
RToSgrp: Residuated totally ordered semigroups

Superclasses

DLrLSgrp: Distributive left-residuated lattice-ordered semigroups

DRLMag: Distributive residuated lattice-ordered magmas

RLSgrp: Residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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32. DRL: Distributive residuated lattices

Definition

A *distributive residuated lattice* is a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that $\wedge, \vee$ are distributive: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	Variety
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, n=2
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 20,\ f_5 = 115,\ f_6 = 899,\ f_7 = 7782,\ f_8 = 80468$

Subclasses

BIRL: Boolean integral residuated lattices

CDRL: Commutative distributive residuated lattices

DIRL: Distributive integral residuated lattices

DIdRL: Distributive idempotent residuated lattices

DInFL: Distributive involutive FL-algebras

GBL: Generalized BL-algebras

RToMon: Residuated totally ordered monoids

Superclasses

DLrLMon: Distributive left-residuated lattice-ordered monoids

DRLSgrp: Distributive residuated lattice-ordered semigroups

RL: Residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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33. DIRL: Distributive integral residuated lattices

Definition

A *distributive integral residuated lattice* is an distributive residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that
x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 49,\ f_6 = 359$

Subclasses

BIdRSgrp: Boolean idempotent residuated semigroups
CDIRL: Commutative distributive integral residuated lattices
DIInFL: Distributive integral involutive FL-algebras
IRToMon: Integral residuated totally ordered monoids

Superclasses

DILrLMon: Distributive integral left-residuated lattice-ordered monoids
DRL: Distributive residuated lattices
IRL: Integral residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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34. DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups

Definition

An *distributive idempotent residuated lattice-ordered semigroup* is a distributive residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that
$\cdot$ is *distributive idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 24,\ f_5 = 124$

Subclasses

BIdRL: Boolean idempotent residuated lattices

CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups
DIdRL: Distributive idempotent residuated lattices
IdRToSgrp: Idempotent residuated totally ordered semigroups

Superclasses

DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups
DRLSgrp: Distributive residuated lattice-ordered semigroups
IdRLSgrp: Idempotent residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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35. DIdRL: Distributive idempotent residuated lattices

Definition

An *distributive idempotent residuated lattice* is a distributive residuated lattice-ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 27,\ f_6 = 96$

Subclasses

CDIdRL: Commutative distributive idempotent residuated lattices
IdRToMon: Idempotent residuated totally ordered monoids

Superclasses

DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids
DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups
DRL: Distributive residuated lattices
IdRL: Idempotent residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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36. DGalLat: Distributive Galois lattices

Definition

A *distributive Galois lattice* is an algebra $\mathbf{P} = \langle P, \leq, \sim, - \rangle$ such that P is a distributive lattice and $\sim, -$ are a pair of unary operations on P that form a
Galois connection: $x \leq {\sim}y \iff y \leq -x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 30,\ f_5 = 126$

Subclasses

BGalLat: Boolean Galois lattices

DInLat: Distributive involutive lattices

GalToLat: Galois chains

Superclasses

DLNUn: Distributive lattice-ordered negated unars

DLUn: Distributive lattice-ordered unars

GalLat: Galois lattices Cont|Po|J|M|L|D|To|B|U|Ind

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37. DInLat: Distributive involutive lattices

Definition

A *distributive involutive lattice* is a distributive Galois lattice $\mathbf{P} = \langle P, \leq, \sim, -\rangle$ such that $\sim, -$ are inverses of each other:

$\sim{-x} = x$

$-{\sim x} = x$

Formal Definition

$x \leq {\sim y} \iff y \leq -x$

$\sim{-x} = x$

$-{\sim x} = x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 1,\ f_6 = 4,\ f_7 = 3,\ f_8 = 11$

Subclasses

BInMag: Boolean involutive magmas

DInLMag: Distributive involutive lattice-ordered magmas

InToLat: Involutive chains

Superclasses

DGalLat: Distributive Galois lattices

InLat: Involutive lattices Cont|Po|J|M|L|D|To|B|U|Ind

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38. DInLMag: Distributive involutive lattice-ordered magmas

Definition

A *distributive involutive lattice-ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that

$\langle A, \leq, \cdot\rangle$ is a distributive lattice-ordered magma,

$\sim, -$ is an involutive pair: $\sim{-x} = x = -{\sim x}$,

$x \cdot y \leq z \iff y \leq {\sim(-z \cdot x)}$ and

$x \cdot y \leq z \iff x \leq -(y \cdot {\sim z})$.

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 42,\ f_5 = 164$

Subclasses

BInSgrp: Boolean involutive semigroups

CyDInLMag: Cyclic distributive involutive lattice-ordered magmas

DInLSgrp: Distributive involutive lattice-ordered semigroups

InToMag: Involutive totally ordered magmas

Superclasses

DInLat: Distributive involutive lattices

DRLMag: Distributive residuated lattice-ordered magmas

InLMag: Involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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39. DInLSgrp: Distributive involutive lattice-ordered semigroups

Definition

An *distributive involutive lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\langle A, \leq, \cdot \rangle$ is an distributive involutive lattice-ordered magma and
$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 63,\ f_6 = 454$

Subclasses

BInFL: Boolean involutive FL-algebras

CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups

DInFL: Distributive involutive FL-algebras

InToSgrp: Involutive totally ordered semigroups

Superclasses

DInLMag: Distributive involutive lattice-ordered magmas

DRLSgrp: Distributive residuated lattice-ordered semigroups

InLSgrp: Involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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40. DInFL: Distributive involutive FL-algebras

Definition

An *distributive involutive FL-algebra* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that $\langle A, \leq, \cdot\rangle$ is an distributive involutive lattice-ordered semigroup that has an identity: $x \cdot 1 = x = 1 \cdot x$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 8,\ f_6 = 43,\ f_7 = 49$

Subclasses

BIInFL: Boolean integral involutive FL-algebras

CyDInFL: Cyclic distributive involutive FL-algebras

DIInFL: Distributive integral involutive FL-algebras

Superclasses

DInLSgrp: Distributive involutive lattice-ordered semigroups

DRL: Distributive residuated lattices

InFL: Involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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41. DIInFL: Distributive integral involutive FL-algebras

Definition

A *distributive integral involutive FL-algebra* is an involutive FL-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ that is integral: $x \leq 1$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 13,\ f_8 = 66$

Subclasses

BCyInMag: Boolean cyclic involutive magmas

CyDIInFL: Cyclic distributive involutive lattice-ordered integral monoids
psMV: Pseudo MV-algebras

Superclasses

DIRL: Distributive integral residuated lattices
DInFL: Distributive involutive FL-algebras
IInFL: Integral involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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42. CyDInLMag: Cyclic distributive involutive lattice-ordered magmas

Definition

A *cyclic distributive involutive lattice-ordered magma* is an inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that $\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 42,\ f_5 = 156$

Subclasses

BCyInSgrp: Boolean cyclic involutive semigroups
CDInLMag: Commutative distributive involutive lattice-ordered magmas
CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups
CyInToMag: Cyclic involutive totally ordered magmas

Superclasses

CyInLMag: Cyclic involutive lattice-ordered magmas
DInLMag: Distributive involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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43. CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups

Definition

A *cyclic distributive involutive lattice-ordered semigroup* is a cyinpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that $\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 55,\ f_6 = 353$

Subclasses

BCyInFL: Boolean cyclic involutive FL-algebras
CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

CyDInFL: Cyclic distributive involutive FL-algebras
CyInToSgrp: Cyclic involutive totally ordered semigroups

Superclasses

CyDInLMag: Cyclic distributive involutive lattice-ordered magmas
CyInLSgrp: Cyclic involutive lattice-ordered semigroups
DInLSgrp: Distributive involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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44. CyDInFL: Cyclic distributive involutive FL-algebras

Definition

A *cyclic distributive involutive FL-algebra* is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that
$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 8,\ f_6 = 43,\ f_7 = 48$

Subclasses

BCyIInFL: Boolean cyclic involutive integral monoids
CDInFL: Commutative distributive involutive FL-algebras
CyDIInFL: Cyclic distributive involutive lattice-ordered integral monoids
LGrp: Lattice-ordered groups

Superclasses

CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups
CyInFL: Cyclic involutive FL-algebras
DInFL: Distributive involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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45. CyDIInFL: Cyclic distributive involutive lattice-ordered integral monoids

Definition

A *cyclic distributive integral involutive FL-algebra* is an inporim $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that
$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

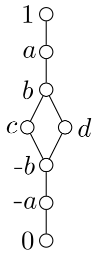
Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 12,\ f_8 = 65$

Small Members (not in any subclass)



InD$_{8,1}$

$\cdot$	a	b	d	c	-b	-a
a	a	d	d	-b	-b	0
b	c	-b	-a	-b	0	0
d	-b	-b	0	-b	0	0
c	c	-a	-a	0	0	0
-b	-b	0	0	0	0	0
-a	0	0	0	0	0	0

CyDIInFL$_{8,1}$

Subclasses

CDIInFL: Commutative distributive integral involutive FL-algebras

Superclasses

CyDInFL: Cyclic distributive involutive FL-algebras

CyIInFL: Cyclic involutive lattice-ordered integral monoids

DIInFL: Distributive integral involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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46. LGrp: Lattice-ordered groups

Definition

A *lattice-ordered group* is an algebra $\mathbf{G} = \langle G, \cdot, ^{-1}, 1, \leq\rangle$ such that

$\langle G, \cdot, ^{-1}, 1\rangle$ is a group

$\langle G, \leq\rangle$ is a lattice

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot x^{-1} = 1$

Examples

Basic Results

Properties

Classtype	Variety
Equational theory	Decidable Holland and McCleary [1979]
Quasiequational theory	Undecidable Glass and Gurevich [1983]
First-order theory	hereditarily undecidable Burris [1985]
Congruence distributive	Yes, see lattices
Congruence n-permutable	Yes, $n = 2$, see groups
Congruence regular	Yes, see groups
Congruence uniform	Yes, see groups
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_6 = 0$

Subclasses

NVLGrp: Normal valued lattice-ordered groups

Superclasses

CyDInFL: Cyclic distributive involutive FL-algebras

ImpLat: Implicative lattices

PoGrp: Partially ordered groups Cont|Po|J|M|L|D|To|B|U|Ind

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47. RepLGrp: Representable lattice-ordered groups

Definition

A *representable lattice-ordered group* is an algebra $\mathbf{G} = \langle G, \cdot, ^{-1}, 1, \leq\rangle$ such that

$\langle G, \cdot, ^{-1}, 1\rangle$ is a group

$\langle G, \leq\rangle$ is a lattice

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot x^{-1} = 1$

Examples

Basic Results

Properties

Classtype	Variety
Equational theory	Decidable Holland and McCleary [1979]
Quasiequational theory	Undecidable Glass and Gurevich [1983]
First-order theory	hereditarily undecidable Burris [1985]
Congruence distributive	Yes, see lattices
Congruence n-permutable	Yes, $n = 2$, see groups
Congruence regular	Yes, see groups
Congruence uniform	Yes, see groups
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_6 = 0$

Subclasses

AbLGrp: Abelian lattice-ordered groups

ToGrp: Totally ordered groups

Superclasses

NVLGrp: Normal valued lattice-ordered groups Cont|Po|J|M|L|D|To|B|U|Ind

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48. CDLSgrp: Commutative distributive lattice-ordered semigroups

Definition

A *commutative distributive lattice-ordered semigroup* is a distributive lattice-ordered semigroup $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 20,\ f_4 = 149,\ f_5 = 1106$

Subclasses

BCMon: Boolean commutative monoids
CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups
CDLMon: Commutative distributive lattice-ordered monoids
CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups
CToSgrp: Commutative totally ordered semigroups

Superclasses

CLSgrp: Commutative lattice-ordered semigroups
DLImpA: Distributive lattice-ordered implication algebras
DLSgrp: Distributive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

49. CDLMon: Commutative distributive lattice-ordered monoids

Definition

A *commutative distributive lattice-ordered monoid* is a distributive lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 31,\ f_5 = 149$

Subclasses

BCIMon: Boolean commutative integral monoids
CDILMon: Commutative distributive integral lattice-ordered monoids
CDIdLMon: Commutative distributive idempotent lattice-ordered monoids
CDRL: Commutative distributive residuated lattices
CToMon: Commutative totally ordered monoids

Superclasses

CDLSgrp: Commutative distributive lattice-ordered semigroups
CLMon: Commutative lattice-ordered monoids
DLMon: Distributive lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

50. CDILMon: Commutative distributive integral lattice-ordered monoids

Definition

A *commutative distributive integral lattice-ordered monoid* is a distributive integral lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 124,\ f_7 = 645$

Subclasses

BCIdSgrp: Boolean commutative idempotent semigroups

CDIRL: Commutative distributive integral residuated lattices

CIToMon: Commutative integral totally ordered monoids

Superclasses

CDLMon: Commutative distributive lattice-ordered monoids

CILMon: Commutative Integral lattice-ordered monoids

DILMon: Distributive integral lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

51. CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups

Definition

A *commutative distributive idempotent lattice-ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that $\langle A, \wedge, \vee, \cdot\rangle$ is an distributive idempotent lattice-ordered semigroup and

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 19,\ f_5 = 68$

Subclasses

BCIdMon: Boolean commutative idempotent monoids

CDIdLMon: Commutative distributive idempotent lattice-ordered monoids
CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups
CIdToSgrp: Commutative idempotent totally ordered semigroups

Superclasses

CDLSgrp: Commutative distributive lattice-ordered semigroups
CIdLSgrp: Commutative idempotent lattice-ordered semigroups
DIdLSgrp: Distributive idempotent lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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52. CDIdLMon: Commutative distributive idempotent lattice-ordered monoids

Definition

A *commutative distributive idempotent lattice-ordered monoid* is a distributive idempotent lattice-ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 4,\ f_4 = 12,\ f_5 = 31,\ f_6 = 90,\ f_7 = 241$

Subclasses

BCDivLat: Boolean commutative division lattices
CDIdRL: Commutative distributive idempotent residuated lattices
CIdToMon: Commutative idempotent totally ordered monoids

Superclasses

CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups
CDLMon: Commutative distributive lattice-ordered monoids
CIdLMon: Commutative idempotent lattice-ordered monoids
DIdLMon: Distributive idempotent lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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53. CDDivLat: Commutative distributive division lattices

Definition

A *commutative distributive division lattice* is a division lattice $\mathbf{P} = \langle P, \leq\rangle$ such that P is a distributive lattice and

Formal Definition

$(x \wedge y)/z = x/z \wedge y/z$
$x \leq z/y \iff y \leq x\backslash z$
$x/y = y\backslash x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 20,\ f_4 = 364$

Subclasses

BCRMag: Boolean commutative residuated magmas

CDRLMag: Commutative distributive residuated lattice-ordered magmas

Superclasses

CDivLat: Commutative division lattices

DDivLat: Distributive division lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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54. CDRLMag: Commutative distributive residuated lattice-ordered magmas

Definition

A *commutative distributive residuated lattice-ordered magma* is a distributive residuated lattice-ordered magma such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 148,\ f_5 = 3554$

Subclasses

CDInLMag: Commutative distributive involutive lattice-ordered magmas

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups

CRToMag: Commutative residuated totally ordered magmas

Superclasses

CDDivLat: Commutative distributive division lattices

CRLMag: Commutative residuated lattice-ordered magmas

DRLMag: Distributive residuated lattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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55. CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups

Definition

A *commutative distributive residuated lattice-ordered semigroup* is a distributive residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \le, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 57,\ f_5 = 392$

Subclasses

CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups

CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

CDRL: Commutative distributive residuated lattices

CRSlSgrp: Commutative residuated semilinear semigroups

Superclasses

CDLSgrp: Commutative distributive lattice-ordered semigroups

CDRLMag: Commutative distributive residuated lattice-ordered magmas

CRLSgrp: Commutative residuated lattice-ordered semigroups

DRLSgrp: Distributive residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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56. CDRL: Commutative distributive residuated lattices

Definition

A *commutative distributive residuated lattice* is a distributive residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 16,\ f_5 = 70,\ f_6 = 399$

Subclasses

CDIRL: Commutative distributive integral residuated lattices

CDIdRL: Commutative distributive idempotent residuated lattices

CDInFL: Commutative distributive involutive FL-algebras

CRSlMon: Commutative residuated semilinear monoids

DunnMon: Dunn monoid

Superclasses

CDLMon: Commutative distributive lattice-ordered monoids

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups

CRL: Commutative residuated lattices

DRL: Distributive residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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57. CDIRL: Commutative distributive integral residuated lattices

Definition

A *distributive lattice-ordered residuated integral monoid* is a distributive residuated lattice-ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety
Congruence distributive	Yes (relatively) Blok and Raftery [1997]
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 26,\ f_6 = 124,\ f_7 = 645$

Subclasses

CDIInFL: Commutative distributive integral involutive FL-algebras

CIRSlMon: Commutative integral residuated semilinear monoids

Superclasses

CDILMon: Commutative distributive integral lattice-ordered monoids

CDRL: Commutative distributive residuated lattices

CIRL: Commutative integral residuated lattices

DIRL: Distributive integral residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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58. CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups

Definition

A *commutative idempotent residuated lattice-ordered semigroup* is an distributive idempotent residuated lattice-ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 25,\ f_6 = 97$

Subclasses

CDIdRL: Commutative distributive idempotent residuated lattices

CIdRSlSgrp: Commutative idempotent residuated semilinear semigroups

Superclasses

CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups

CIdRLSgrp: Commutative idempotent residuated lattice-ordered semigroups

DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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59. CDIdRL: Commutative distributive idempotent residuated lattices

Definition

A *commutative idempotent residuated lattice* is an idmpotent residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, /\rangle$ such that $\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 15,\ f_6 = 44,\ f_7 = 115$

Subclasses

BCInMag: Boolean commutative involutive magmas

CIdRSlMon: Commutative idempotent residuated semilinear monoids

Superclasses

CDIdLMon: Commutative distributive idempotent lattice-ordered monoids

CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups

CDRL: Commutative distributive residuated lattices

CIdRL: Commutative idempotent residuated lattices

DIdRL: Distributive idempotent residuated lattices

DunnMon: Dunn monoid Cont|Po|J|M|L|D|To|B|U|Ind

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60. CDInLMag: Commutative distributive involutive lattice-ordered magmas

Definition

A *commutative distributive involutive lattice-ordered magma* is a inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that $\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 38,\ f_5 = 90,\ f_6 = 858$

Subclasses

CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

CInToMag: Commutative involutive totally ordered magmas

Superclasses

CDRLMag: Commutative distributive residuated lattice-ordered magmas

CInLMag: Commutative involutive lattice-ordered magmas

CyDInLMag: Cyclic distributive involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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61. CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

Definition

A *commutative distributive involutive lattice-ordered semigroup* is a inpo-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 53,\ f_6 = 330$

Subclasses

CDInFL: Commutative distributive involutive FL-algebras

CInSlSgrp: Commutative involutive semilinear semigroups

Superclasses

CDInLMag: Commutative distributive involutive lattice-ordered magmas

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups

CInLSgrp: Commutative involutive lattice-ordered semigroups

CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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62. CDInFL: Commutative distributive involutive FL-algebras

Definition

A *commutative distributive involutive FL-algebra* is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 9,\ f_5 = 8,\ f_6 = 42,\ f_7 = 46$

Subclasses

AbLGrp: Abelian lattice-ordered groups

CDIInFL: Commutative distributive integral involutive FL-algebras

CInSlMon: Commutative involutive semilinear monoids

DmMon: De Morgan monoids

Superclasses

CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

CDRL: Commutative distributive residuated lattices

CInFL: Commutative involutive FL-algebras

CyDInFL: Cyclic distributive involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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63. CDIInFL: Commutative distributive integral involutive FL-algebras

Definition

A *commutative distributive integral involutive FL-algebra* is an in-porim $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

$x \cdot 1 = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 12,\ f_7 = 12,\ f_8 = 60,\ f_9 = 73$

Subclasses

Superclasses

CDIRL: Commutative distributive integral residuated lattices

CDInFL: Commutative distributive involutive FL-algebras

CIInFL: Commutative integral involutive FL-algebras

CyDIInFL: Cyclic distributive involutive lattice-ordered integral monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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64. AbLGrp: Abelian lattice-ordered groups

Definition

An *abelian lattice-ordered group* is a lattice-ordered group $\mathbf{A} = \langle A, \cdot, ^{-1}, 1, \leq \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x^{-1} \cdot x = 1$

$x \cdot x^{-1} = 1$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	hereditarily undecidable Burris [1985]
Locally finite	No
Congruence distributive	yes (see lattices)
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$ (see groups)
Congruence regular	Yes, (see groups)
Congruence uniform	Yes, (see groups)
Amalgamation property	Yes
Strong amalgamation property	no Cherri and Powell [1993]

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_3 = 0,\ f_4 = 0,\ f_5 = 0,\ f_6 = 0$

Subclasses

AbToGrp: Abelian totally ordered groups

LRng: Lattice-ordered rings

Superclasses

AbPoGrp: Abelian partially ordered groups

CDInFL: Commutative distributive involutive FL-algebras

CInSlMon: Commutative involutive semilinear monoids

CInToMon: Commutative involutive totally ordered monoids

RepLGrp: Representable lattice-ordered groups

Cont|Po|J|M|L|D|To|B|U|Ind

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65. GBL: Generalized BL-algebras

Definition

A *generalized BL-algebra* is a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, / \rangle$ such that

$x \wedge y = y \cdot (y \backslash x \wedge e)$, $x \wedge y = (x/y \wedge e) \cdot y$

Properties

Classtype	Variety
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 10,\ f_6 = 23,\ f_7 = 49,\ f_8 = 111$

Subclasses

BLA: Basic logic algebras

GMV: Generalized MV-algebras

Superclasses

DRL: Distributive residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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66. GMV: Generalized MV-algebras

Definition

A *generalized MV-algebra* is a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, / \rangle$ such that

$x \vee y = x/(y\backslash x \wedge e)$, $x \vee y = (x/y \wedge e)\backslash y$

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No

Finite Members

Subclasses

MV: MV-algebras

Superclasses

GBL: Generalized BL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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67. psMV: Pseudo MV-algebras

Definition

A *pseudo MV-algebra*[(GI2001)] (or *psMV-algebra* for short) is a structure $\mathbf{A} = \langle A, \oplus, ^-, ^\sim, 0, 1 \rangle$ such that

$(x \oplus y) \oplus z = x \oplus (y \oplus z)$

$x \oplus 0 = x$

$x \oplus 1 = 1$

$(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$

$(x \oplus y^{\sim})^{-} \oplus x = y \oplus (x^{-} \oplus y)^{\sim}$
$x \oplus (y^{-} \oplus x)^{\sim} = y \oplus (x^{-} \oplus y)^{\sim}$
$x^{-\sim} = x$
$0^{-} = 1$

Basic Results

$0 + x = x$, $1 + x = 1$, $x^{\sim -} = x$, $0^{\sim} = 1$ and axiom A7 in[(GI2001)] follow from the above axioms.
Pseudo MV-algebras are term-equivalent to divisible involutive residuated lattices.
Every psMV-algebra is obtained from an interval in a lattice-ordered group[(Dvu2002)].
Every finite psMV-algebra is commutative.
Every commutative psMV-algebra is an MV-algebra.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence e-regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 3,\ f_9 = 2,\ f_{10} = 2$

Subclasses

MV: MV-algebras

Superclasses

DIInFL: Distributive integral involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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68. WaHp: Wajsberg hoops

Definition

A *Wajsberg hoop* is a hoop $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ such that
$(x \to y) \to y = (y \to x) \to x$
Remark: Lattice operations are term-definable by $x \wedge y = x \cdot (x \to y)$ and $x \vee y = (x \to y) \to y$.

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	No
Congruence distributive	Yes
Congruence modular	Yes
Congruence regular	Yes

Finite Members

Subclasses

MV: MV-algebras

Superclasses

Hp: Hoops

Cont|Po|J|M|L|D|To|B|U|Ind

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69. BrA: Brouwerian algebras

Definition

A *Brouwerian algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, 1, \rightarrow\rangle$ such that

$\langle A, \wedge, \vee, 1\rangle$ is a distributive lattice with top

$\rightarrow$ gives the residual of $\wedge$: $x \wedge y \leq z \Longleftrightarrow y \leq x \rightarrow z$

Definition

A *Brouwerian algebra* is a BL-algebra $\mathbf{A} = \langle A, \wedge, \vee, 1, \cdot, \rightarrow\rangle$ such that

$x \wedge y = x \cdot y$

Properties

Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8,\ f_8 = 15,\ f_9 = 26,\ f_{10} = 47,\ f_{11} = 82,\ f_{12} = 151,$
$f_{13} = 269,\ f_{14} = 494,\ f_{15} = 891,\ f_{16} = 1639,\ f_{17} = 2978,\ f_{18} = 5483,\ f_{19} = 10006,\ f_{20} = 18428$

Values known up to size 49 Erné et al. [2002]

Subclasses

GBA: Generalized Boolean algebras

HA: Heyting algebras

Superclasses

BrSlat: Brouwerian semilattices

Cont|Po|J|M|L|D|To|B|U|Ind

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70. GBA: Generalized Boolean algebras

Definition

A *generalized Boolean algebra* is a Brouwerian algebra $\mathbf{A} = \langle A, \wedge, \vee, 1, \rightarrow\rangle$ such that

$x \vee y = (x \rightarrow y) \rightarrow y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Locally finite	Yes
Residual size	2
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence e-regular	Yes, $e = 1$
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0$

Subclasses

BA: Boolean algebras

Superclasses

BrA: Brouwerian algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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71. BoolLat: Boolean lattices

Definition

A *Boolean lattice* is a bounded distributive lattice $\mathbf{L} = \langle L, \vee, 0, \wedge, 1\rangle$ such that every element has a complement: $\exists y(x \vee y = 1$ and $x \wedge y = 0)$

Examples

Example 1: $\langle \mathcal{P}(S), \cup, \emptyset, \cap, S\rangle$, the collection of subsets of a set S, with union, empty set, intersection, and the whole set S.

Properties

Classtype	first-order
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Locally finite	Yes

Finite Members

Any finite member is a power of the 2-element Boolean lattice.

Subclasses

BA: Boolean algebras

Superclasses

bDLat: Bounded distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

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72. CRSlSgrp: Commutative residuated semilinear semigroups

Definition

A *commutative residuated semilinear semigroup* is a residuated semilinear semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 57,\ f_5 = 392$

Subclasses

CIdRSlSgrp: Commutative idempotent residuated semilinear semigroups

CInSlSgrp: Commutative involutive semilinear semigroups

CRSlMon: Commutative residuated semilinear monoids

CRToSgrp: Commutative residuated totally ordered semigroups

Superclasses

CDRLSgrp: Commutative distributive residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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73. CRSlMon: Commutative residuated semilinear monoids

Definition

A *commutative residuated semilinear monoid* is a residuated semilinear monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$1 \leq x \backslash y \vee y \backslash x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 12,\ f_5 = 47,\ f_6 = 220$

Subclasses

CIRSlMon: Commutative integral residuated semilinear monoids
CIdRSlMon: Commutative idempotent residuated semilinear monoids
CInSlMon: Commutative involutive semilinear monoids
CRToMon: Commutative residuated totally ordered monoids

Superclasses

CDRL: Commutative distributive residuated lattices
CRSlSgrp: Commutative residuated semilinear semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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74. CIRSlMon: Commutative integral residuated semilinear monoids

Definition

A *commutative integral residuated semilinear monoid* is a residuated semilinear monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y = y \cdot x$
$1 \leq x \backslash y \vee y \backslash x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 23,\ f_6 = 99,\ f_7 = 464$

Subclasses

CIRToMon: Commutative integral residuated totally ordered monoids
IMTL: Involutive monoidal t-norm logic algebras
MTLA: Monoidal t-norm logic algebras

Superclasses

CDIRL: Commutative distributive integral residuated lattices
CRSlMon: Commutative residuated semilinear monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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75. MTLA: Monoidal t-norm logic algebras

Definition

A *monoidal t-norm logic algebra* is a FLew-algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \to, 0 \rangle$ such that

$\cdot$ is *prelinear*: $(x \to y) \vee (y \to x) = 1$

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	No
Congruence uniform	No

Finite Members

Subclasses

BLA: Basic logic algebras

Superclasses

CIRSlMon: Commutative integral residuated semilinear monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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76. BLA: Basic logic algebras

Definition

A *basic logic algebra* or *BL-algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \cdot, \to\rangle$ such that

$\langle A, \vee, 0, \wedge, 1\rangle$ is a bounded lattice

$\langle A, \cdot, 1\rangle$ is a commutative monoid

$\to$ gives the residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \to z$

prelinearity: $(x \to y) \vee (y \to x) = 1$

BL: $x \cdot (x \to y) = x \wedge y$

Remark: The BL identity implies that the lattice is distributive.

Definition

A *basic logic algebra* is an FL_e-algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \cdot, \to\rangle$ such that

linearity: $(x \to y) \vee (y \to x) = 1$

BL: $x \cdot (x \to y) = x \wedge y$

Remark: The BL identity implies that the identity element 1 is the top of the lattice.

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$
Congruence uniform	No
Congruence extension property	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 10,\ f_6 = 23,\ f_7 = 49,\ f_8 = 111$

The number of subdirectly irreducible BL-algebras of size n is 2^{n-2}.

Subclasses

HA: Heyting algebras

MV: MV-algebras

Superclasses

GBL: Generalized BL-algebras

77. MV: MV-algebras

Definition

An *MV-algebra* (short for *multivalued logic algebra*) is an algebra $\mathbf{A} = \langle A, +, 0, \neg\rangle$ such that

$\langle A, +, 0\rangle$ is a commutative monoid

$\neg\neg x = x$

$x + \neg 0 = \neg 0$

$\neg(\neg x + y) + y = \neg(\neg y + x) + x$

Remark: This is the definition from Cignoli et al. [2000]

Definition

An *MV-algebra* is an algebra $\mathbf{A} = \langle A, +, 0, \cdot, 1, \neg\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a commutative monoid

$\neg$ is a DeMorgan involution for $+, \cdot$: $\neg\neg x = x$, $x + y = \neg(\neg x \cdot \neg y)$

$\neg 0 = 1$, $0 \cdot x = 0$, $\neg(\neg x + y) + y = \neg(\neg y + x) + x$

Definition

An *MV-algebra* is a basic logic algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \cdot, \to\rangle$ that satisfies

MV: $x \vee y = (x \to y) \to y$

Definition

A *Wajsberg algebra* is an algebra $\mathbf{A} = \langle A, \to, \neg, 1\rangle$ such that

$1 \to x = x$

$(x \to y) \to ((y \to z) \to (x \to z) = 1$

$(x \to y) \to y = (y \to x) \to x$

$(\neg x \to \neg y) \to (y \to x) = 1$

Remark: Wajsberg algebras are term-equivalent to MV-algebras via $x \to y = \neg x + y$, $1 = \neg 0$ and $x + y = \neg x \to y$, $0 = \neg 1$.

Definition

A *bounded Wajsberg hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \to, 0, 1\rangle$ such that

$\langle A, \cdot, \to, 1\rangle$ is a hoop

$(x \to y) \to y = (y \to x) \to x$

$0 \to x = 1$

Remark: Bounded Wajsberg hoops are term-equivalent to Wajsberg algebras via $x \cdot y = \neg(x \to \neg y)$, $0 = \neg 1$, and $\neg x = x \to 0$. See [(BP1994)] for details.

Definition

A *lattice implication algebra* is an algebra $\mathbf{A} = \langle A, \to, -, 1\rangle$ such that

$x \to (y \to z) = y \to (x \to z)$

$1 \to x = x$

$x \to 1 = 1$

$x \to y = -y \to -x$

$(x \to y) \to y = (y \to x) \to x$

Remark: Lattice implication algebras are term-equivalent to MV-algebras via $x + y = -x \to y$, $0 = -1$, and $\neg x = -x$.

Properties

Classtype	Variety
Equational theory	Decidable
Universal theory	Decidable (FEP[(BF2000)])
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$
Congruence extension property	Yes
Equationally def. pr. cong.	No
Amalgamation property	yes [(Mu1987)]

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 3$

The number of algebras with n elements is given by the number of ways of factoring n into a product with nontrivial factors, see A001055

Subclasses

BA: Boolean algebras

Superclasses

BLA: Basic logic algebras

GMV: Generalized MV-algebras

ImpLat: Implicative lattices

WaHp: Wajsberg hoops

psMV: Pseudo MV-algebras

qMV: Quasi-MV-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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78. HA: Heyting algebras

Definition

A *Heyting algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \to\rangle$ such that

$\langle A, \vee, 0, \wedge, 1\rangle$ is a bounded distributive lattice

$\to$ gives the residual of $\wedge$: $x \wedge y \leq z \iff y \leq x \to z$

Definition

A *Heyting algebra* is a FLew-algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \cdot, \to\rangle$ such that

$x \wedge y = x \cdot y$

Examples

Example 1: The open sets of any topological space $\mathbf{X}$ form a Heyting algebra under the operations of union $\cup$, empty set $\emptyset$, intersection $\cap$, whole space X, and the operation $U \to V =$ interior of $(X - U) \cup V$.

Example 2: Any frame can be expanded to a unique Heyting algebra by defining $x \to y = \bigvee\{z : x \wedge z \leq y\}$.

Basic Results

Any finite distributive lattice is the reduct of a unique Heyting algebra. More generally the same result holds for any complete and completely distributive lattice.

A Heyting algebra is subdirectly irreducible if and only if it has a unique coatom.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8,\ f_8 = 15,\ f_9 = 26,\ f_{10} = 47,\ f_{11} = 82,\ f_{12} = 151,\ f_{13} = 269,\ f_{14} = 494,\ f_{15} = 891,\ f_{16} = 1639,\ f_{17} = 2978,\ f_{18} = 5483,\ f_{19} = 10006,\ f_{20} = 18428$

Values known up to size 49 Erné et al. [2002]

Subclasses

GödA: Gödel algebras

Superclasses

BCKLat: BCK-lattices

BLA: Basic logic algebras

BrA: Brouwerian algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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79. GödA: Gödel algebras

Definition

A *Gödel algebra* is a Heyting algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \to\rangle$ such that

$(x \to y) \vee (y \to x) = 1$

Remark: Gödel algebras are also called *linear Heyting algebras* since subdirectly irreducible Gödel algebras are linearly ordered Heyting algebras.

Definition

A *Gödel algebra* is a representable FLew-algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \cdot, \to\rangle$ such that

$x \wedge y = x \cdot y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Residual size	countable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 3,\ f_9 = 1,\ f_{10} = 2$

Subclasses

BA: Boolean algebras

BCIdRSgrp: Boolean commutative idempotent residuated semigroups

Superclasses

HA: Heyting algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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80. CIdRSlSgrp: Commutative idempotent residuated semilinear semigroups

Definition

A *commutative idempotent residuated semilinear semigroup* is an idempotent residuated semilinear semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 25,\ f_6 = 97$

Subclasses

CIdRSlMon: Commutative idempotent residuated semilinear monoids

CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups

Superclasses

CDIdRLSgrp: Commutative distributive idempotent residuated lattice-ordered semigroups

CRSlSgrp: Commutative residuated semilinear semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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81. CIdRSlMon: Commutative idempotent residuated semilinear monoids

Definition

A *commutative idempotent residuated semilinear monoid* is an idmpotent residuated semilinear monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

$1 \leq x\backslash y \vee y\backslash x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 9,\ f_6 = 20,\ f_7 = 38$

Subclasses

BCInSgrp: Boolean commutative involutive semigroups

CIdRToMon: Commutative idempotent residuated totally ordered monoids

Superclasses

CDIdRL: Commutative distributive idempotent residuated lattices

CIdRSlSgrp: Commutative idempotent residuated semilinear semigroups

CRSlMon: Commutative residuated semilinear monoids Cont|Po|J|M|L|D|To|B|U|Ind

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82. CInSlSgrp: Commutative involutive semilinear semigroups

Definition

A *commutative involutive semilinear semigroup* is an insl-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 29,\ f_5 = 53,\ f_6 = 330$

Subclasses

CInSlMon: Commutative involutive semilinear monoids

CInToSgrp: Commutative involutive totally ordered semigroups

Superclasses

CDInLSgrp: Commutative distributive involutive lattice-ordered semigroups

CRSlSgrp: Commutative residuated semilinear semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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83. CInSlMon: Commutative involutive semilinear monoids

Definition

A *commutative involutive semilinear monoid* is an insl-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$1 \le -(-x \cdot y) \vee -(-y \cdot x)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 8,\ f_6 = 20,\ f_7 = 36,\ f_8 = 90$

Subclasses

AbLGrp: Abelian lattice-ordered groups

CInToMon: Commutative involutive totally ordered monoids

IMTL: Involutive monoidal t-norm logic algebras

Superclasses

CDInFL: Commutative distributive involutive FL-algebras

CInSlSgrp: Commutative involutive semilinear semigroups

CRSlMon: Commutative residuated semilinear monoids

Cont|Po|J|M|L|D|To|B|U|Ind

84. DunnMon: Dunn monoid

Definition

A *Dunn monoid* is a commutative distributive residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \to\rangle$ such that

$\cdot$ is square-increasing: $x \le x^2$

Remark: Here $x^2 = x \cdot x$. These algebras were first defined by J.M.Dunn in [(Du1966)] and were named by R.K. Meyer[(Me1972)].

Properties

Classtype	Variety
Equational theory	Undecidable[(Ur1984)]
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

Subclasses

CDIdRL: Commutative distributive idempotent residuated lattices

DmMon: De Morgan monoids

Superclasses

CDRL: Commutative distributive residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

85. IMTL: Involutive monoidal t-norm logic algebras

Definition

An *involutive monoidal t-norm logic algebra*, or IMTL-algebra, is a commutative involutive semilinear monoid $\mathbf{A} = \langle A, \le, \cdot, \sim, -\rangle$ such that

$\cdot$ is integral: $x \le 1$.

Formal Definition

$--x = x$

$x \cdot y \le z \iff y \le -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

$x \cdot 1 = x$

$x \le 1$

$1 \le -(-x \cdot y) \vee -(-y \cdot x)$

Definition

An *m-zeroid* (or IMTL-algebra with dual signature) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, +, 0, -\rangle$ such that

$\langle A, +\rangle$ is a commutative semigroup

$\langle A, \wedge, \vee\rangle$ is a lattice

$-x = x$

$x + 0 = 0$

$x + -x = 0$

$x \leq y \iff 0 = -x + y$

$x + (y \vee z) = (x + y) \vee (x + z)$

Basic Results

All subdirectly irreducible algebras are linearly ordered.

The lattice is always bounded, with top element 0.

The bottom element -0 is the identity of $+$.

The dual operation $x \cdot y = -(-y + -x)$ is the fusion of a commutative integral involutive semilinear residuated lattice. In fact, m-zeroids are precisely the duals of these residuated lattices, which are also known as involutive IMTL algebras.

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence e-regular	Yes, $e = 1$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 3,\ f_5 = 3,\ f_6 = 8,\ f_7 = 12,\ f_8 = 35$

Subclasses

IMTLChn: Involutive monoidal t-norm logic chains

Superclasses

CIRSlMon: Commutative integral residuated semilinear monoids

CInSlMon: Commutative involutive semilinear monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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86. ImpLat: Implicative lattices

Definition

An *implicative lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow\rangle$ such that

$\langle A, \wedge, \vee\rangle$ is a distributive lattice and

$\rightarrow$ is a (semi-classical) implication:

$x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$

$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

$(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes

Finite Members

Subclasses

LGrp: Lattice-ordered groups

MV: MV-algebras

Superclasses

DLImpA: Distributive lattice-ordered implication algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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87. KLA: Kleene logic algebras

Definition

A *Kleene logic algebra* is a De Morgan algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg\rangle$ that satisfies
$x \wedge \neg x \leq y \vee \neg y$.
Remark: Also called Kleene algebras, but this name more commonly refers to the algebraic models of regular languages.

Examples

Example 1: Let $\{0 < a < 1\}$ be the 3-element lattice with $0' = 1, a' = a, b' = b$.

Basic Results

The algebra in Example 1 generates the variety of Kleene logic algebras

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence extension property	Yes
Locally finite	Yes
Residual size	3

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 3,\ f_7 = 2,\ f_8 = 6,\ f_9 = 4,\ f_{10} = 10$

Subclasses

BA: Boolean algebras

Superclasses

DmA: De Morgan algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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88. NVLGrp: Normal valued lattice-ordered groups

Definition

A *normal valued lattice-ordered group* (or *normal valued ℓ-group*) is a lattice-ordered group $\mathbf{L} = \langle L, \wedge, \vee, \cdot, ^{-1}, e\rangle$ that satisfies
$(x \vee x^{-1})(y \vee y^{-1}) \leq (y \vee y^{-1})^2(x \vee x^{-1})^2$

Basic Results

The variety of normal valued ℓ-groups is the largest proper subvariety of lattice-ordered groups Holland [1976].

Properties

Classtype	Variety
First-order theory	hereditarily undecidable Burris [1985]
Locally finite	No
Congruence distributive	yes (see lattices)
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$ (see groups)
Congruence regular	Yes, (see groups)
Congruence uniform	Yes, (see groups)

Finite Members

None

Subclasses

RepLGrp: Representable lattice-ordered groups

Superclasses

LGrp: Lattice-ordered groups

Cont|Po|J|M|L|D|To|B|U|Ind

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89. LA$_n$: Lukasiewicz algebras of order n

Definition

A *Lukasiewicz algebra of order* n is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \sigma_0, \dots, \sigma_{n-1}\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a De Morgan algebra

1. σ_i is a lattice homomorphism: $\sigma_i(x \vee y) = \sigma_i(x) \vee \sigma_i(y)$ and $\sigma_i(x \wedge y) = \sigma_i(x) \wedge \sigma_i(y)$
2. $\sigma_i(x) \vee \neg(\sigma_i(x)) = 1$, $\sigma_i(x) \wedge \neg(\sigma_i(x)) = 0$
3. $\sigma_i(\sigma_j(x)) = \sigma_j(x)$ for $1 \le j \le n-1$
4. $\sigma_i(\neg x) = \neg(\sigma_{n-i}(x))$
5. $\sigma_i(x) \wedge \sigma_j(x) = \sigma_i(x)$ for $i \le j \le n-1$
6. $x \vee \sigma_{n-1}(x) = \sigma_{n-1}(x)$, $x \wedge \sigma_1(x) = \sigma_1(x)$
7. $y \wedge (x \vee \neg(\sigma_i(x)) \vee \sigma_{i+1}(y)) = y$ for $i \ne n-1$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Locally finite	Yes
Residual size	n

Finite Members

Subclasses

BA: Boolean algebras

Superclasses

DmA: De Morgan algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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90. LRng: Lattice-ordered rings

Definition

A *lattice-ordered ring* (or *ℓ-ring*) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, +, -, 0, \cdot\rangle$ such that

$\langle L, \wedge, \vee\rangle$ is a lattice

$\langle L, +, -, 0, \cdot\rangle$ is a ring

$+$ is order-preserving: $x \le y \implies x + z \le y + z$

$\uparrow 0$ is closed under $\cdot$: $0 \le x, y \implies 0 \le x \cdot y$

Formal Definition

Basic Results

The lattice reducts of lattice-ordered rings are distributive lattices.

Properties

Classtype	Variety
Congruence distributive	Yes, see lattices
Congruence n-permutable	Yes, $n = 2$, see groups
Congruence regular	Yes, see groups
Congruence uniform	Yes, see groups

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 5,\ f_7 = 8$

Subclasses

CLRng: Commutative lattice-ordered rings

FRng: Function rings

ToRng: Totally ordered rings

Superclasses

AbLGrp: Abelian lattice-ordered groups

Cont|Po|J|M|L|D|To|B|U|Ind

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91. CLRng: Commutative lattice-ordered rings

Definition

A *commutative lattice-ordered ring* is a lattice-ordered ring $\mathbf{A} = \langle A, \wedge, \vee, +, -, 0, \cdot \rangle$ such that

$\cdot$ is *commutative*: $xy = yx$

Properties

Congruence distributive	yes
Congruence modular	yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	yes
Congruence uniform	yes

Finite Members

Subclasses

CToRng: Commutative totally ordered rings

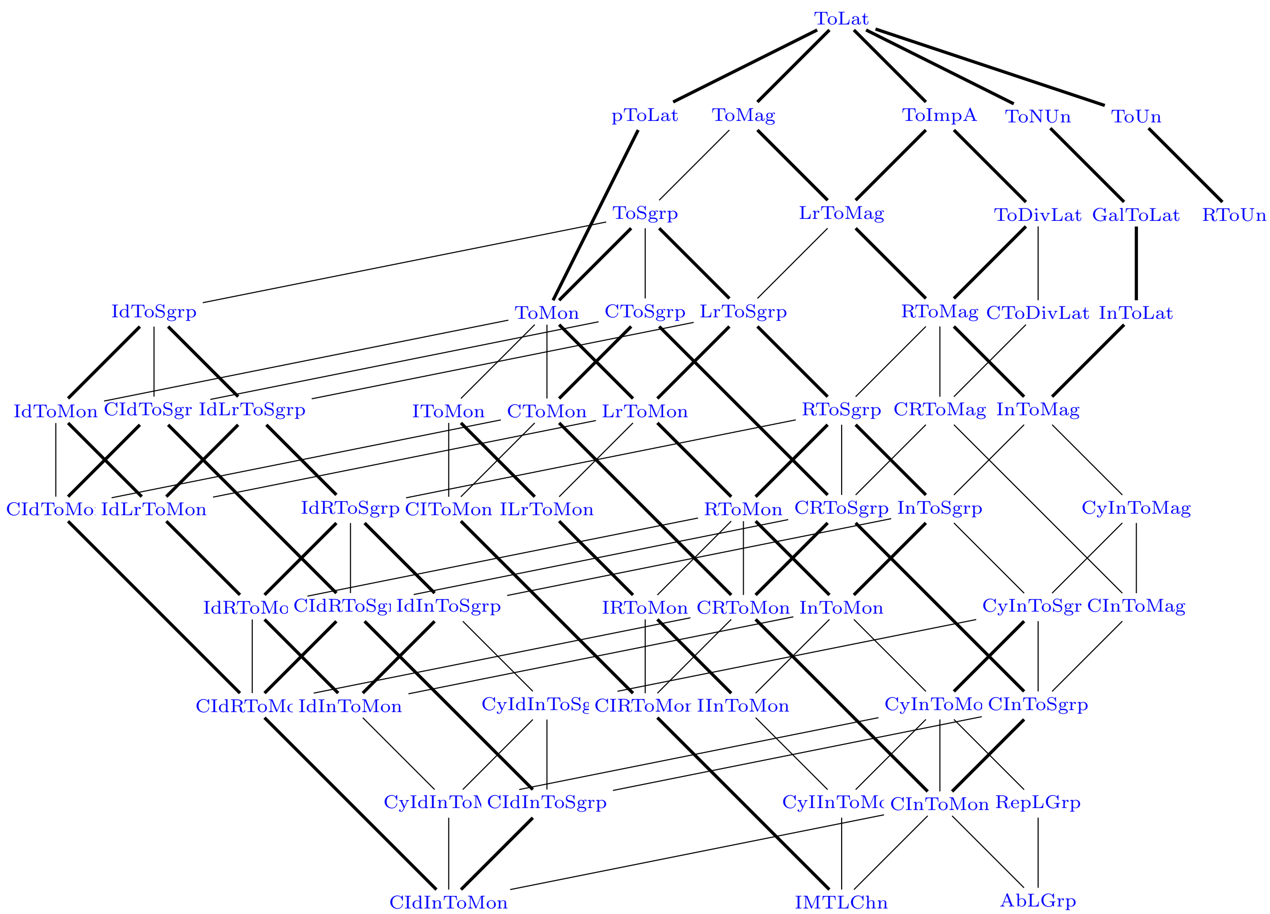
Superclasses

LRng: Lattice-ordered rings

Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 7

Totally ordered algebras



1. ToLat: Totally ordered lattices

Formal Definition

A *totally ordered lattice* is a lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ such that

$\wedge$ is conservative: $x \wedge y = x$ or $x \wedge y = y$

Examples

$\mathbf{C}_n = \langle\{0, 1, \ldots, n-1\}, \wedge, \vee\rangle$, the n-element chain with $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

Any linearly ordered poset with the same operations as in the previous example.

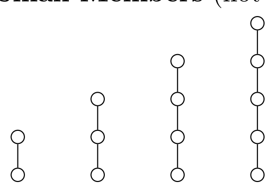
Properties

Classtype	Universal class
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes
Locally finite	Yes
Residual size	2

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_n = 1$ for $n > 1$

Small Members (not in any subclass)



$\mathbf{C}_2$ $\mathbf{C}_3$ $\mathbf{C}_4$ $\mathbf{C}_5$

Subclasses

ToImpA: Totally ordered implication algebras

ToMag: Totally ordered magmas

ToNUn: Totally ordered negated unars

ToUn: Totally ordered unars

pDLat: Pointed distributive lattices

pToLat: Pointed totally ordered lattices

Superclasses

DLat: Distributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

2. pToLat: Pointed totally ordered lattices

Definition

A *pointed distributive lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, c\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a totally ordered lattice and c is a constant operation on A.

Formal Definition

$c = c$

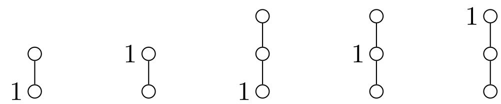
Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 4,\ f_n = n$

Small Members (not in any subclass)



$\mathbf{pC}_{2,1}$ $\mathbf{pC}_{2,2}$ $\mathbf{pC}_{3,1}$ $\mathbf{pC}_{3,2}$ $\mathbf{pC}_{3,3}$

Subclasses

ToMon: Totally ordered monoids

Superclasses

ToLat: Totally ordered lattices

pDLat: Pointed distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

3. ToMag: Totally ordered magmas

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$x \cdot (y \wedge z) = x \cdot y \wedge x \cdot z$

$(x \wedge y) \cdot z = x \cdot z \wedge y \cdot z$

Properties

Classtype	universal class

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

LrToMag: Left-residuated totally ordered magmas

ToSgrp: Totally ordered semigroups

Superclasses

DLMag: Distributive lattice-ordered magmas

ToLat: Totally ordered lattices Cont|Po|J|M|L|D|To|B|U|Ind

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4. ToSgrp: Totally ordered semigroups

Definition

A *totally ordered semigroup* is an algebra $\langle C, \wedge, \vee, \cdot\rangle$ such that

$\langle C, \wedge, \vee\rangle$ is a totally ordered lattice,

$\langle C, \cdot\rangle$ is a semigroup and

$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$.

Formal Definition

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$x \wedge y = x$ or $x \wedge y = y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	universal class

Finite Members

Subclasses

CToSgrp: Commutative totally ordered semigroups

IdToSgrp: Idempotent totally ordered semigroups

LrToSgrp: Left-residuated totally ordered semigroups

ToMon: Totally ordered monoids

Superclasses

DLSgrp: Distributive lattice-ordered semigroups

ToMag: Totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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5. ToMon: Totally ordered monoids

Definition

A *totally ordered monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a monoid

$\langle G, \leq\rangle$ is a distributive lattice

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 34,\ f_5 = 184,\ f_6 = 1218,\ f_7 = 9742,\ f_8 = 92882,\ f_9 = 1053248,\ f_{10} = 14592054$

Subclasses

CToMon: Commutative totally ordered monoids

IToMon: Integral totally ordered monoids

IdToMon: Idempotent totally ordered monoids

LrToMon: Left-residuated totally ordered monoids

Superclasses

DLMon: Distributive lattice-ordered monoids

ToSgrp: Totally ordered semigroups

pToLat: Pointed totally ordered lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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6. IToMon: Integral totally ordered monoids

Definition

An *integral totally ordered monoid* is a totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 44,\ f_6 = 308,\ f_7 = 2641,\ f_8 = 27120,\ f_9 = 332507,\ f_{10} = 5035455$

Subclasses

CIToMon: Commutative integral totally ordered monoids

ILrToMon: Integral left-residuated totally ordered monoids

Superclasses

DILMon: Distributive integral lattice-ordered monoids

ToMon: Totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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7. IdToSgrp: Idempotent totally ordered semigroups

Definition

An *idempotent totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that

$\langle A, \wedge, \vee, \cdot \rangle$ is a totally ordered semigroup and

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 17,\ f_4 = 82,\ f_5 = 422$

Subclasses

CIdToSgrp: Commutative idempotent totally ordered semigroups

IdLrToSgrp: Idempotent left-residuated totally ordered semigroups

IdToMon: Idempotent totally ordered monoids

Superclasses

DIdLSgrp: Distributive idempotent lattice-ordered semigroups

ToSgrp: Totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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8. IdToMon: Idempotent totally ordered monoids

Definition

An *idempotent totally ordered monoid* is a totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1 \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 16,\ f_5 = 44,\ f_6 = 120$

Subclasses

CIdToMon: Commutative idempotent totally ordered monoids

IdLrToMon: Idempotent left-residuated totally ordered monoids

Superclasses

DIdLMon: Distributive idempotent lattice-ordered monoids

IdToSgrp: Idempotent totally ordered semigroups

ToMon: Totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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9. ToImpA: Totally ordered implication algebras

Formal Definition

$x \le y \implies y \to z \le x \to z$

$x \le y \implies z \to x \le z \to y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 175$

Subclasses

LrToMag: Left-residuated totally ordered magmas

ToDivLat: Totally ordered division lattices

Superclasses

DLImpA: Distributive lattice-ordered implication algebras

ToLat: Totally ordered lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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10. LrToMag: Left-residuated totally ordered magmas

Definition

A *left-residuated totally ordered magma* is an algebra $\mathbf{A} = \langle A, \le, \cdot, \backslash, \rangle$ such that

$\langle A, \le\rangle$ is a distributive lattice,

$\langle A, \cdot\rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x\backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$x \cdot y \le z \iff y \le x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 50,\ f_4 = 4116$

Subclasses

LrToSgrp: Left-residuated totally ordered semigroups

RToMag: Residuated totally ordered magmas

Superclasses

DLrLMag: Distributive left-residuated lattice-ordered magmas

ToImpA: Totally ordered implication algebras
ToMag: Totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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11. LrToSgrp: Left-residuated totally ordered semigroups

Definition

A *left-residuated totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a distributive lattice,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 18,\ f_4 = 144,\ f_5 = 1370$

Subclasses

IdLrToSgrp: Idempotent left-residuated totally ordered semigroups
LrToMon: Left-residuated totally ordered monoids
RToSgrp: Residuated totally ordered semigroups

Superclasses

DLrLSgrp: Distributive left-residuated lattice-ordered semigroups
LrToMag: Left-residuated totally ordered magmas
ToSgrp: Totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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12. LrToMon: Left-residuated totally ordered monoids

Definition

A *left-residuated totally ordered monoid* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a distributive lattice,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 17,\ f_5 = 92,\ f_6 = 609$

Subclasses

ILrToMon: Integral left-residuated totally ordered monoids
IdLrToMon: Idempotent left-residuated totally ordered monoids
RToMon: Residuated totally ordered monoids

Superclasses

DLrLMon: Distributive left-residuated lattice-ordered monoids
LrToSgrp: Left-residuated totally ordered semigroups
ToMon: Totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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13. ILrToMon: Integral left-residuated totally ordered monoids

Definition

An *integral left-residuated totally ordered monoid* is a left-residuated totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ for which
$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 44,\ f_6 = 308$

Subclasses

IRToMon: Integral residuated totally ordered monoids

Superclasses

DILrLMon: Distributive integral left-residuated lattice-ordered monoids
IToMon: Integral totally ordered monoids
LrToMon: Left-residuated totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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14. IdLrToSgrp: Idempotent left-residuated totally ordered semigroups

Definition

An *idempotent left-residuated totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that
$\langle A, \wedge, \vee, \cdot \rangle$ is a left-residuated totally ordered semigroup and
$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 30,\ f_5 = 144,\ f_6 = 740$

Subclasses

IdLrToMon: Idempotent left-residuated totally ordered monoids
IdRToSgrp: Idempotent residuated totally ordered semigroups

Superclasses

DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups
IdToSgrp: Idempotent totally ordered semigroups
LrToSgrp: Left-residuated totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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15. IdLrToMon: Idempotent left-residuated totally ordered monoids

Definition

An *idempotent left-residuated totally ordered monoid* is a left-residuated totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \le z \iff y \le x\backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 8,\ f_5 = 22,\ f_6 = 60,\ f_7 = 164$

Subclasses

IdRToMon: Idempotent residuated totally ordered monoids

Superclasses

DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids
IdLrToSgrp: Idempotent left-residuated totally ordered semigroups
IdToMon: Idempotent totally ordered monoids
LrToMon: Left-residuated totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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16. RToUn: Residuated totally-ordered unars

Definition

A *residuated totally-ordered unar* (also called an *rto-unar* for short) is a residuated lattice-ordered unar $\langle C, \wedge, \vee, f, g\rangle$ such that $\langle C, \wedge, \vee\rangle$ is a chain.

Formal Definition

$f(x) \le y \iff x \le g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets. In particular, $f(x \vee y) = f(x) \vee f(y)$ and $g(x \wedge y) = g(x) \wedge g(y)$.

Properties

Classtype	po-variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

Subclasses

InToMon: Involutive totally ordered monoids

Superclasses

DRLUn: Distributive residuated lattice-ordered unars

ToUn: Totally ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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17. ToDivLat: Totally ordered division lattices

Definition

A *totally ordered division lattice* is a division lattice $\mathbf{C} = \langle C, \wedge, \vee, \backslash, / \rangle$ such that $\langle C, \wedge, \vee \rangle$ is a totally ordered lattice.

Formal Definition

$x \leq z/y \iff y \leq x\backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 216$

Subclasses

CToDivLat: Commutative division chains

RToMag: Residuated totally ordered magmas

Superclasses

DDivLat: Distributive division lattices

ToImpA: Totally ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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18. RToMag: Residuated totally ordered magmas

Definition

A *residuated totally ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\langle A, \leq \rangle$ is a distributive lattice,

$\langle A, \cdot \rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x\backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 20,\ f_4 = 980$

Subclasses

CRToMag: Commutative residuated totally ordered magmas

InToMag: Involutive totally ordered magmas

RToSgrp: Residuated totally ordered semigroups

Superclasses

DRLMag: Distributive residuated lattice-ordered magmas

LrToMag: Left-residuated totally ordered magmas

ToDivLat: Totally ordered division lattices Cont|Po|J|M|L|D|To|B|U|Ind

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19. RToSgrp: Residuated totally ordered semigroups

Definition

A *residuated totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\langle A, \leq \rangle$ is a distributive lattice,

$\langle A, \cdot \rangle$ is a semigroup and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \leq z \iff x \leq z/y$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 101,\ f_5 = 1003$

Subclasses

CRToSgrp: Commutative residuated totally ordered semigroups

IdRToSgrp: Idempotent residuated totally ordered semigroups

InToSgrp: Involutive totally ordered semigroups

RToMon: Residuated totally ordered monoids

Superclasses

DRLSgrp: Distributive residuated lattice-ordered semigroups

LrToSgrp: Left-residuated totally ordered semigroups

RToMag: Residuated totally ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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20. RToMon: Residuated totally ordered monoids

Definition

A *residuated totally ordered monoid* is a totally ordered monoid $\mathbf{L} = \langle L, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\wedge, \vee$ are distributive: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	Variety
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, n=2
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 15,\ f_5 = 84,\ f_6 = 575$

Subclasses

CRToMon: Commutative residuated totally ordered monoids

IRToMon: Integral residuated totally ordered monoids

IdRToMon: Idempotent residuated totally ordered monoids

InToMon: Involutive totally ordered monoids

Superclasses

DRL: Distributive residuated lattices

LrToMon: Left-residuated totally ordered monoids

RToSgrp: Residuated totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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21. IRToMon: Integral residuated totally ordered monoids

Definition

An *integral residuated totally ordered monoid* is a residuated totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 8,\ f_5 = 44,\ f_6 = 308$

Subclasses

CIRToMon: Commutative integral residuated totally ordered monoids

IInToMon: Integral involutive totally ordered monoids

Superclasses

DIRL: Distributive integral residuated lattices

ILrToMon: Integral left-residuated totally ordered monoids

RToMon: Residuated totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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22. IdRToSgrp: Idempotent residuated totally ordered semigroups

Definition

An *idempotent residuated totally ordered semigroup* is a residuated totally ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 4,\ f_4 = 17,\ f_5 = 82$

Subclasses

CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups

IdRToMon: Idempotent residuated totally ordered monoids

Superclasses

DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups

IdLrToSgrp: Idempotent left-residuated totally ordered semigroups

RToSgrp: Residuated totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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23. IdRToMon: Idempotent residuated totally ordered monoids

Definition

An *idempotent residuated totally ordered monoid* is a residuated totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x\backslash z$
$x \cdot y \leq z \iff x \leq z/y$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 16,\ f_6 = 44,\ f_7 = 120$

Subclasses

CIdRToMon: Commutative idempotent residuated totally ordered monoids

Superclasses

DIdRL: Distributive idempotent residuated lattices
IdLrToMon: Idempotent left-residuated totally ordered monoids
IdRToSgrp: Idempotent residuated totally ordered semigroups
RToMon: Residuated totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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24. ToUn: Totally ordered unars

Definition

A *totally ordered unar* is an algebra $\mathbf{P} = \langle P, \leq, f\rangle$ such that P is a distributive lattice and f is a unary operation on P that is
order-preserving: $x \leq y \implies f(x) \leq f(y)$

Formal Definition

$x \leq y \implies f(x) \leq f(y)$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 35,\ f_5 = 126,\ f_6 = 462$

Subclasses

RToUn: Residuated totally-ordered unars

Superclasses

DLUn: Distributive lattice-ordered unars
ToLat: Totally ordered lattices Cont|Po|J|M|L|D|To|B|U|Ind

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25. ToNUn: Totally ordered negated unars

Definition

A *totally ordered negated unar* is an algebra $\mathbf{C} = \langle C, \wedge, \vee, \sim\rangle$ such that $\langle C, \wedge, \vee\rangle$ is a chain and $\sim$ is a unary operation on C that is
order-reversing: $x \leq y \implies {\sim}y \leq {\sim}x$

Formal Definition

$x \leq y \implies {\sim}y \leq {\sim}x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 35,\ f_5 = 126,\ f_6 = 462$

Subclasses

GalToLat: Galois chains

Superclasses

DLNUn: Distributive lattice-ordered negated unars

ToLat: Totally ordered lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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26. GalToLat: Galois chains

Definition

A *Galois chain* is an algebra $\mathbf{P} = \langle P, \leq, \sim, -\rangle$ such that P is a distributive lattice and $\sim, -$ are a pair of unary operations on P that form a

Galois connection: $x \leq {\sim}y \iff y \leq -x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 20,\ f_5 = 70,\ f_6 = 252,\ f_7 = 924$

Subclasses

InToLat: Involutive chains

Superclasses

DGalLat: Distributive Galois lattices

ToNUn: Totally ordered negated unars

Cont|Po|J|M|L|D|To|B|U|Ind

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27. InToLat: Involutive chains

Definition

An *involutive chain* is a Galois chain $\mathbf{P} = \langle P, \leq, \sim, -\rangle$ such that $\sim, -$ are inverses of each other:

${\sim}{-}x = x$

$-{\sim}x = x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

${\sim}{-}x = x$

$-{\sim}x = x$

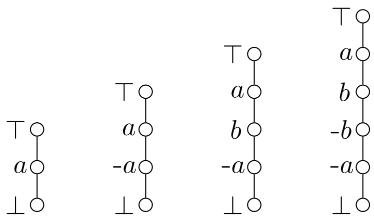
Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 1$

Small Members (not in any subclass)



InC$_3$ **InC$_4$** **InC$_5$** **InC$_6$**

Subclasses

InToMag: Involutive totally ordered magmas

Superclasses

DInLat: Distributive involutive lattices

GalToLat: Galois chains

Cont|Po|J|M|L|D|To|B|U|Ind

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28. InToMag: Involutive totally ordered magmas

Definition

An *involutive totally ordered magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\langle A, \leq, \cdot \rangle$ is a totally ordered magma,

$\sim, -$ is an involutive pair: $\sim -x = x = -\sim x$,

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$ and

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$.

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 22,\ f_5 = 142$

Subclasses

CyInToMag: Cyclic involutive totally ordered magmas

InToSgrp: Involutive totally ordered semigroups

Superclasses

DInLMag: Distributive involutive lattice-ordered magmas

InToLat: Involutive chains

RToMag: Residuated totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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29. InToSgrp: Involutive totally ordered semigroups

Definition

An *involutive totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\langle A, \leq, \cdot \rangle$ is an involutive totally ordered magma and

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$\sim -x = x$

$-{\sim}x = x$

$x \cdot y \leq z \iff y \leq {\sim}(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot {\sim}z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 14,\ f_5 = 43,\ f_6 = 147,\ f_7 = 578$

Subclasses

CyInToSgrp: Cyclic involutive totally ordered semigroups

InToMon: Involutive totally ordered monoids

Superclasses

DInLSgrp: Distributive involutive lattice-ordered semigroups

InToMag: Involutive totally ordered magmas

RToSgrp: Residuated totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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30. InToMon: Involutive totally ordered monoids

Definition

An *involutive totally ordered monoid* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that $\langle A, \leq, \cdot\rangle$ is an involutive totally ordered semigroup that has an identity: $x \cdot 1 = x = 1 \cdot x$

Formal Definition

${\sim}{-}x = x$

$-{\sim}x = x$

$x \cdot y \leq z \iff y \leq {\sim}(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot {\sim}z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 4,\ f_5 = 8,\ f_6 = 17,\ f_7 = 38$

Subclasses

CyInToMon: Cyclic involutive totally ordered monoids

IInToMon: Integral involutive totally ordered monoids

Superclasses

InToSgrp: Involutive totally ordered semigroups

RToMon: Residuated totally ordered monoids

RToUn: Residuated totally-ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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31. IInToMon: Integral involutive totally ordered monoids

Definition

An *integral involutive totally ordered monoid* is an involutive totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ that is

integral: $x \leq 1$

Formal Definition

$\sim{-x} = x$

$-{\sim x} = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 7,\ f_7 = 12,\ f_8 = 35$

Subclasses

CyIInToMon: Cyclic integral involutive totally ordered monoids

Superclasses

IRToMon: Integral residuated totally ordered monoids

InToMon: Involutive totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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32. CyInToMag: Cyclic involutive totally ordered magmas

Definition

A *cyclic involutive totally ordered magma* is an insl-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 22,\ f_5 = 138$

Subclasses

CInToMag: Commutative involutive totally ordered magmas

CyInToSgrp: Cyclic involutive totally ordered semigroups

Superclasses

CyDInLMag: Cyclic distributive involutive lattice-ordered magmas

InToMag: Involutive totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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33. CyInToSgrp: Cyclic involutive totally ordered semigroups

Definition

A *cyclic involutive totally ordered semigroup* is a cyinsl-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 14,\ f_5 = 39,\ f_6 = 119$

Subclasses

CInToSgrp: Commutative involutive totally ordered semigroups

CyInToMon: Cyclic involutive totally ordered monoids

Superclasses

CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups

CyInToMag: Cyclic involutive totally ordered magmas

InToSgrp: Involutive totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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34. CyInToMon: Cyclic involutive totally ordered monoids

Definition

A *cyclic involutive totally ordered monoid* is an insl-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 4,\ f_5 = 8,\ f_6 = 17,\ f_7 = 38,\ f_8 = 91$

Subclasses

CInToMon: Commutative involutive totally ordered monoids

CyIInToMon: Cyclic integral involutive totally ordered monoids

ToGrp: Totally ordered groups

Superclasses

CyInToSgrp: Cyclic involutive totally ordered semigroups

InToMon: Involutive totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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35. CyIInToMon: Cyclic integral involutive totally ordered monoids

Definition

A *cyclic integral involutive totally ordered monoid* is an inporim $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that

$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \le z \iff y \le -(-z \cdot x)$

$x \cdot y \le z \iff x \le -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \le 1$

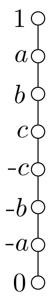
Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 7,\ f_7 = 12,\ f_8 = 35$

Small Members (not in any subclass)



InTo$_8$

$\cdot$	a	b	c	-c	-b	-a
a	a	c	c	-c	-b	0
b	b	-b	-b	-a	0	0
c	c	-b	-b	0	0	0
-c	-b	-b	0	0	0	0
-b	-b	0	0	0	0	0
-a	0	0	0	0	0	0

CyIInToMon$_{8,1}$

$\cdot$	a	b	c	-c	-b	-a
a	a	b	c	-b	-b	0
b	c	-b	-b	-b	0	0
c	c	-b	-b	0	0	0
-c	-c	-a	0	0	0	0
-b	-b	0	0	0	0	0
-a	0	0	0	0	0	0

CyIInToMon$_{8,2}$

$\cdot$	a	b	c	-c	-b	-a
a	a	c	c	-c	-b	0
b	b	c	c	-a	0	0
c	c	c	c	0	0	0
-c	-b	-b	0	0	0	0
-b	-b	0	0	0	0	0
-a	0	0	0	0	0	0

CyIInToMon$_{8,3}$

$\cdot$	a	b	c	-c	-b	-a
a	a	b	c	-b	-b	0
b	c	c	c	-b	0	0
c	c	c	c	0	0	0
-c	-c	-a	0	0	0	0
-b	-b	0	0	0	0	0
-a	0	0	0	0	0	0

CyIInToMon$_{8,4}$

Subclasses

IMTLChn: Involutive monoidal t-norm logic chains

Superclasses

CyInToMon: Cyclic involutive totally ordered monoids

IInToMon: Integral involutive totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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36. ToGrp: Totally ordered groups

Definition

A *totally ordered group* is a lattice-ordered group $\langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x^{-1} \cdot x = 1$

$x \cdot x^{-1} = 1$

Examples

Properties

Classtype	Variety
Equational theory	Decidable Holland and McCleary [1979]
Quasiequational theory	Undecidable Glass and Gurevich [1983]
First-order theory	hereditarily undecidable Burris [1985]
Congruence distributive	Yes, see lattices
Congruence n-permutable	Yes, $n = 2$, see groups
Congruence regular	Yes, see groups
Congruence uniform	Yes, see groups
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_n = 0$ for $n > 1$

Subclasses

AbToGrp: Abelian totally ordered groups

Superclasses

CyInToMon: Cyclic involutive totally ordered monoids

RepLGrp: Representable lattice-ordered groups

Cont|Po|J|M|L|D|To|B|U|Ind

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37. CToSgrp: Commutative totally ordered semigroups

Definition

A *commutative totally ordered semigroup* is a totally ordered semigroup $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 20,\ f_4 = 114,\ f_5 = 710,\ f_6 = 4726,\ f_7 = 33157,\ f_8 = 243048,\ f_9 = 1850817,$
$f_{10} = 14590692$

Subclasses

CIdToSgrp: Commutative idempotent totally ordered semigroups

CRToSgrp: Commutative residuated totally ordered semigroups

CToMon: Commutative totally ordered monoids

Superclasses

CDLSgrp: Commutative distributive lattice-ordered semigroups

ToSgrp: Totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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38. CToMon: Commutative totally ordered monoids

Definition

A *commutative totally ordered monoid* is a totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 6,\ f_4 = 22,\ f_5 = 92,\ f_6 = 426$

Subclasses

CIToMon: Commutative integral totally ordered monoids
CIdToMon: Commutative idempotent totally ordered monoids
CRToMon: Commutative residuated totally ordered monoids

Superclasses

CDLMon: Commutative distributive lattice-ordered monoids
CToSgrp: Commutative totally ordered semigroups
ToMon: Totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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39. CIToMon: Commutative integral totally ordered monoids

Definition

A *commutative integral totally ordered monoid* is a integral totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 22,\ f_6 = 94,\ f_7 = 451$

Subclasses

CIRToMon: Commutative integral residuated totally ordered monoids

Superclasses

CDILMon: Commutative distributive integral lattice-ordered monoids
CToMon: Commutative totally ordered monoids
IToMon: Integral totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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40. CIdToSgrp: Commutative idempotent totally ordered semigroups

Definition

A *commutative idempotent totally ordered semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that $\langle A, \wedge, \vee, \cdot \rangle$ is an idempotent totally ordered semigroup and
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 14,\ f_5 = 42,\ f_6 = 132$

Subclasses

CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups
CIdToMon: Commutative idempotent totally ordered monoids

Superclasses

CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups
CToSgrp: Commutative totally ordered semigroups
IdToSgrp: Idempotent totally ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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41. CIdToMon: Commutative idempotent totally ordered monoids

Definition

A *commutative idempotent totally ordered monoid* is an idempotent totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1 \rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 4,\ f_4 = 8,\ f_5 = 16,\ f_6 = 32,\ f_7 = 64$

Subclasses

CIdRToMon: Commutative idempotent residuated totally ordered monoids

Superclasses

CDIdLMon: Commutative distributive idempotent lattice-ordered monoids
CIdToSgrp: Commutative idempotent totally ordered semigroups
CToMon: Commutative totally ordered monoids
IdToMon: Idempotent totally ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

42. CToDivLat: Commutative division chains

Definition

A *commutative totally ordered division lattice* is a commutative division lattice $\mathbf{C} = \langle C, \wedge, \vee, \backslash, /\rangle$ such that $\langle C, \wedge, \vee\rangle$ is a totally ordered lattice.

Formal Definition

$(x \wedge y)/z = x/z \wedge y/z$

$x \leq z/y \iff y \leq x\backslash z$

$x/y = y\backslash x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 20,\ f_4 = 294$

Subclasses

CRToMag: Commutative residuated totally ordered magmas

Superclasses

ToDivLat: Totally ordered division lattices

Cont|Po|J|M|L|D|To|B|U|Ind

43. CRToMag: Commutative residuated totally ordered magmas

Definition

A *commutative residuated totally ordered magma* is a residuated totally ordered magma such that $\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 10,\ f_4 = 112,\ f_5 = 2772$

Subclasses

CInToMag: Commutative involutive totally ordered magmas

CRToSgrp: Commutative residuated totally ordered semigroups

Superclasses

CDRLMag: Commutative distributive residuated lattice-ordered magmas

CToDivLat: Commutative division chains

RToMag: Residuated totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

44. CRToSgrp: Commutative residuated totally ordered semigroups

Definition

A *commutative residuated totally ordered semigroup* is a residuated totally ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 8,\ f_4 = 41,\ f_5 = 241$

Subclasses

CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups

CInToSgrp: Commutative involutive totally ordered semigroups

CRToMon: Commutative residuated totally ordered monoids

Superclasses

CRSlSgrp: Commutative residuated semilinear semigroups

CRToMag: Commutative residuated totally ordered magmas

CToSgrp: Commutative totally ordered semigroups

RToSgrp: Residuated totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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45. CRToMon: Commutative residuated totally ordered monoids

Definition

A *commutative residuated totally ordered monoid* is a residuated totally ordered monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 11,\ f_5 = 46,\ f_6 = 213$

Subclasses

CIRToMon: Commutative integral residuated totally ordered monoids

CIdRToMon: Commutative idempotent residuated totally ordered monoids

CInToMon: Commutative involutive totally ordered monoids

Superclasses

CRSlMon: Commutative residuated semilinear monoids

CRToSgrp: Commutative residuated totally ordered semigroups

CToMon: Commutative totally ordered monoids

RToMon: Residuated totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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46. CIRToMon: Commutative integral residuated totally ordered monoids

Definition

A *commutative integral residuated totally ordered monoid* is a residuated totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 6,\ f_5 = 22,\ f_6 = 94$

Subclasses

IMTLChn: Involutive monoidal t-norm logic chains

Superclasses

CIRSlMon: Commutative integral residuated semilinear monoids

CIToMon: Commutative integral totally ordered monoids

CRToMon: Commutative residuated totally ordered monoids

IRToMon: Integral residuated totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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47. CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups

Definition

A *commutative idempotent residuated totally ordered semigroup* is an idempotent residuated totally ordered semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 5,\ f_5 = 14,\ f_6 = 42$

Subclasses

CIdRToMon: Commutative idempotent residuated totally ordered monoids

Superclasses

CIdRSlSgrp: Commutative idempotent residuated semilinear semigroups

CIdToSgrp: Commutative idempotent totally ordered semigroups

CRToSgrp: Commutative residuated totally ordered semigroups

IdRToSgrp: Idempotent residuated totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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48. CIdRToMon: Commutative idempotent residuated totally ordered monoids

Definition

A *commutative idempotent residuated totally ordered monoid* is an idmpotent residuated totally ordered monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 4,\ f_5 = 8,\ f_6 = 16,\ f_7 = 32$

Subclasses

Superclasses

CIdRSlMon: Commutative idempotent residuated semilinear monoids

CIdRToSgrp: Commutative idempotent residuated totally ordered semigroups

CIdToMon: Commutative idempotent totally ordered monoids

CRToMon: Commutative residuated totally ordered monoids

IdRToMon: Idempotent residuated totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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49. CInToMag: Commutative involutive totally ordered magmas

Definition

A *commutative involutive totally ordered magma* is a insl-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 18,\ f_5 = 72,\ f_6 = 384$

Subclasses

CInToSgrp: Commutative involutive totally ordered semigroups

Superclasses

CDInLMag: Commutative distributive involutive lattice-ordered magmas

CRToMag: Commutative residuated totally ordered magmas

CyInToMag: Cyclic involutive totally ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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50. CInToSgrp: Commutative involutive totally ordered semigroups

Definition

A *commutative involutive totally ordered semigroup* is an insl-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 14,\ f_5 = 37,\ f_6 = 107$

Subclasses

CInToMon: Commutative involutive totally ordered monoids

Superclasses

CInSlSgrp: Commutative involutive semilinear semigroups

CInToMag: Commutative involutive totally ordered magmas

CRToSgrp: Commutative residuated totally ordered semigroups

CyInToSgrp: Cyclic involutive totally ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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51. CInToMon: Commutative involutive totally ordered monoids

Definition

A *commutative involutive totally ordered monoid* is an insl-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 4,\ f_5 = 8,\ f_6 = 17,\ f_7 = 36,\ f_8 = 81$

Subclasses

AbLGrp: Abelian lattice-ordered groups
IMTLChn: Involutive monoidal t-norm logic chains

Superclasses

CInSlMon: Commutative involutive semilinear monoids
CInToSgrp: Commutative involutive totally ordered semigroups
CRToMon: Commutative residuated totally ordered monoids
CyInToMon: Cyclic involutive totally ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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52. IMTLChn: Involutive monoidal t-norm logic chains

Definition

A *involutive monoidal t-norm logic chain* is an integral involutive to-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$
$x \cdot 1 = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 3,\ f_6 = 7,\ f_7 = 12,\ f_8 = 31,\ f_9 = 59$

Subclasses

TrivA: Trivial algebras

Superclasses

CIRToMon: Commutative integral residuated totally ordered monoids
CInToMon: Commutative involutive totally ordered monoids
CyIInToMon: Cyclic integral involutive totally ordered monoids
IMTL: Involutive monoidal t-norm logic algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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53. AbToGrp: Abelian totally ordered groups

Definition

An *abelian totally ordered group* is a totally ordered group $\mathbf{A} = \langle A, \wedge, \vee, \cdot, ^{-1}, 1 \rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x^{-1} \cdot x = 1$

$x \cdot x^{-1} = 1$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	hereditarily undecidable Burris [1985]
Locally finite	No
Congruence distributive	yes (see lattices)
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$ (see groups)
Congruence regular	Yes, (see groups)
Congruence uniform	Yes, (see groups)
Amalgamation property	Yes
Strong amalgamation property	no Cherri and Powell [1993]

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_n = 0$ for $n > 1$

Subclasses

TrivA: Trivial algebras

Superclasses

AbLGrp: Abelian lattice-ordered groups

ToGrp: Totally ordered groups

Cont|Po|J|M|L|D|To|B|U|Ind

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54. ToRng: Totally ordered rings

Definition

A *totally ordered ring* is an algebra $\mathbf{A} = \langle A, +, -, 0, \cdot, 1, \leq\rangle$ such that

$\langle A, +, -, 0, \cdot, 1\rangle$ is a ring

$\langle A, \leq\rangle$ is a linear order

$+$ is *order-preserving*: $x \leq y \implies x + z \leq y + z$

$\cdot$ is *order-preserving* for positive elements: $x \leq y$ and $0 \leq z \implies xz \leq yz$

Properties

Classtype	Universal

Finite Members

Subclasses

ToFld: Totally ordered fields

Superclasses

LRng: Lattice-ordered rings

Cont|Po|J|M|L|D|To|B|U|Ind

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55. CToRng: Commutative totally ordered rings

Definition

A *commutative totally ordered ring* is an totally ordered ring $\mathbf{A} = \langle A, +, -, 0, \cdot, \leq\rangle$ such that

$\cdot$ is *commutative*: $xy = yx$

Properties

Finite Members

Subclasses

ToFld: Totally ordered fields

Superclasses

CLRng: Commutative lattice-ordered rings Cont|Po|J|M|L|D|To|B|U|Ind

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56. ToFld: Totally ordered fields

Definition

An *ordered field* is an algebra $\mathbf{F} = \langle F, +, -, 0, \cdot, 1, \leq\rangle$ such that

$\langle F, +, -, 0, \cdot, 1\rangle$ is a field

$\langle F, \leq\rangle$ is a linear order

$+$ is *order-preserving*: $x \leq y \implies x + z \leq y + z$

$\cdot$ is *order-preserving* for positive elements: $x \leq y$ and $0 \leq z \implies xz \leq yz$

Properties

Classtype	Universal

Finite Members

None

Subclasses

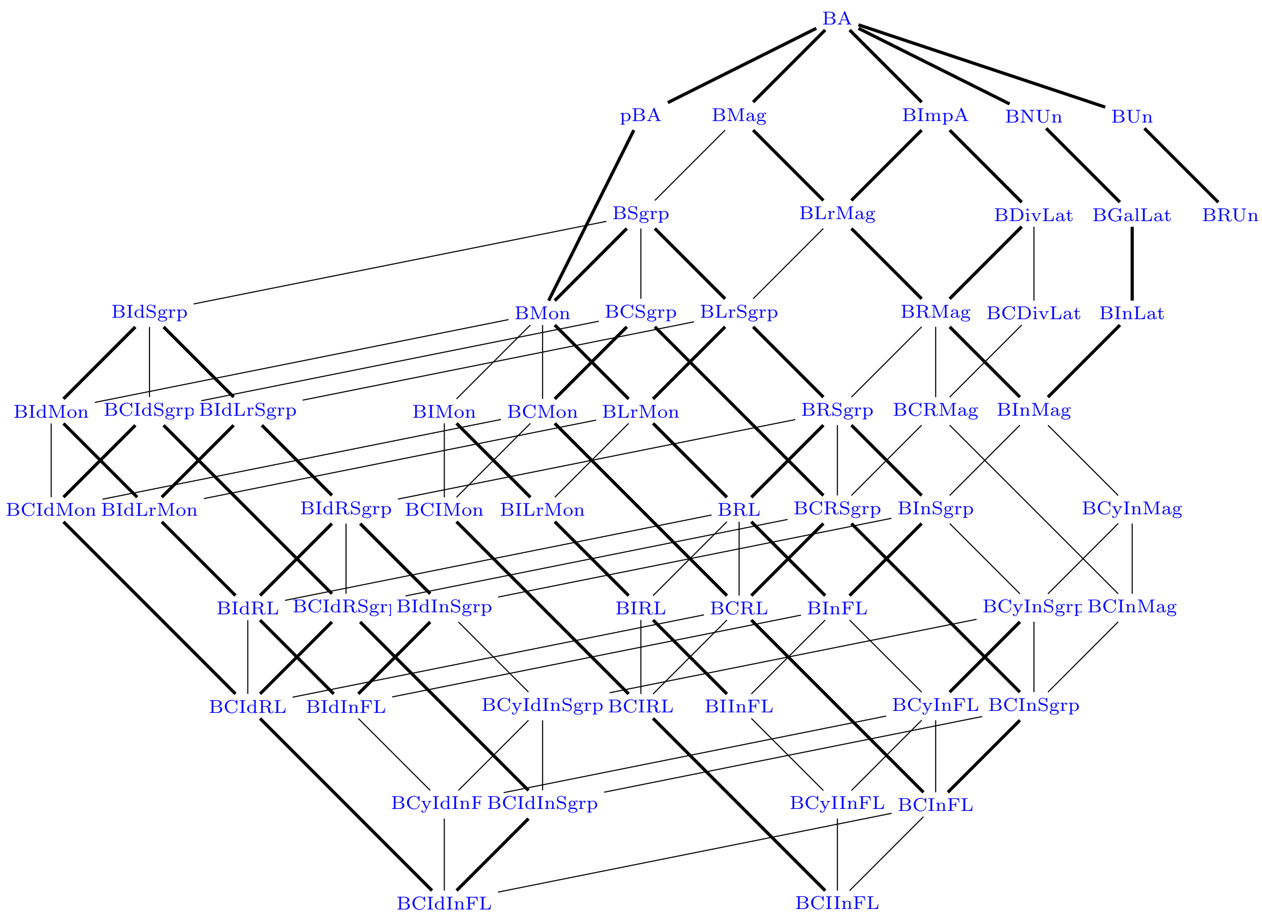
Superclasses

CToRng: Commutative totally ordered rings

ToRng: Totally ordered rings Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 8

Boolean-ordered algebras



1. BA: Boolean algebras

Definition

A *Boolean algebra* is a complemented lattice $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice

Formal Definition

Examples

Example 1: $\langle \mathcal{P}(S), \cup, \emptyset, \cap, S, -\rangle$, the collection of subsets of a sets S, with union, intersection, and set complementation.

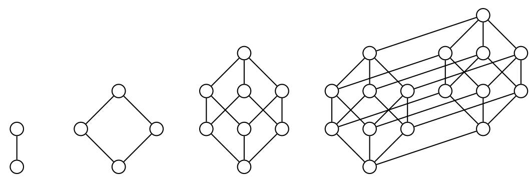
Properties

Classtype	Variety
Equational theory	NPTIME
Quasiequational theory	Decidable
First-order theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes
Locally finite	Yes
Residual size	2

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1,\ f_9 = 0,\ f_{2^n} = 1,\ f_k = 0$ if $k \neq 2^n$

Small Members (not in any subclass)



B_1 B_2 B_3 B_4

Subclasses

BImpA: Boolean implication algebras

BMag: Boolean magmas

BNUn: Boolean negated unars

BUn: Boolean unars

pBA: Pointed Boolean algebras

Superclasses

BoolLat: Boolean lattices

CRng_1: Commutative rings with identity

CplmModLat: Complemented modular lattices

DLat: Distributive lattices

DblStAlg: Double Stone algebras

GBA: Generalized Boolean algebras

GödA: Gödel algebras

KLA: Kleene logic algebras

LA_n: Lukasiewicz algebras of order n

MV: MV-algebras

ModOLat: Modular ortholattices

bDLat: Bounded distributive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pBA: Pointed Boolean algebras

Definition

A *pointed Boolean algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, -, 0, 1, c\rangle$ such that $\langle A, \wedge, \vee, -, 0, 1\rangle$ is a Boolean algebra and c is a constant operation on A.

Formal Definition

$c = c$

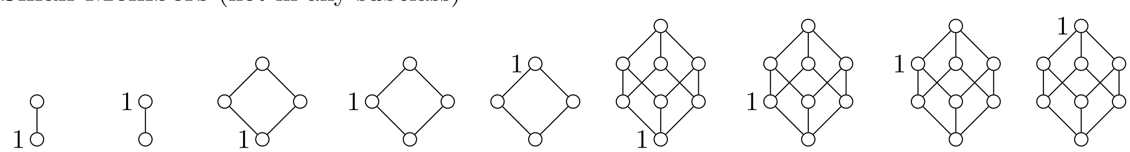
Properties

Classtype	Variety
Equational theory	NPTIME
Quasiequational theory	Decidable
First-order theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes
Locally finite	Yes
Residual size	2

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 3,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1,\ f_9 = 0,\ f_{2^n} = n + 1,\ f_k = 0$ if $k \neq 2^n$

Small Members (not in any subclass)



$pB_{1,1}$ $pB_{1,2}$ $pB_{2,1}$ $pB_{2,2}$ $pB_{2,3}$ $pB_{3,1}$ $pB_{3,2}$ $pB_{3,3}$ $pB_{3,4}$

Subclasses

BMon: Boolean monoids

Superclasses

BA: Boolean algebras

pDLat: Pointed distributive lattices Cont|Po|J|M|L|D|To|B|U|Ind

3. BUn: Boolean unars

Formal Definition

A *Boolean unar* is an algebra $\langle B, \wedge, \vee, \neg, \bot, \top, f\rangle$ such that $\langle B, \wedge, \vee, \neg, \bot, \top\rangle$ is a Boolean algebra and f is a unary operation on B that is

join-preserving: $f(x \vee y) = f(x) \vee f(y)$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 0,\ f_4 = 15,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 147,\ f_9 = 0$

Subclasses

BRMod: Boolean modules over a relation algebra

BRUn: Boolean residuated unars
CA_2: Cylindric algebras of dimension 2
MA: Modal algebras

Superclasses

BA: Boolean algebras
DLUn: Distributive lattice-ordered unars

Cont|Po|J|M|L|D|To|B|U|Ind

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4. BNUn: Boolean negated unars

Formal Definition

A *Boolean negated unar* is an algebra $\langle B, \wedge, \vee, \neg, \bot, \top, \sim\rangle$ such that $\langle B, \wedge, \vee, \neg, \bot, \top\rangle$ is a Boolean algebra and $\sim$ is a unary operation on B that is

join-reversing: $\sim(x \vee y) = \sim y \wedge \sim x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 0,\ f_4 = 15,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 147,\ f_9 = 0$

Subclasses

BGalLat: Boolean Galois lattices

Superclasses

BA: Boolean algebras
DLNUn: Distributive lattice-ordered negated unars

Cont|Po|J|M|L|D|To|B|U|Ind

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5. MA: Modal algebras

Definition

A *modal algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \diamond\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra

$\diamond$ is *join-preserving*: $\diamond(x \vee y) = \diamond x \vee \diamond y$

$\diamond$ is *normal*: $\diamond 0 = 0$

Remark: Modal algebras provide algebraic models for modal logic. The operator $\diamond$ is the *possibility operator*, and the *necessity operator* $\Box$ is defined as $\Box x = \neg \diamond \neg x$.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No
Discriminator variety	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

Subclasses

MonA: Monadic algebras

TA: Tense algebras

Superclasses

BUn: Boolean unars

Cont|Po|J|M|L|D|To|B|U|Ind

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6. TA: Tense algebras

Definition

A *tense algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \diamond_f, \diamond_p\rangle$ such that both

$\langle A, \vee, 0, \wedge, 1, \neg, \diamond_f\rangle$ and $\langle A, \vee, 0, \wedge, 1, \neg, \diamond_p\rangle$ are modal algebras

$\diamond_p$ and $\diamond_f$ are *conjugates*: $x \wedge \diamond_p y = 0$ iff $\diamond_f x \wedge y = 0$

Remark: Tense algebras provide algebraic models for logic of tenses. The two possibility operators $\diamond_p$ and $\diamond_f$ are intuitively interpreted as *at some past instance* and *at some future instance*.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No
Discriminator variety	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

Subclasses

TrivA: Trivial algebras

Superclasses

MA: Modal algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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7. MonA: Monadic algebras

Definition

A *monadic algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, f\rangle$ of type $\langle 2, 0, 2, 0, 1, 1\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra

f is a *unary closure operator*: $f(x \vee y) = f(x) \vee f(y)$, $f(0) = 0$, $x \leq f(x) = f(f(x))$

f is *self conjugated*: $f(x) \wedge y = 0 \iff x \wedge f(y) = 0$

Properties

Classtype	Variety
Equational theory	Decidable
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes

Finite Members

Subclasses

TrivA: Trivial algebras

Superclasses

MA: Modal algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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8. BMag: Boolean magmas

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 0,\ f_4 = 1176$

Subclasses

BLrMag: Boolean left-residuated magmas

BSgrp: Boolean semigroups

Superclasses

BA: Boolean algebras

DLMag: Distributive lattice-ordered magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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9. BSgrp: Boolean semigroups

Definition

A *Boolean semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that

$\langle A, \cdot \rangle$ is a semigroup

$\langle G, \leq \rangle$ is a Boolean algebra

$\cdot$ is *orderpreserving*: $x \leq y \implies x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 0,\ f_4 = 93,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

BCSgrp: Boolean commutative semigroups

BIdSgrp: Boolean idempotent semigroups

BLrSgrp: Boolean left-residuated semigroups

BMon: Boolean monoids

Superclasses

BMag: Boolean magmas

DLSgrp: Distributive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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10. BMon: Boolean monoids

Definition

A *Boolean monoid* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\langle A, \cdot, 1\rangle$ is a monoid

$\langle G, \leq\rangle$ is a Boolean algebra

$\cdot$ is *orderpreserving*: $x \leq y \implies wxz \leq wyz$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 11,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 383$

Subclasses

BCMon: Boolean commutative monoids

BIMon: Boolean integral monoids

BIdMon: Boolean idempotent monoids

BLrMon: Boolean left-residuated monoids

Superclasses

BSgrp: Boolean semigroups

DLMon: Distributive lattice-ordered monoids

pBA: Pointed Boolean algebras Cont|Po|J|M|L|D|To|B|U|Ind

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11. BIMon: Boolean integral monoids

Definition

A *Boolean integral monoid* is a Boolean monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0$

Subclasses

BCIMon: Boolean commutative integral monoids
BILrMon: Boolean integral left-residuated monoids

Superclasses

BMon: Boolean monoids
DILMon: Distributive integral lattice-ordered monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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12. BIdSgrp: Boolean idempotent semigroups

Definition

An *Boolean idempotent semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that
$\langle A, \wedge, \vee, \cdot \rangle$ is a Boolean semigroup and
$\cdot$ is *Boolean idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 0,\ f_4 = 18,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 88,\ f_9 = 0,\ f_{10} = 0$

Subclasses

BCIdSgrp: Boolean commutative idempotent semigroups
BIdLrSgrp: Boolean idempotent left-residuated semigroups
BIdMon: Boolean idempotent monoids

Superclasses

BSgrp: Boolean semigroups
DIdLSgrp: Distributive idempotent lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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13. BIdMon: Boolean idempotent monoids

Definition

An *Boolean idempotent monoid* is a Boolean monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1 \rangle$ such that
$\cdot$ is *Boolean idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 6,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 24$

Subclasses

BCIdMon: Boolean commutative idempotent monoids
BIdLrMon: Boolean idempotent left-residuated monoids

Superclasses

BIdSgrp: Boolean idempotent semigroups
BMon: Boolean monoids
DIdLMon: Distributive idempotent lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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14. BImpA: Boolean implication algebras

Formal Definition

$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
$(x \vee y) \to z = (x \to z) \wedge (y \to z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 6,\ f_3 = 0,\ f_4 = 1176,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

BDivLat: Boolean division lattices
BLrMag: Boolean left-residuated magmas

Superclasses

BA: Boolean algebras
DLImpA: Distributive lattice-ordered implication algebras Cont|Po|J|M|L|D|To|B|U|Ind

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15. BLrMag: Boolean left-residuated magmas

Definition

A *Boolean left-residuated magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a Boolean algebra,
$\langle A, \cdot \rangle$ is a magma and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 0,\ f_4 = 325,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses
BLrSgrp: Boolean left-residuated semigroups
BRMag: Boolean residuated magmas
Superclasses
BImpA: Boolean implication algebras
BMag: Boolean magmas
DLrLMag: Distributive left-residuated lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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16. BLrSgrp: Boolean left-residuated semigroups

Definition
A *Boolean left-residuated semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a Boolean algebra,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
Formal Definition
$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
Properties

Classtype	variety

Finite Members
$f_1 = 1,\ f_2 = 3,\ f_3 = 0,\ f_4 = 39,\ f_5 = 0$
Subclasses
BIdLrSgrp: Boolean idempotent left-residuated semigroups
BLrMon: Boolean left-residuated monoids
BRSgrp: Boolean residuated semigroups
Superclasses
BLrMag: Boolean left-residuated magmas
BSgrp: Boolean semigroups
DLrLSgrp: Distributive left-residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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17. BLrMon: Boolean left-residuated monoids

Definition
A *Boolean left-residuated monoid* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ such that
$\langle A, \leq \rangle$ is a Boolean algebra,
$\langle A, \cdot, 1 \rangle$ is a monoid and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \leq z \iff y \leq x \backslash z$
Formal Definition
$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 6,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 90$

Subclasses

BILrMon: Boolean integral left-residuated monoids

BIdLrMon: Boolean idempotent left-residuated monoids

BRL: Boolean residuated lattices

Superclasses

BLrSgrp: Boolean left-residuated semigroups

BMon: Boolean monoids Cont|Po|J|M|L|D|To|B|U|Ind

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18. BILrMon: Boolean integral left-residuated monoids

Definition

A *Boolean left-residuated integral monoid* is a Boolean left-residuated monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, \rangle$ for which $x \leq 1$.

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0$

Subclasses

BIRL: Boolean integral residuated lattices

Superclasses

BIMon: Boolean integral monoids

BLrMon: Boolean left-residuated monoids

DLrLMon: Distributive left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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19. BIdLrSgrp: Boolean idempotent left-residuated semigroups

Definition

An *Boolean idempotent left-residuated semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot \rangle$ such that
$\langle A, \wedge, \vee, \cdot \rangle$ is a Boolean left-residuated semigroup and
$\cdot$ is *Boolean idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 10,\ f_5 = 0,\ f_6 = 0$

Subclasses

BIdLrMon: Boolean idempotent left-residuated monoids
BIdRSgrp: Boolean idempotent residuated semigroups

Superclasses

BIdSgrp: Boolean idempotent semigroups
BLrSgrp: Boolean left-residuated semigroups
DILrLMon: Distributive integral left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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20. BIdLrMon: Boolean idempotent left-residuated monoids

Definition

An *Boolean idempotent left-residuated monoid* is a Boolean left-residuated monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 3,\ f_5 = 0,\ f_6 = 0$

Subclasses

BIdRL: Boolean idempotent residuated lattices

Superclasses

BIdLrSgrp: Boolean idempotent left-residuated semigroups
BIdMon: Boolean idempotent monoids
BLrMon: Boolean left-residuated monoids
DIdLrLSgrp: Distributive idempotent left-residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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21. BRUn: Boolean residuated unars

Formal Definition

A *boolean residuated unar* (also called a *br-unar* for short) is an algebra of the form $\langle B, \wedge, \vee, \neg, \top, \bot, f, g\rangle$ such that $\langle B, \wedge, \vee, \neg, \top, \bot\rangle$ is a Boolean algebra and

$f(x) \le y \iff x \le g(y)$.

Basic Results

Both f and g are order preserving. More specifically, f preserves all existing joins and g preserves all existing meets. In particular, $f(x \vee y) = f(x) \vee f(y)$ and $g(x \wedge y) = g(x) \wedge g(y)$.

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 10,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 104$

Subclasses

BInFL: Boolean involutive FL-algebras

Superclasses

BUn: Boolean unars

DIdLrLMon: Distributive idempotent left-residuated lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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22. BDivLat: Boolean division lattices

Formal Definition

A *Boolean division lattice* is an algebra $\langle B, \wedge, \vee, \neg, \top, \bot, \backslash, /\rangle$ such that $\langle B, \wedge, \vee, \neg, \top, \bot\rangle$ is a Boolean algebra,

$x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z$,

$(x \wedge y)/z = x/z \wedge y/z$ and

$x \le z/y \iff y \le x\backslash z$

Basic Results

In any Boolean division lattice $x/(y \vee z) = x/y \wedge x/z$ since $w \le x/(y \vee z) \iff y \vee z \le w\backslash x \iff y \le w\backslash x$ and $z \le w\backslash x \iff w \le x/y$ and $w \le x/z \iff w \le x/y \wedge x/z$.

Similarly, $(x \vee y)\backslash z = x\backslash z \wedge y\backslash z$.

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 0,\ f_4 = 325$

Subclasses

BCDivLat: Boolean commutative division lattices

BRMag: Boolean residuated magmas

Superclasses

BImpA: Boolean implication algebras

DRLUn: Distributive residuated lattice-ordered unars Cont|Po|J|M|L|D|To|B|U|Ind

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23. BRMag: Boolean residuated magmas

Definition

A *Boolean residuated magma* is an algebra $\mathbf{A} = \langle A, \le, \cdot, \backslash, /\rangle$ such that

$\langle A, \le\rangle$ is a Boolean algebra,

$\langle A, \cdot\rangle$ is a magma and

$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x\backslash z$

$/$ is the right residual of $\cdot$: $x \cdot y \le z \iff x \le z/y$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$x \cdot y \le z \iff y \le x \backslash z$
$x \cdot y \le z \iff x \le z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 136,\ f_5 = 0$

Subclasses

BCRMag: Boolean commutative residuated magmas
BInMag: Boolean involutive magmas
BRSgrp: Boolean residuated semigroups
NA: Nonassociative relation algebras

Superclasses

BDivLat: Boolean division lattices
BLrMag: Boolean left-residuated magmas
DDivLat: Distributive division lattices Cont|Po|J|M|L|D|To|B|U|Ind

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24. BRSgrp: Boolean residuated semigroups

Definition

A *Boolean residuated semigroup* is an algebra $\mathbf{A} = \langle A, \le, \cdot, \backslash, / \rangle$ such that
$\langle A, \le \rangle$ is a Boolean algebra,
$\langle A, \cdot \rangle$ is a semigroup and
$\backslash$ is the left residual of $\cdot$: $x \cdot y \le z \iff y \le x \backslash z$
$/$ is the right residual of $\cdot$: $x \cdot y \le z \iff x \le z/y$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$x \cdot y \le z \iff y \le x \backslash z$
$x \cdot y \le z \iff x \le z/y$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 28,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCRSgrp: Boolean commutative residuated semigroups
BIdRSgrp: Boolean idempotent residuated semigroups
BInSgrp: Boolean involutive semigroups
BRL: Boolean residuated lattices

Superclasses

BLrSgrp: Boolean left-residuated semigroups
BRMag: Boolean residuated magmas
DRLMag: Distributive residuated lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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25. BRL: Boolean residuated lattices

Definition

A *Boolean residuated lattice* is a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that

$\wedge, \vee$ are distributive: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	Variety
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, n=2
Congruence regular	No
Congruence e-regular	Yes
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCRL: Boolean commutative residuated lattices

BIRL: Boolean integral residuated lattices

BIdRL: Boolean idempotent residuated lattices

BInFL: Boolean involutive FL-algebras

Superclasses

BLrMon: Boolean left-residuated monoids

BRSgrp: Boolean residuated semigroups

DRLSgrp: Distributive residuated lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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26. BIRL: Boolean integral residuated lattices

Definition

A *Boolean integral residuated lattice* is an Boolean residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

x is *integral*: $x \leq 1$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIRL: Boolean commutative integral residuated lattices

BIInFL: Boolean integral involutive FL-algebras

SeqA: Sequential algebras

Superclasses

BILrMon: Boolean integral left-residuated monoids

BRL: Boolean residuated lattices

DRL: Distributive residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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27. BIdRSgrp: Boolean idempotent residuated semigroups

Definition

An *Boolean idempotent residuated semigroup* is a Boolean residuated semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that $\cdot$ is *Boolean idempotent*: $x \cdot x = x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x\backslash z$

$x \cdot y \leq z \iff x \leq z/y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 7,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 26$

Subclasses

BCIdRSgrp: Boolean commutative idempotent residuated semigroups

BIdRL: Boolean idempotent residuated lattices

Superclasses

BIdLrSgrp: Boolean idempotent left-residuated semigroups

BRSgrp: Boolean residuated semigroups

DIRL: Distributive integral residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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28. BIdRL: Boolean idempotent residuated lattices

Definition

An *Boolean idempotent residuated lattice* is a Boolean residuated monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that $\cdot$ is *idempotent*: $x \cdot x = x$

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$
$x \le y \implies z \cdot x \le z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \le z \iff y \le x \backslash z$
$x \cdot y \le z \iff x \le z / y$
$x \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 2,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIdRL: Boolean commutative idempotent residuated lattices

Superclasses

BIdLrMon: Boolean idempotent left-residuated monoids
BIdRSgrp: Boolean idempotent residuated semigroups
BRL: Boolean residuated lattices
DIdRLSgrp: Distributive idempotent residuated lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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29. BGalLat: Boolean Galois lattices

Definition

A *Boolean Galois lattice* is an algebra $\mathbf{P} = \langle P, \le, \sim, - \rangle$ such that P is a Boolean algebra and $\sim, -$ are a pair of unary operations on P that form a
Galois connection: $x \le {\sim}y \iff y \le -x$

Formal Definition

$x \le {\sim}y \iff y \le -x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 10,\ f_5 = 0,\ f_6 = 0$

Subclasses

BInLat: Boolean involutive lattices

Superclasses

BNUn: Boolean negated unars
DGalLat: Distributive Galois lattices Cont|Po|J|M|L|D|To|B|U|Ind

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30. BInLat: Boolean involutive lattices

Definition

A *Boolean involutive lattice* is a Boolean Galois lattice $\mathbf{P} = \langle P, \le, \sim, - \rangle$ such that $\sim, -$ are inverses of each other:
${\sim}{-}x = x$
$-{\sim}x = x$

Formal Definition

$x \leq {\sim}y \iff y \leq -x$

${\sim}{-x} = x$

$-{\sim}x = x$

Properties

Classtype	variety
Universal theory	Decidable
First-order theory	Undecidable

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 2,\ f_5 = 0,\ f_6 = 0$

Subclasses

BInMag: Boolean involutive magmas

Superclasses

BGalLat: Boolean Galois lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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31. BInMag: Boolean involutive magmas

Definition

A *Boolean involutive magma* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\langle A, \leq, \cdot \rangle$ is a Boolean magma,
${\sim}, -$ is an involutive pair: ${\sim}{-x} = x = -{\sim}x$,
$x \cdot y \leq z \iff y \leq {\sim}(-z \cdot x)$ and
$x \cdot y \leq z \iff x \leq -(y \cdot {\sim}z)$.

Formal Definition

${\sim}{-x} = x$

$-{\sim}x = x$

$x \cdot y \leq z \iff y \leq {\sim}(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot {\sim}z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 20,\ f_5 = 0$

Subclasses

BCyInMag: Boolean cyclic involutive magmas

BInSgrp: Boolean involutive semigroups

Superclasses

BInLat: Boolean involutive lattices

BRMag: Boolean residuated magmas

DInLat: Distributive involutive lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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32. BInSgrp: Boolean involutive semigroups

Definition

An *Boolean involutive semigroup* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\langle A, \leq, \cdot \rangle$ is an Boolean involutive magma and
$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 15,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCyInSgrp: Boolean cyclic involutive semigroups

BInFL: Boolean involutive FL-algebras

Superclasses

BInMag: Boolean involutive magmas

BRSgrp: Boolean residuated semigroups

DInLMag: Distributive involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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33. BInFL: Boolean involutive FL-algebras

Definition

An *Boolean involutive FL-algebra* is an algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that $\langle A, \leq, \cdot \rangle$ is an Boolean involutive semigroup that has an identity: $x \cdot 1 = x = 1 \cdot x$

Formal Definition

$\sim -x = x$

$-\sim x = x$

$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 25$

Subclasses

BCyInFL: Boolean cyclic involutive FL-algebras

BIInFL: Boolean integral involutive FL-algebras

Superclasses

BInSgrp: Boolean involutive semigroups

BRL: Boolean residuated lattices

BRUn: Boolean residuated unars

DInLSgrp: Distributive involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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34. BIInFL: Boolean integral involutive FL-algebras

Definition
A *Boolean integral involutive FL-algebra* is an involutive FL-algebra $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ that is integral: $x \leq 1$

Formal Definition
$\sim -x = x$
$-\sim x = x$
$x \cdot y \leq z \iff y \leq \sim(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot \sim z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members
$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1$

Subclasses
BCyIInFL: Boolean cyclic involutive integral monoids

Superclasses
BIRL: Boolean integral residuated lattices
BInFL: Boolean involutive FL-algebras
DInFL: Distributive involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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35. BCyInMag: Boolean cyclic involutive magmas

Definition
A *cyclic distributive involutive magma* is an inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, -\rangle$ such that $\sim, -$ are cyclic: $\sim x = -x$

Formal Definition
$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

Properties

Classtype	variety

Finite Members
$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 20,\ f_5 = 0$

Subclasses
BCInMag: Boolean commutative involutive magmas
BCyInSgrp: Boolean cyclic involutive semigroups

Superclasses
BInMag: Boolean involutive magmas
DIInFL: Distributive integral involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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36. BCyInSgrp: Boolean cyclic involutive semigroups

Definition

A *cyclic distributive involutive semigroup* is a cyinpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that $\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 15,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCInSgrp: Boolean commutative involutive semigroups
BCyInFL: Boolean cyclic involutive FL-algebras

Superclasses

BCyInMag: Boolean cyclic involutive magmas
BInSgrp: Boolean involutive semigroups
CyDInLMag: Cyclic distributive involutive lattice-ordered magmas Cont|Po|J|M|L|D|To|B|U|Ind

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37. BCyInFL: Boolean cyclic involutive FL-algebras

Definition

A *cyclic distributive involutive FL-algebra* is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, - \rangle$ such that $\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y \leq z \iff x \leq -(y \cdot -z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

BCInFL: Boolean commutative involutive FL-algebras
BCyIInFL: Boolean cyclic involutive integral monoids

Superclasses

BCyInSgrp: Boolean cyclic involutive semigroups
BInFL: Boolean involutive FL-algebras
CyDInLSgrp: Cyclic distributive involutive lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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38. BCyIInFL: Boolean cyclic involutive integral monoids

Definition

A *cyclic distributive integral involutive FL-algebra* is an inporim $\mathbf{A} = \langle A, \leq, \cdot, 1, \sim, -\rangle$ such that
$\sim, -$ are cyclic: $\sim x = -x$

Formal Definition

$--x = x$

$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$x \cdot y \leq z \iff x \leq -(y \cdot -z)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1$

Subclasses

BCIInFL: Boolean commutative integral involutive FL-algebras

Superclasses

BCyInFL: Boolean cyclic involutive FL-algebras

BIInFL: Boolean integral involutive FL-algebras

CyDInFL: Cyclic distributive involutive FL-algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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39. BCSgrp: Boolean commutative semigroups

Definition

A *commutative distributive semigroup* is a Boolean semigroup $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that
$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 0,\ f_4 = 35,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1237,\ f_9 = 0$

Subclasses

BCIdSgrp: Boolean commutative idempotent semigroups

BCMon: Boolean commutative monoids

BCRSgrp: Boolean commutative residuated semigroups

BSlat: Boolean semilattices

Superclasses

BSgrp: Boolean semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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40. BCMon: Boolean commutative monoids

Definition

A *commutative distributive monoid* is a Boolean monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 9,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

BCIMon: Boolean commutative integral monoids

BCIdMon: Boolean commutative idempotent monoids

BCRL: Boolean commutative residuated lattices

Superclasses

BCSgrp: Boolean commutative semigroups

BMon: Boolean monoids

CDLSgrp: Commutative distributive lattice-ordered semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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41. BCIMon: Boolean commutative integral monoids

Definition

A *commutative distributive integral monoid* is a Boolean integral monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \leq 1$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIRL: Boolean commutative integral residuated lattices

Superclasses

BCMon: Boolean commutative monoids

BIMon: Boolean integral monoids

CDLMon: Commutative distributive lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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42. BSlat: Boolean semilattices

Definition

A *Boolean semilattice* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \cdot\rangle$ such that

$\mathbf{A}$ is in the variety generated by complex algebras of semilattices

Let $\mathbf{S} = \langle S, \cdot\rangle$ be a semilattice. The *complex algebra* of $\mathbf{S}$ is $Cm(\mathbf{S}) = \langle P(S), \cup, \emptyset, \cap, S, -, \cdot\rangle$, where $\langle P(S), \cup, \emptyset, \cap, S, -\rangle$ is the Boolean algebra of subsets of S, and

$X \cdot Y = \{x \cdot y \mid x \in X,\ y \in Y\}$.

Properties

Classtype	Variety
Finitely axiomatizable	open
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence extension property	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

TrivA: Trivial algebras

Superclasses

BCSgrp: Boolean commutative semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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43. BCIdSgrp: Boolean commutative idempotent semigroups

Definition

A *commutative distributive idempotent semigroup* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot\rangle$ such that

$\langle A, \wedge, \vee, \cdot\rangle$ is an Boolean idempotent semigroup and

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$

$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 13$

Subclasses

BCIdMon: Boolean commutative idempotent monoids

BCIdRSgrp: Boolean commutative idempotent residuated semigroups

Superclasses

BCSgrp: Boolean commutative semigroups
BIdSgrp: Boolean idempotent semigroups
CDILMon: Commutative distributive integral lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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44. BCIdMon: Boolean commutative idempotent monoids

Definition
A *commutative distributive idempotent monoid* is a Boolean idempotent monoid $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1\rangle$ such that $\cdot$ is *commutative*: $x \cdot y = y \cdot x$
Formal Definition
$x \cdot (y \vee z) = x \cdot y \vee x \cdot z$
$(x \vee y) \cdot z = x \cdot z \vee y \cdot z$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot x = x$
$x \cdot y = y \cdot x$
Properties

Classtype	variety

Finite Members
$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 4,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 9$
Subclasses
BCIdRL: Boolean commutative idempotent residuated lattices
Superclasses
BCIdSgrp: Boolean commutative idempotent semigroups
BCMon: Boolean commutative monoids
BIdMon: Boolean idempotent monoids
CDIdLSgrp: Commutative distributive idempotent lattice-ordered semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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45. BCDivLat: Boolean commutative division lattices

Definition
A *commutative distributive division lattice* is a division lattice $\mathbf{P} = \langle P, \leq\rangle$ such that P is a Boolean algebra and
Formal Definition
$(x \wedge y)/z = x/z \wedge y/z$ and
$x \leq z/y \iff y \leq x\backslash z$
$x/y = y\backslash x$
Properties

Classtype	variety

Finite Members
$f_1 = 1,\ f_2 = 3,\ f_3 = 0\ f_4 = 70,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$
Subclasses
BCRMag: Boolean commutative residuated magmas
Superclasses
BDivLat: Boolean division lattices

CDIdLMon: Commutative distributive idempotent lattice-ordered monoids Cont|Po|J|M|L|D|To|B|U|Ind

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46. BCRMag: Boolean commutative residuated magmas

Definition

A *commutative distributive residuated magma* is a Boolean residuated magma such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 36,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCInMag: Boolean commutative involutive magmas

BCRSgrp: Boolean commutative residuated semigroups

Superclasses

BCDivLat: Boolean commutative division lattices

BRMag: Boolean residuated magmas

CDDivLat: Commutative distributive division lattices Cont|Po|J|M|L|D|To|B|U|Ind

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47. BCRSgrp: Boolean commutative residuated semigroups

Definition

A *commutative distributive residuated semigroup* is a Boolean residuated semigroup $\mathbf{A} = \langle A, \le, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \le y \implies x \cdot z \le y \cdot z$

$x \le y \implies z \cdot x \le z \cdot y$

$x \cdot y \le z \iff y \le x \backslash z$

$x \cdot y \le z \iff x \le z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 16,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIdRSgrp: Boolean commutative idempotent residuated semigroups

BCInSgrp: Boolean commutative involutive semigroups

BCRL: Boolean commutative residuated lattices

Superclasses

BCRMag: Boolean commutative residuated magmas

BCSgrp: Boolean commutative semigroups

BRSgrp: Boolean residuated semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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48. BCRL: Boolean commutative residuated lattices

Definition

A *commutative distributive residuated lattice* is a Boolean residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ such that $\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0$

Subclasses

BCIRL: Boolean commutative integral residuated lattices

BCIdRL: Boolean commutative idempotent residuated lattices

BCInFL: Boolean commutative involutive FL-algebras

Superclasses

BCMon: Boolean commutative monoids

BCRSgrp: Boolean commutative residuated semigroups

BRL: Boolean residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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49. BCIRL: Boolean commutative integral residuated lattices

Definition

A *Boolean residuated integral monoid* is a Boolean residuated monoid $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that x is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$
$x \leq y \implies z \cdot x \leq z \cdot y$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \leq 1$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIInFL: Boolean commutative integral involutive FL-algebras

Superclasses

BCIMon: Boolean commutative integral monoids

BCRL: Boolean commutative residuated lattices

BIRL: Boolean integral residuated lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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50. BCIdRSgrp: Boolean commutative idempotent residuated semigroups

Definition

A *commutative idempotent residuated semigroup* is an Boolean idempotent residuated semigroup $\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$.

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$x \cdot y \leq z \iff y \leq x \backslash z$

$x \cdot y \leq z \iff x \leq z / y$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 3,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCIdRL: Boolean commutative idempotent residuated lattices

Superclasses

BCIdSgrp: Boolean commutative idempotent semigroups

BCRSgrp: Boolean commutative residuated semigroups

BIdRSgrp: Boolean idempotent residuated semigroups

GödA: Gödel algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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51. BCIdRL: Boolean commutative idempotent residuated lattices

Definition

A *commutative idempotent residuated lattice* is an idmpotent residuated lattice $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ such that

$\cdot$ is *commutative*: $x \cdot y = y \cdot x$

Formal Definition

$x \leq y \implies x \cdot z \leq y \cdot z$

$x \leq y \implies z \cdot x \leq z \cdot y$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y \leq z \iff y \leq x \backslash z$
$x \cdot y \leq z \iff x \leq z / y$
$x \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 2,\ f_5 = 0,\ f_6 = 0$

Subclasses

Superclasses

BCIdMon: Boolean commutative idempotent monoids
BCIdRSgrp: Boolean commutative idempotent residuated semigroups
BCRL: Boolean commutative residuated lattices
BIdRL: Boolean idempotent residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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52. BCInMag: Boolean commutative involutive magmas

Definition

A *commutative distributive involutive magma* is a inpo-magma $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 20,\ f_5 = 0$

Subclasses

BCInSgrp: Boolean commutative involutive semigroups

Superclasses

BCRMag: Boolean commutative residuated magmas
BCyInMag: Boolean cyclic involutive magmas
CDIdRL: Commutative distributive idempotent residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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53. BCInSgrp: Boolean commutative involutive semigroups

Definition

A *commutative distributive involutive semigroup* is a inpo-semigroup $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 0,\ f_4 = 15,\ f_5 = 0,\ f_6 = 0$

Subclasses

BCInFL: Boolean commutative involutive FL-algebras

Superclasses

BCInMag: Boolean commutative involutive magmas
BCRSgrp: Boolean commutative residuated semigroups
BCyInSgrp: Boolean cyclic involutive semigroups
CIdRSlMon: Commutative idempotent residuated semilinear monoids Cont|Po|J|M|L|D|To|B|U|Ind

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54. BCInFL: Boolean commutative involutive FL-algebras

Definition

A *commutative distributive involutive FL-algebra* is an inpo-monoid $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot y = y \cdot x$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 5,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0$

Subclasses

BCIInFL: Boolean commutative integral involutive FL-algebras

Superclasses

BCInSgrp: Boolean commutative involutive semigroups
BCRL: Boolean commutative residuated lattices
BCyInFL: Boolean cyclic involutive FL-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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55. BCIInFL: Boolean commutative integral involutive FL-algebras

Definition

A *commutative distributive integral involutive FL-algebra* is an in-porim $\mathbf{A} = \langle A, \leq, \cdot, \sim, - \rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$--x = x$
$x \cdot y \leq z \iff y \leq -(-z \cdot x)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$
$x \cdot 1 = x$
$x \leq 1$

Properties

Classtype	variety

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1,\ f_9 = 0$

Subclasses

TrivA: Trivial algebras

Superclasses

BCIRL: Boolean commutative integral residuated lattices
BCInFL: Boolean commutative involutive FL-algebras
BCyIInFL: Boolean cyclic involutive integral monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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56. $\mathbf{CA}_2$: Cylindric algebras of dimension 2

Definition

A *cylindric algebra* of dimension $\alpha = 2$ is a Boolean algebra with operators $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, -, c_i, d_{ij} : i, j < \alpha\rangle$ such that for all $i, j < \alpha$

the c_i are increasing: $x \leq c_i x$
the c_i semi-distribute over $\wedge$: $c_i(x \wedge c_i y) = c_i x \wedge c_i y$
the c_i commute: $c_i c_j x = c_j c_i x$
the diagonals d_{ii} equal the top element: $d_{ii} = 1$
$d_{ij} = c_k(d_{ik} \wedge d_{kj})$ for $k \neq i, j$
$c_i(d_{ij} \wedge x) \wedge c_i(d_{ij} \wedge -x) = 0$ for $i \neq j$

Properties

Classtype	Variety
Equational theory	Undecidable for $\alpha \geq 3$, decidable otherwise
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes

Finite Members

Subclasses

TrivA: Trivial algebras

Superclasses

BUn: Boolean unars

Cont|Po|J|M|L|D|To|B|U|Ind

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57. SeqA: Sequential algebras

Definition

A *sequential algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \circ, e, \triangleright, \triangleleft\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra
$\langle A, \circ, e\rangle$ is a monoid

$\triangleright$ is the *right-conjugate* of $\circ$: $(x \circ y) \wedge z = 0 \iff (x \triangleright z) \wedge y = 0$
$\triangleleft$ is the *left-conjugate* of $\circ$: $(x \circ y) \wedge z = 0 \iff (z \triangleleft y) \wedge x = 0$
$\triangleright, \triangleleft$ are *balanced*: $x \triangleright e = e \triangleleft x$
$\circ$ is *euclidean*: $x \cdot (y \triangleleft z) \leq (x \cdot y) \triangleleft z$

Properties

Classtype	Variety
Equational theory	Undecidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Discriminator variety	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

Subclasses

RA: Relation algebras

Superclasses

BIRL: Boolean integral residuated lattices Cont|Po|J|M|L|D|To|B|U|Ind

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58. NA: Nonassociative relation algebras

Definition

A *nonassociative relation algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \circ, ^{\smile}, e\rangle$ such that
$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra
e is an *identity* for $\circ$: $x \circ e = x$, $e \circ x = x$
$\circ$ is *join-preserving*: $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$
$^{\smile}$ is an *involution*: $x^{\smile\smile} = x$, $(x \circ y)^{\smile} z = y^{\smile} \circ x^{\smile}$
$^{\smile}$ is *join-preserving*: $(x \vee y)^{\smile} z = x^{\smile} \vee y^{\smile}$
$\circ$ is residuated: $x^{\smile} \circ (\neg(x \circ y)) \leq \neg y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Discriminator variety	No

Finite Members

Subclasses

RA: Relation algebras

Superclasses

BRMag: Boolean residuated magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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59. RA: Relation algebras

Definition

A *relation algebra* is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, \circ, ^{\smile}, e\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra

$\langle A, \circ, e\rangle$ is a monoid

$\circ$ is *join-preserving*: $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$

$^{\smile}$ is an *involution*: $x^{\smile\smile} = x$, $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$

$^{\smile}$ is *join-preserving*: $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$

$\circ$ is residuated: $x^{\smile} \circ (\neg(x \circ y)) \leq \neg y$

Examples

Example 1: $\langle \mathcal{P}(U^2), \cup, \emptyset, \cap, U^2, -, \circ, ^{\smile}, id_U\rangle$ the full relation algebra of binary relations on a set U.

Example 2: $\langle \mathcal{P}(G), \cup, \emptyset, \cap, G, -, \circ, ^{\smile}, \{e\}\rangle$ the group relation algebra of a group $\langle G, *, ^{-1}, e\rangle$, where $X \circ Y = \{x * y : x \in X, y \in Y\}$ and $X^{\smile} = \{x^{-1} : x \in X\}$.

Properties

Classtype	Variety
Equational theory	Undecidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Discriminator variety	Yes
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 3,\ f_5 = 0,\ f_6 = 0$

Subclasses

IRA: Integral relation algebras

Superclasses

NA: Nonassociative relation algebras

SeqA: Sequential algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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60. IRA: Integral relation algebras

Definition

An *integral relation algebra* is a relation algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1,', \circ, ^{\smile}, e\rangle$ in which *the identity element* e *is* 0 *or an atom*: $e = x \vee y \implies x = 0$ or $y = 0$

Examples

For any group $\mathbf{G} = \langle G, *, ^{-1}, e\rangle$, construct the integral relation algebra $\mathcal{R}(G) = \langle \mathcal{P}(G), \cup, \emptyset, \cap, G,', \circ, ^{\smile}, \{e\}\rangle$, where $X \circ Y = \{x * y : x \in X, y \in Y\}$ and $X^{\smile} = \{x^{-1} : x \in X\}$ for $X, Y \subseteq G$.

Basic Results

Every nontrivial integral relation algebra is simple.

Every simple commutative relation algebra is integral.

Every group relation algebra is integral.

Properties

Classtype	Universal
Equational theory	Undecidable
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	No
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 2,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 10,\ f_{16} = 102,\ f_{32} = 4412,\ f_{64} = 4886349$

For $n \neq 2^k$, the number of algebras is 0.

Subclasses

TrivA: Trivial algebras

Superclasses

RA: Relation algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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61. BRMod: Boolean modules over a relation algebra

Definition

A *Boolean module over a relation algebra* $\mathbf{R}$ is an algebra $\mathbf{A} = \langle A, \vee, 0, \wedge, 1, \neg, f_r\ (r \in R)\rangle$ such that

$\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra

f_r is *join-preserving*: $f_r(x \vee y) = f_r(x) \vee f_r(y)$

$f_{r\vee s}(x) = f_r(x) \vee f_s(x)$

$f_r(f_s(x)) = f_{r\circ s}(x)$

$f_{1'}$ is the identity map: $f_{1'}(x) = x$

$f_0(x) = 0$

$f_{r^\smile}(\neg(f_r(x))) \leq \neg x$

Remark: Since f_r is order-preserving, the last identity is equivalent to the condition that $f_{r^\smile}$ and f_r are conjugate operators. It follows that f_r is *normal*: $f_r(0) = 0$.

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

Subclasses

TrivA: Trivial algebras

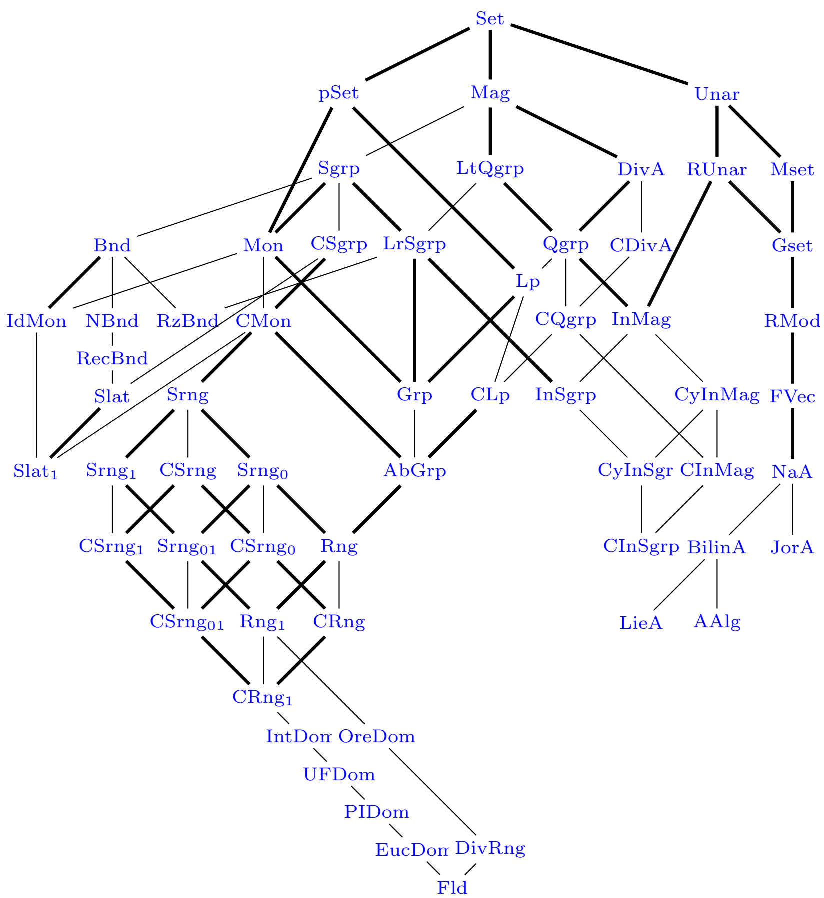
Superclasses

BUn: Boolean unars

Cont|Po|J|M|L|D|To|B|U|Ind

CHAPTER 9

Unordered algebras



1. Set: The category of sets

Definition

A *set* is an algebra $\mathbf{A} = \langle A \rangle$ with no operations or relations defined on A.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Locally finite	Yes
Residual size	2
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_n = 1$ for all n.

Subclasses

Mag: Magmas

Unar: Unary Algebras

pSet: The category of pointed sets

Superclasses

Pos: Partially ordered sets

Cont|Po|J|M|L|D|To|B|U|Ind

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2. pSet: The category of pointed sets

Definition

A *pointed set* is an algebra $\mathbf{A} = \langle A, c\rangle$ with a constant operation c.

This category can also be considered the category of sets with partial functions as morphisms. All elements that map to c are considered undefined.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Locally finite	Yes
Residual size	2
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_n = 1$ for all n.

Subclasses

Mon: Monoids

Superclasses

Set: The category of sets

pPos: Pointed posets

Cont|Po|J|M|L|D|To|B|U|Ind

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3. Unar: Unary Algebras

Definition

A *unar* is an algebra $\langle A, f\rangle$ such that f is a unary operation on A.

Examples

Example 1: The free unary algebra on one generator is isomorphic to the natural numbers $\mathbb{N}$. The number 0 is the generator x, and the operation f is the successor function, i.e., $f(n) = n + 1$.

The free unary algebra on X generators is a union of $|X|$ disjoint copies of the one-generated free algebra.

Basic Results

Monounary algebras are equivalent to directed graphs in which every vertex has exactly one outgoing edge.

One-generated monounary algebras are either isomorphic to the free one-generated algebra or they are finite and contain a path of length l from the generator to a cycle of length k (where $l \geq 0$ and $k \geq 1$).

The variety of monounary algebras has countably many subvarieties, each determined by an equation of the form $f^m(x) = f^n(x)$.

Let $j > k \geq 0$ and $m > n \geq 0$. Then $\mathrm{Mod}(f^j(x) = f^k(x) \subseteq \mathrm{Mod}(f^m(x) = f^n(x)$ if and only if $k \leq n$ and $(j - k)|(m - n)$.

Hence the lattice of nontrivial subvarieties of monounary algebras is isomorphic to $(\mathbb{N}, \leq) \times (\mathbb{N}, |)$, which is itself isomorphic to the lattice of divisibility of the natural numbers. The variety $\mathrm{Mod}(x = y)$ of trivial subvarieties is the unique element below the variety $\mathrm{Mod}(f(x) = x)$ (which is term-equivalent to the variety of sets).

Properties

Classtype	Variety
Equational theory	Undecidable if $I| > 2$
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

Depends on I

Subclasses

Mset: M-sets

Superclasses

PoUn: Partially ordered unars

Set: The category of sets

Cont|Po|J|M|L|D|To|B|U|Ind

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4. AAlg: Associative algebras

Definition

An *associative algebra* is a nonassociative algebra $\mathbf{A} = \langle A, +, -, 0, \cdot, s_r\ (r \in F)\rangle$ where $\mathbf{F}$ is a field such that $\cdot$ is associative: $(xy)z = x(yz)$

Properties

Finite Members

Subclasses

Superclasses

BilinA: Bilinear algebras Cont|Po|J|M|L|D|To|B|U|Ind

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5. BCI: BCI-algebras

Formal Definition

A *BCI-algebra* is an algebra $\langle A, \cdot, 0\rangle$ of type $\langle 2, 0\rangle$ such that

(1): $((x \cdot y) \cdot (x \cdot z)) \cdot (z \cdot y) = 0$

(2): $(x \cdot (x \cdot y)) \cdot y = 0$

(3): $x \cdot x = 0$

(4): $x \cdot y = 0$ and $y \cdot x = 0 \implies x = y$

(5): $x \cdot 0 = 0 \implies x = 0$

Properties

Classtype	Quasivariety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 22,\ f_5 = 118,\ f_6 = 974$

Subclasses

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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6. RtQgrp: Right quasigroups

Formal Definition

A *right quasigroup* is an algebra $\mathbf{A} = \langle A, \cdot, /\rangle$ such that

$(y/x) \cdot x = y$

$(x \cdot y)/y = x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 44,\ f_4 = 14022$

See also https://oeis.org/A193623

Subclasses

Qgrp: Quasigroups

RtLp:

Superclasses

Mag: Magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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7. Qgrp: Quasigroups

Formal Definition

A *quasigroup* is an algebra $\langle A, \cdot, \backslash, / \rangle$ of type $\langle 2, 2, 2 \rangle$ such that

$(y/x) \cdot x = y$

$x \cdot (x \backslash y) = y$

$(x \cdot y)/y = x$

$x \backslash (x \cdot y) = y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 5,\ f_4 = 35,\ f_5 = 1411$

Subclasses

Lp: Loops

MouQgrp: Moufang quasigroups

Superclasses

RtQgrp: Right quasigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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8. MouQgrp: Moufang quasigroups

Definition

A *Moufang quasigroup* is a quasigroup $\langle A, \cdot, \backslash, / \rangle$ such that

$\cdot$ satisfies the Moufang law: $ye = y \implies ((xy)z)x = x(y((ez)x))$

Formal Definition

$(y/x) \cdot x = y$

$x \cdot (x \backslash y) = y$

$(x \cdot y)/y = x$

$x \backslash (x \cdot y) = y$

$y \cdot 1 = y \implies ((x \cdot y) \cdot z) \cdot x = x \cdot (y \cdot ((1 \cdot z) \cdot x))$

Properties

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 5,\ f_4 = 29,\ f_5 = 1351$

Subclasses

MouLp: Moufang loops

Superclasses

Qgrp: Quasigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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9. Lp: Loops

Definition

A *loop* is a quasigroup $\langle A, \cdot, \backslash, /, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that

1 is an identity for $\cdot$: $x \cdot 1 = x$, $1 \cdot x = x$

Formal Definition

$(y/x) \cdot x = y$

$x \cdot (x\backslash y) = y$

$(x \cdot y)/y = x$

$x\backslash(x \cdot y) = y$

$x \cdot 1 = x$

$1 \cdot x = x$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 6,\ f_6 = 109,\ f_7 = 23746,\ f_8 = 106228849,\ f_9 = 9365022303540,\ f_{10} = 20890436195945769617,\ f_{11} = 1478157455158044452849321016$

Subclasses

LNeofld: Left neofields

MouLp: Moufang loops

Superclasses

Qgrp: Quasigroups

RtLp:

Cont|Po|J|M|L|D|To|B|U|Ind

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10. MouLp: Moufang loops

Definition

A *Moufang loop* is a loop $\mathbf{A} = \langle A, \cdot, \backslash, /, e\rangle$ such that

$((xy)z)x = x(y(zx))$, $y(x(yz)) = ((yx)y)z$, $(yx)(zy) = (y(xz))y$

Formal Definition

$(y/x) \cdot x = y$
$x \cdot (x\backslash y) = y$
$(x \cdot y)/y = x$
$x\backslash(x \cdot y) = y$
$x \cdot 1 = x$
$1 \cdot x = x$
$((x \cdot y) \cdot z) \cdot x = x \cdot (y \cdot (z \cdot x))$
$y \cdot (x \cdot (y \cdot z)) = ((y \cdot x) \cdot y) \cdot z$
$(y \cdot x) \cdot (z \cdot y) = (y \cdot (x \cdot z)) \cdot y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 5,\ f_9 = 2,\ f_{10} = 2,\ f_{11} = 1$

Subclasses

Grp: Groups

Superclasses

Lp: Loops

MouQgrp: Moufang quasigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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11. Shell: Shells

Formal Definition

A *shell* is an algebra $\mathbf{S} = \langle S, +, 0, \cdot, 1\rangle$ of type $\langle 2, 0, 2, 0\rangle$ such that

0 is an identity for $+$: $0 + x = x$, $x + 0 = x$

1 is an identity for $\cdot$: $1 \cdot x = x$, $x \cdot 1 = x$

0 is a zero for $\cdot$: $0 \cdot x = 0$, $x \cdot 0 = 0$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 243$

Subclasses

Srng_{01}: Semirings with identity and zero

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

12. Mag: Magmas

Definition

A *magma* is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ where $\cdot$ is any binary operation on A.

Examples

Example 1: $\langle \mathbb{N}, \hat{} \rangle$ is the exponentiation magma of the natural numbers, where $0\hat{}0 = 1$. It is not associative nor commutative, and does not have a (two-sided) identity.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1$, $f_2 = 10$, $f_3 = 3330$, $f_4 = 178981952$, $f_5 = 2483527537094825$, $f_6 = 14325590003318891522275680$

See also https://oeis.org/A001329

Subclasses

BCI: BCI-algebras

CnjMag: Conjugative magmas

Dtoid: Directoids

MedMag: Medial magmas

OrdA: Order algebras

Qnd: Quandles

QtMag: Quasitrivial magmas

RtQgrp: Right quasigroups

Sgrp: Semigroups

Shell: Shells

Superclasses

Set: The category of sets Cont|Po|J|M|L|D|To|B|U|Ind

13. Bnd: Bands

Definition

A *band* is a semigroup $\langle B, \cdot \rangle$ such that

$\cdot$ is idempotent: $x \cdot x = x$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Locally finite	Yes
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 10,\ f_4 = 46,\ f_5 = 251,\ f_6 = 1682,\ f_7 = 13213$

Subclasses

NBnd: Normal bands

Superclasses

OrdA: Order algebras

RegSgrp: Regular semigroups

Sgrp: Semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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14. NBnd: Normal bands

Definition

A *normal band* is a band $\mathbf{B} = \langle B, \cdot \rangle$ such that

$\cdot$ is normal: $x \cdot y \cdot z \cdot x = x \cdot z \cdot y \cdot x$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y \cdot z \cdot x = x \cdot z \cdot y \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Locally finite	Yes

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 8\ f_4 = 30,\ f_5 = 114,\ f_6 = 536$

Subclasses

RecBnd: Rectangular bands

Superclasses

Bnd: Bands

Cont|Po|J|M|L|D|To|B|U|Ind

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15. RecBnd: Rectangular bands

Definition

A *rectangular band* is a band $\mathbf{B} = \langle B, \cdot \rangle$ such that

$\cdot$ is rectangular: $x \cdot y \cdot x = x$.

Definition

A *rectangular band* is a band $\mathbf{B} = \langle B, \cdot \rangle$ such that

$x \cdot y \cdot z = x \cdot z$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x = x$

$x \cdot y \cdot x = x$

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Locally finite	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 2,\ f_4 = 3,\ f_5 = 2,\ f_6 = 4,\ f_7 = 2,\ f_8 = 4,\ f_9 = 3,\ f_{10} = 4$

Subclasses

Superclasses

NBnd: Normal bands

SkLat: Skew lattices

Cont|Po|J|M|L|D|To|B|U|Ind

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16. SkLat: Skew lattices

Definition

A *skew lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee \rangle$ such that

$\langle A, \wedge \rangle$ is a band,

$\langle A, \vee \rangle$ is a band,

and the following absorption laws hold: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$, $(x \vee y) \wedge y = y = (x \wedge y) \vee y$.

Formal Definition

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$

$x \wedge x = x$

$(x \vee y) \vee z = x \vee (y \vee z)$

$x \vee x = x$

$x \wedge (x \vee y) = x$

$x \vee (x \wedge y) = x$

$(x \vee y) \wedge y = y$

$(x \wedge y) \vee y = y$

Properties

Classtype	Variety

Finite Members

Subclasses

Lat: Lattices

RecBnd: Rectangular bands

Superclasses

Cont|Po|J|M|L|D|To|B|U|Ind

17. Sgrp: Semigroups

Formal Definition

A *semigroup* is an algebra $\langle S, \cdot\rangle$, where $\cdot$ is an infix binary operation, called the *semigroup product*, such that $\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Examples

Example 1: $\langle X^X, \circ\rangle$, the collection of functions on a sets X, with composition.

Example 2: $\langle \Sigma^+, \cdot\rangle$, the collection of nonempty strings over Σ, with concatenation.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1$, $f_2 = 5$, $f_3 = 24$, $f_4 = 188$, $f_5 = 1915$, $f_6 = 28634$, $f_7 = 1627672$, $f_8 = 3684030417$, $f_9 = 105978177936292$

See also https://oeis.org/A027851

Subclasses

Bnd: Bands

CSgrp: Commutative semigroups

LtCanSgrp: Left cancellative semigroups

Mon: Monoids

RegSgrp: Regular semigroups

Sgrp_0: Semigroups with zero

Superclasses

Mag: Magmas

Cont|Po|J|M|L|D|To|B|U|Ind

18. Sgrp_0: Semigroups with zero

Definition

A *semigroup with zero* is a semigroup $\langle S, \cdot, 0\rangle$ of type $\langle 2, 0\rangle$ such that

0 is a zero for $\cdot$: $x \cdot 0 = 0$, $0 \cdot x = 0$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 0 = 0$

$0 \cdot x = 0$

Properties

Classtype	Variety
Equational theory	Decidable in PTIME
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 12,\ f_4 = 90,\ f_5 = 960$

Subclasses

Srng_0: Semirings with zero

Superclasses

Sgrp: Semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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19. RegSgrp: Regular semigroups

Definition

An element x of a semigroup S is said to be *regular* if exists y in S such that $xyx = x$.

Definition

A *regular semigroup* is a semigroup $\mathbf{S} = \langle S, \cdot \rangle$ such that each element is regular.

Definition

A *regular semigroup* is an algebra $\mathbf{S} = \langle S, \cdot \rangle$, where $\cdot$ is an infix binary operation, called the *semigroup product*, such that

$\cdot$ is associative: $(xy)z = x(yz)$

each element is *regular*: $\exists y(xyx = x)$

Definition

We say that y is an *inverse* of an element x in a semigroup S if $x = xyx$ and $y = yxy$.

Examples

Example 1: $\langle T_X, \circ \rangle$, the *full transformation semigroup* of functions on X, with composition.

$\langle End(V), \circ \rangle$, the *endomorphism monoid* of a vector space V, with composition.

Basic Results

If x is a regular element of a semigroup (say $x = xyx$), then x has an inverse, namely yxy, since $x = x(yxy)x$ and $yxy = (yxy)x(yxy)$.

Properties

Classtype	First-order
Locally finite	No
Congruence distributive	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 9,\ f_4 = 42,\ f_5 = 206,\ f_6 = 1352,\ f_7 = 10168,\ f_8 = 91073,\ f_9 = 925044$

(the opposite of a semigroup S is identified with S in the table above, see https://oeis.org/A001427)

Subclasses

Bnd: Bands

InvSgrp: Inverse semigroups

Superclasses

Sgrp: Semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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20. InvSgrp: Inverse semigroups

Definition

An *inverse semigroup* is an algebra $\mathbf{S} = \langle S, \cdot, ^{-1} \rangle$ such that

$\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$^{-1}$ is an inverse: $xx^{-1}x = x$ and $(x^{-1})^{-1} = x$

idempotents commute: $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x^{-1} \cdot x = x$

$(x^{-1})^{-1} = x$

$x \cdot x^{-1} \cdot y \cdot y^{-1} = y \cdot y^{-1} \cdot x \cdot x^{-1}$

Examples

Example 1: $\langle I_X, \circ, ^{-1} \rangle$, the *symmetric inverse semigroup* of all one-to-one partial functions on a set X, with composition and function inverse. Every inverse semigroup can be embedded in a symmetric inverse semigroup.

Basic Results

$x * x = x \implies \exists y\ x = y * y^{-1}$

$\forall x \exists y\ xx^{-1} = y^{-1}y$

Properties

Classtype	Variety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1$, $f_2 = 2$, $f_3 = 5$, $f_4 = 16$, $f_5 = 52$, $f_6 = 208$, $f_7 = 911$, $f_8 = 4637$, $f_9 = 26422$, $f_{10} = 169163$, $f_{11} = 1198651$, $f_{12} = 9324047$, $f_{13} = 78860687$, $f_{14} = 719606005$, $f_{15} = 7035514642$

https://oeis.org/A001428

Subclasses

CInvSgrp: Commutative inverse semigroups

CliffSgrp: Clifford semigroups

Superclasses

RegSgrp: Regular semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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21. Mon: Monoids

Definition

A *monoid* is a semigroup $\langle M, \cdot, 1\rangle$, such that

1 is an identity for $\cdot$: $1 \cdot x = x$, $x \cdot 1 = x$.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

Examples

Example 1: $\langle X^X, \circ, id_X\rangle$, the collection of functions on a sets X, with composition, and identity map.

Example 2: $\langle M(V)_n, \cdot, I_n\rangle$, the collection of $n \times n$ matrices over a vector space V, with matrix multiplication and identity matrix.

Example 3: $\langle \Sigma^*, \cdot, \lambda\rangle$, the collection of strings over a set Σ, with concatenation and the empty string. This is the free monoid generated by Σ.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Undecidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 35,\ f_5 = 228,\ f_6 = 2237,\ f_7 = 31559$

Subclasses

CMon: Commutative monoids

LtCanMon:

Superclasses

Sgrp: Semigroups

pSet: The category of pointed sets

Cont|Po|J|M|L|D|To|B|U|Ind

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22. CanSgrp: Cancellative semigroups

Definition

A *cancellative semigroup* is a semigroup $\mathbf{S} = \langle S, \cdot\rangle$ such that

$\cdot$ is left cancellative: $z \cdot x = z \cdot y \implies x = y$

$\cdot$ is right cancellative: $x \cdot z = y \cdot z \implies x = y$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$z \cdot x = z \cdot y \implies x = y$

$x \cdot z = y \cdot z \implies x = y$

Examples

Example 1: $\langle \mathbb{N}, + \rangle$, the natural numbers, with additition.

Properties

Classtype	Quasivariety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 5,\ f_9 = 2,\ f_{10} = 2,\ f_{11} = 1$

Subclasses

CanCSgrp: Cancellative commutative semigroups

CanMon: Cancellative monoids

Superclasses

LtCanSgrp: Left cancellative semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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23. CanMon: Cancellative monoids

Definition

A *cancellative monoid* is a monoid $\mathbf{M} = \langle M, \cdot, e \rangle$ such that

$\cdot$ is left cancellative: $z \cdot x = z \cdot y \implies x = y$

$\cdot$ is right cancellative: $x \cdot z = y \cdot z \implies x = y$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$z \cdot x = z \cdot y \implies x = y$

$x \cdot z = y \cdot z \implies x = y$

Examples

Example 1: $\langle \mathbb{N}, +, 0 \rangle$, the natural numbers, with addition and zero.

Basic Results

All free monoids are cancellative.

All finite (left or right) cancellative monoids are reducts of groups.

Properties

Classtype	Quasivariety
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 2,\ f_7 = 1,\ f_8 = 5,\ f_9 = 2,\ f_{10} = 2,\ f_{11} = 1$

Subclasses

CanCMon: Cancellative commutative monoids

Grp: Groups

Superclasses

CanSgrp: Cancellative semigroups

LtCanMon:

Cont|Po|J|M|L|D|To|B|U|Ind

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24. Grp: Groups

Definition

A *group* is an algebra $\langle G, \cdot, ^{-1}, 1\rangle$, where $^{-1}$ is a postfix unary operation, called the *group inverse*, such that $\langle G, \cdot, 1\rangle$ is a monoid and
$^{-1}$ gives a right-inverse: $x \cdot x^{-1} = 1$.
Remark: It follows that $^{-1}$ gives a left inverse: $x^{-1}x = 1$. Also, it suffices to assume $\cdot$ has a right identity $x1 = x$, then $1x = x$ follows as well.

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot x^{-1} = 1$

Examples

Example 1: $\langle S_X, \circ, ^{-1}, id_X\rangle$, the collection of permutations of a sets X, with composition, inverse, and identity map.

Example 2: The general linear group $\langle GL_n(V), \cdot, ^{-1}, I_n\rangle$, the collection of invertible $n \times n$ matrices over a vector space V, with matrix multiplication, inverse, and identity matrix.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Undecidable
First-order theory	Undecidable
Congruence distributive	no ($\mathbb{Z}_2 \times \mathbb{Z}_2$)
Congruence modular	Yes
Congruence n-permutable	Yes, n=2, $p(x, y, z) = xy^{-1}z$ is a Mal'cev term
Congruence regular	Yes
Congruence uniform	Yes
Congruence types	1=permutational
Congruence extension property	no, consider a non-simple subgroup of a simple group
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes
Locally finite	No
Residual size	Unbounded

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 1$, $f_6 = 2$, $f_7 = 1$, $f_8 = 5$, $f_9 = 2$, $f_{10} = 2$, $f_{11} = 1$, $f_{12} = 5$, $f_{13} = 1$, $f_{14} = 2$, $f_{15} = 1$, $f_{16} = 14$, $f_{17} = 1$, $f_{18} = 5$

Information about small groups up to size 2000: http://www.tu-bs.de/ hubesche/small.html

Subclasses

AbGrp: Abelian groups

NRng: Near-rings

NlGrp: Nilpotent groups
pGrp: P-groups

Superclasses

CanMon: Cancellative monoids
CliffSgrp: Clifford semigroups
MouLp: Moufang loops

Cont|Po|J|M|L|D|To|B|U|Ind

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25. AbpGrp: Abelian p-groups

Definition

An *Abelian p-group* is a p-group $\langle A, +, -, 0\rangle$ such that
$\cdot$ is *commutative*: $x + y = y + x$

Properties

Classtype	higher-order
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

Subclasses

BGrp: Boolean groups

Superclasses

pGrp: P-groups

Cont|Po|J|M|L|D|To|B|U|Ind

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26. CMag: Commutative magmas

Definition

A *commutative magma* is a magma $\langle A, \cdot\rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$.

Examples

Example 1: $\langle \mathbb{N}, |\cdot|\rangle$ is the distance magma of the natural numbers, where the binary operation is $|x - y|$.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 129,\ f_4 = 43968,\ f_5 = 254429900,\ f_6 = 30468670170912$

See also https://oeis.org/A001425

Subclasses

CSgrp: Commutative semigroups

Superclasses

CnjMag: Conjugative magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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27. CSgrp: Commutative semigroups

Definition

A *commutative semigroup* is a semigroup $\langle S, \cdot\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot y = y \cdot x$

Examples

Example 1: $\langle \mathbb{N}, +\rangle$, the natural numbers, with addition.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Decidable
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 12,\ f_4 = 58,\ f_5 = 325,\ f_6 = 2143,\ f_7 = 17291$

Subclasses

CMon: Commutative monoids

CanCSgrp: Cancellative commutative semigroups

qMV: Quasi-MV-algebras

Superclasses

CMag: Commutative magmas

Sgrp: Semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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28. LtCanSgrp: Left cancellative semigroups

Definition

A *left cancellative semigroup* is a semigroup $\mathbf{S} = \langle S, \cdot\rangle$ such that

$\cdot$ is left cancellative: $z \cdot x = z \cdot y \implies x = y$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$z \cdot x = z \cdot y \implies x = y$

Examples

Example 1: $\langle \mathbb{N}, + \rangle$, the natural numbers, with additition.

Properties

Classtype	Quasivariety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 2,\ f_4 = 4,\ f_5 = 2,\ f_6 = 5,\ f_7 = 2,\ f_8 = 9$

Subclasses

CanSgrp: Cancellative semigroups

LtCanMon:

Superclasses

Sgrp: Semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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29. CanCSgrp: Cancellative commutative semigroups

Definition

A *cancellative commutative semigroup* is a commutative semigroup $\mathbf{S} = \langle S, \cdot \rangle$ such that

$\cdot$ is *cancellative*: $x \cdot z = y \cdot z \implies x = y$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot z = y \cdot z \implies x = y$

$x \cdot y = y \cdot x$

Examples

Example 1: $\langle \mathbb{N}, + \rangle$, the natural numbers, with additition.

Properties

Classtype	Quasivariety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1$

Subclasses

CanCMon: Cancellative commutative monoids

Superclasses

CSgrp: Commutative semigroups

CanSgrp: Cancellative semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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30. CInvSgrp: Commutative inverse semigroups

Definition

A *commutative inverse semigroup* is an inverse semigroup $\langle S, \cdot, ^{-1} \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot x^{-1} \cdot x = x$

$(x^{-1})^{-1} = x$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	Yes
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1$, $f_2 = 2$, $f_3 = 5$, $f_4 = 16$, $f_5 = 51$, $f_6 = 201$, $f_7 = 877$, $f_8 = 4443$, $f_9 = 25284$, $f_{10} = 161698$, $f_{11} = 1145508$, $f_{12} = 8910291$, $f_{13} = 75373563$, $f_{14} = 687950735$, $f_{15} = 6727985390$

Subclasses

AbGrp: Abelian groups

Superclasses

InvSgrp: Inverse semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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31. CMon: Commutative monoids

Definition

A *commutative monoid* is a monoid $\mathbf{M} = \langle M, \cdot, e \rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot y = y \cdot x$

Examples

Example 1: $\langle \mathbb{N}, +, 0\rangle$, the natural numbers, with addition and zero. The finitely generated free commutative monoids are direct products of this one.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 19,\ f_5 = 78,\ f_6 = 421,\ f_7 = 2637$

Subclasses

CanCMon: Cancellative commutative monoids

Srng: Semirings

Superclasses

CSgrp: Commutative semigroups

Mon: Monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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32. CanCMon: Cancellative commutative monoids

Definition

A *cancellative commutative monoid* is a cancellative monoid $\mathbf{M} = \langle M, \cdot, e\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot z = y \cdot z \implies x = y$

$x \cdot y = y \cdot x$

Examples

Example 1: $\langle \mathbb{N}, +, 0\rangle$, the natural numbers, with addition and zero.

Basic Results

All commutative free monoids are cancellative.

All finite commutative (left or right) cancellative monoids are reducts of abelian groups.

Properties

Classtype	Quasivariety
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 2,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1$

Subclasses

AbGrp: Abelian groups

Superclasses

CMon: Commutative monoids

CanCSgrp: Cancellative commutative semigroups

CanMon: Cancellative monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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33. AbGrp: Abelian groups

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0\rangle$, the integers, with addition, unary subtraction, and zero. The variety of abelian groups is generated by this algebra.

Example 2: $\mathbb{Z}_n = \langle \mathbb{Z}/n\mathbb{Z}, +_n, -_n, 0 + n\mathbb{Z}\rangle$, integers mod n.

Example 3: Any one-generated subgroup of a group.

Basic Results

The free abelian group on n generators is $\mathbb{Z}^n$.

Classification of finitely generated abelian groups: Every n-generated abelian group is isomorphic to a direct product of $\mathbb{Z}_{p_i^{k_i}}$ for $i = 1, \ldots, m$ and $n - m$ copies of $\mathbb{Z}$, where the p_i are (not necessarily distinct) primes and $m \geq 0$.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Decidable
First-order theory	Decidable Szmielew [1949]
Locally finite	No
Residual size	ω
Congruence distributive	no ($\mathbb{Z}_2 \times \mathbb{Z}_2$)
Congruence n-permutable	Yes, $n = 2$, $p(x, y, z) = x - y + z$
Congruence regular	Yes, congruences are determined by subalgebras
Congruence uniform	Yes
Congruence types	permutational
Congruence extension property	Yes, if $K \leq H \leq G$ then $K \leq G$
Definable principal congruences	No
Equationally def. pr. cong.	No
Amalgamation property	Yes
Strong amalgamation property	Yes

Finite Members

$f_1 = 1$, $f_2 = 1$, $f_3 = 1$, $f_4 = 2$, $f_5 = 1$, $f_6 = 1$, $f_7 = 1$, $f_8 = 3$, $f_9 = 2$, $f_{10} = 1$, $f_{11} = 1$, $f_{12} = 2$, $f_{13} = 1$, $f_{14} = 1$

See A000688

Subclasses

BGrp: Boolean groups

Rng: Rings

Superclasses

CInvSgrp: Commutative inverse semigroups

CanCMon: Cancellative commutative monoids

Grp: Groups
NlGrp: Nilpotent groups Cont|Po|J|M|L|D|To|B|U|Ind

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34. BGrp: Boolean groups

Definition

A *Boolean group* is a monoid $\mathbf{M} = \langle M, \cdot, e\rangle$ such that

every element has order 2: $x \cdot x = e$.

Examples

Example 1: $\langle\{0, 1\}, +, 0\rangle$, the two-element group with addition-mod-2. This algebra generates the variety of Boolean groups.

Properties

Classtype	Variety
Equational theory	Decidable in polynomial time
Quasiequational theory	Decidable
First-order theory	Decidable
Locally finite	Yes
Residual size	2
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 0,\ f_4 = 1,\ f_5 = 0,\ f_6 = 0,\ f_7 = 0,\ f_8 = 1$

Subclasses

TrivA: Trivial algebras

Superclasses

AbGrp: Abelian groups
AbpGrp: Abelian p-groups Cont|Po|J|M|L|D|To|B|U|Ind

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35. NFld: Near-fields

Definition

A *near-field* is a near-ring with identity $\mathbf{N} = \langle N, +, -, 0, \cdot, 1\rangle$ such that

$\mathbf{N}$ is non-trivial: $0 \neq 1$

every non-zero element has a multiplicative inverse: $x \neq 0 \implies \exists y(x \cdot y = 1)$

Remark: The inverse of x is unique, and is usually denoted by x^{-1}.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = 0$ and $x \cdot 0 = 0$.

Properties

Classtype	first-order
Locally finite	No
Residual size	Unbounded
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

Subclasses

Fld: Fields

Superclasses

Rng_1: Rings with identity

Cont|Po|J|M|L|D|To|B|U|Ind

36. NRng: Near-rings

Definition

A *near-ring* is an algebra $\langle N, +, -, 0, \cdot\rangle$ of type $\langle 2, 1, 0, 2\rangle$ such that
$\langle N, +, -, 0\rangle$ is a group
$\langle N, \cdot\rangle$ is a semigroup
$\cdot$ right-distributes over $+$: $(x + y) \cdot z = x \cdot z + y \cdot z$

Formal Definition

$(x + y) + z = x + (y + z)$
$x + 0 = x$
$x + (-x) = 0$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$(x + y) \cdot z = x \cdot z + y \cdot z$

Examples

Example 1: $\langle \mathbb{R}^{\mathbb{R}}, +, -, 0, \cdot\rangle$, the near-ring of functions on the real numbers with pointwise addition, subtraction, zero, and composition.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = 0$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 5,\ f_4 = 35,\ f_5 = 10,\ f_6 = 99,\ f_7 = 24,\ f_8 = 3856,\ f_9 = 486$

Subclasses

NRng_1: Near-rings with identity

Rng: Rings

Superclasses

Grp: Groups

Cont|Po|J|M|L|D|To|B|U|Ind

37. $\mathbf{NRng_1}$: Near-rings with identity

Definition

A *near-ring with identity* is an algebra $\mathbf{N} = \langle N, +, -, 0, \cdot, 1\rangle$ of type $\langle 2, 1, 0, 2, 0\rangle$ such that

$\langle N, +, -, 0, \cdot\rangle$ is a near-ring

1 is a *multiplicative identity*: $x \cdot 1 = x$ and $1 \cdot x = x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$x + (-x) = 0$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$(x + y) \cdot z = x \cdot z + y \cdot z$

Examples

Example 1: $\langle \mathbb{R}^{\mathbb{R}}, +, -, 0, \cdot, 1\rangle$, the near-ring of functions on the real numbers with pointwise addition, subtraction, zero, composition, and the identity function.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = 0$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 6,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1,\ f_8 = 53,\ f_9 = 11,\ f_{10} = 1$

Subclasses

Rng_1: Rings with identity

Superclasses

NRng: Near-rings Cont|Po|J|M|L|D|To|B|U|Ind

38. Neofld: Neofields

Definition

A *neofield* is an algebra $\mathbf{F} = \langle F, +, \backslash, /, 0, \cdot, 1, ^{-1}\rangle$ of type $\langle 2, 2, 2, 0, 2, 0, 1\rangle$ such that

$\langle F, +, \backslash, /, 0\rangle$ is a loop

$\langle F - \{0\}, \cdot, 1, ^{-1}\rangle$ is a group

$\cdot$ distributes over $+$: $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$

Properties

Finite Members

Subclasses

DivRng: Division rings

Superclasses

LNeofld: Left neofields

Cont|Po|J|M|L|D|To|B|U|Ind

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39. Srng: Semirings

Definition

A *semiring* is an algebra $\mathbf{S} = \langle S, +, \cdot \rangle$ of type $\langle 2, 2 \rangle$ such that

$\langle S, \cdot \rangle$ is a semigroup

$\langle S, + \rangle$ is a commutative semigroup

$\cdot$ distributes over $+$: $x \cdot (y + z) = x \cdot y + x \cdot z$, $(y + z) \cdot x = y \cdot x + z \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 10,\ f_3 = 132,\ f_4 = 2341$

Subclasses

CSrng: Commutative semirings

Srng$_0$: Semirings with zero

Srng$_1$: Semirings with identity

Superclasses

CMon: Commutative monoids

Cont|Po|J|M|L|D|To|B|U|Ind

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40. Srng$_1$: Semirings with identity

Definition

A *semiring with identity* is an algebra $\mathbf{S} = \langle S, +, \cdot, 1 \rangle$ of type $\langle 2, 2, 0 \rangle$ such that

$\langle S, + \rangle$ is a commutative semigroup

$\langle S, \cdot, 1 \rangle$ is a monoid

$\cdot$ distributes over $+$: $x \cdot (y + z) = x \cdot y + x \cdot z$, $(y + z) \cdot x = y \cdot x + z \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 22,\ f_4 = 169,\ f_5 = 1819$

Subclasses

Sfld: Semifields

Srng_{01}: Semirings with identity and zero

Superclasses

Srng: Semirings Cont|Po|J|M|L|D|To|B|U|Ind

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41. $\mathbf{Srng_0}$: Semirings with zero

Definition

A *semiring with zero* is an algebra $\mathbf{S} = \langle S, +, 0, \cdot \rangle$ of type $\langle 2, 0, 2 \rangle$ such that

$\langle S, +, 0 \rangle$ is a commutative monoid

$\langle S, \cdot \rangle$ is a semigroup

0 is a zero for $\cdot$: $0 \cdot x = 0,\ x \cdot 0 = 0$

$\cdot$ distributes over $+$: $x \cdot (y + z) = x \cdot y + x \cdot z,\ (y + z) \cdot x = y \cdot x + z \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$x + 0 = x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$0 \cdot x = 0$

$x \cdot 0 = 0$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(x + y) \cdot z = x \cdot z + y \cdot z$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 22,\ f_4 = 283$

Subclasses

IdSrng_0: Idempotent semirings with zero

Superclasses

Sgrp$_0$: Semigroups with zero

Srng: Semirings Cont|Po|J|M|L|D|To|B|U|Ind

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42. Srng$_{01}$: Semirings with identity and zero

Definition

A *semiring with identity and zero* is an algebra $\langle S,+,0,\cdot,1\rangle$ of type $\langle 2,0,2,0\rangle$ such that

$\langle S,+,0\rangle$ is a commutative monoid

$\langle S,\cdot,1\rangle$ is a monoid

0 is a zero for $\cdot$: $0\cdot x=0$, $x\cdot 0=0$

$\cdot$ distributes over $+$: $x\cdot(y+z)=x\cdot y+x\cdot z$, $(y+z)\cdot x=y\cdot x+z\cdot x$

Formal Definition

$(x+y)+z=x+(y+z)$

$x+y=y+x$

$x+0=x$

$(x\cdot y)\cdot z=x\cdot(y\cdot z)$

$x\cdot 1=x$

$1\cdot x=x$

$0\cdot x=0$

$x\cdot 0=0$

$x\cdot(y+z)=x\cdot y+x\cdot z$

$(x+y)\cdot z=x\cdot z+y\cdot z$

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No

Finite Members

$f_1=1,\ f_2=2,\ f_3=6,\ f_4=40,\ f_5=295,\ f_6=3246$

Subclasses

IdSrng$_{01}$: Idempotent semirings with identity and zero

Rng$_1$: Rings with identity

Superclasses

Shell: Shells

Srng$_1$: Semirings with identity Cont|Po|J|M|L|D|To|B|U|Ind

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43. Rng: Rings

Definition

A *ring* is an algebra $\mathbf{R}=\langle R,+,-,0,\cdot\rangle$ of type $\langle 2,1,0,2\rangle$ such that

$\langle R,+,-,0\rangle$ is an abelian group

$\langle R,\cdot\rangle$ is a semigroup

$\cdot$ distributes over $+$: $x\cdot(y+z)=x\cdot y+x\cdot z$, $(y+z)\cdot x=y\cdot x+z\cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot \rangle$, the ring of integers with addition, subtraction, zero, and multiplication.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = 0$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 2,\ f_4 = 11,\ f_5 = 2,\ f_6 = 4$

Subclasses

CRng: Commutative rings

Rng_1: Rings with identity

Superclasses

AbGrp: Abelian groups

NRng: Near-rings

Cont|Po|J|M|L|D|To|B|U|Ind

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44. $\mathbf{Rng_1}$: Rings with identity

Definition

A *ring with identity* is an algebra $\mathbf{R} = \langle R, +, -, 0, \cdot, 1 \rangle$ of type $\langle 2, 1, 0, 2, 0 \rangle$ such that

$\langle R, +, -, 0, \cdot \rangle$ is a ring

1 is an identity for $\cdot$: $x \cdot 1 = x$, $1 \cdot x = x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot, 1\rangle$, the ring of integers with addition, subtraction, zero, multiplication, and one.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = 0$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 4,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1,\ f_8 = 11,\ f_9 = 4,\ f_{10} = 1$

Subclasses

CRng_1: Commutative rings with identity

NFld: Near-fields

OreDom: Ore domains

RegRng: Regular rings

Superclasses

NRng_1: Near-rings with identity

Rng: Rings

Srng_{01}: Semirings with identity and zero

Cont|Po|J|M|L|D|To|B|U|Ind

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45. RegRng: Regular rings

Definition

A *regular ring* is a ring with identity $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ such that every element has a pseudo-inverse: $\forall x \exists y (x \cdot y \cdot x = x)$

Properties

Classtype	first-order
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

Subclasses

CRegRng: Commutative regular rings

DivRng: Division rings

Superclasses

Rng_1: Rings with identity

Cont|Po|J|M|L|D|To|B|U|Ind

46. CRegRng: Commutative regular rings

Definition

A *commutative regular ring* is a regular ring $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ such that $\cdot$ is commutative: $x \cdot y = y \cdot x$

Properties

Classtype	first-order
Locally finite	No
Residual size	Unbounded
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

Subclasses

Fld: Fields

Superclasses

RegRng: Regular rings Cont|Po|J|M|L|D|To|B|U|Ind

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47. CSrng: Commutative semirings

Definition

A *commutative semiring* is a semiring $\langle S, +, \cdot\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

$x \cdot y = y \cdot x$

Properties

Classtype	Variety

Finite Members

Subclasses

CSrng_0: Commutative semirings with zero

CSrng_1: Commutative semirings with identity

Superclasses

Srng: Semirings Cont|Po|J|M|L|D|To|B|U|Ind

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48. CSrng_1: Commutative semirings with identity

Definition

A *commutative semiring with identity* is a semiring with identity $\langle S, +, \cdot, 1\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$
$1 \cdot x = x$
$x \cdot (y + z) = x \cdot y + x \cdot z$
$(y + z) \cdot x = y \cdot x + z \cdot x$
$x \cdot y = y \cdot x$

Properties

Classtype	Variety

Finite Members

Subclasses

CSrng$_{01}$: Commutative semirings with identity and zero

Superclasses

CSrng: Commutative semirings

Cont|Po|J|M|L|D|To|B|U|Ind

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49. CSrng$_0$: Commutative semirings with zero

Definition

A *commutative semiring with zero* is a semiring with zero $\langle S, +, 0, \cdot\rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$
$x + y = y + x$
$x + 0 = x$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$0 \cdot x = 0$
$x \cdot 0 = 0$
$x \cdot (y + z) = x \cdot y + x \cdot z$
$(x + y) \cdot z = x \cdot z + y \cdot z$
$x \cdot y = y \cdot x$

Properties

Classtype	Variety

Finite Members

Subclasses

CSrng$_{01}$: Commutative semirings with identity and zero

Superclasses

CSrng: Commutative semirings

Cont|Po|J|M|L|D|To|B|U|Ind

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50. CSrng$_{01}$: Commutative semirings with identity and zero

Definition

A *commutative semiring with identity and zero* is a semiring with identity and zero $\langle S, +, 0, \cdot, 1\rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$
$x + y = y + x$
$x + 0 = x$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$
$1 \cdot x = x$
$0 \cdot x = 0$
$x \cdot 0 = 0$
$x \cdot (y + z) = x \cdot y + x \cdot z$
$(x + y) \cdot z = x \cdot z + y \cdot z$
$x \cdot y = y \cdot x$

Properties

Classtype	Variety

Finite Members

Subclasses

Superclasses

CSrng_0: Commutative semirings with zero
CSrng_1: Commutative semirings with identity Cont|Po|J|M|L|D|To|B|U|Ind

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51. CRng: Commutative rings

Definition

A *commutative ring* is a ring $\mathbf{R} = \langle R, +, -, 0, \cdot\rangle$ such that
$\cdot$ is commutative: $x \cdot y = y \cdot x$
Remark: $Idl(R) = \{allidealsofR\}$
I is an ideal if $a, b \in I \implies a + b \in I$
and $\forall r \in R\ (r \cdot I \subseteq I)$

Formal Definition

$(x + y) + z = x + (y + z)$
$x + 0 = x$
$-x + x = 0$
$x + y = y + x$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$
$x \cdot (y + z) = x \cdot y + x \cdot z$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot\rangle$, the ring of integers with addition, subtraction, zero, and multiplication.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = x$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 2,\ f_4 = 9,\ f_5 = 2,\ f_6 = 4$

Subclasses

CRng$_1$: Commutative rings with identity

Fld: Fields

Superclasses

Rng: Rings

Cont|Po|J|M|L|D|To|B|U|Ind

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52. CRng$_1$: Commutative rings with identity

Definition

A *commutative ring with identity* is a ring with identity $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ such that $\cdot$ is commutative: $x \cdot y = y \cdot x$

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$x \cdot y = y \cdot x$

$x \cdot (y + z) = x \cdot y + x \cdot z$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot, 1\rangle$, the ring of integers with addition, subtraction, zero, multiplication, and one.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = x$ and $x \cdot 0 = 0$.

Properties

Classtype	Variety
Equational theory	Decidable
First-order theory	Undecidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 4,\ f_5 = 1,\ f_6 = 1,\ f_7 = 1,\ f_8 = 10,\ f_9 = 4,\ f_{10} = 1$

Subclasses

BA: Boolean algebras

IntDom: Integral Domain

Superclasses

CRng: Commutative rings

Rng$_1$: Rings with identity

Cont|Po|J|M|L|D|To|B|U|Ind

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53. IntDom: Integral Domain

Definition

An *integral domain* is a commutative ring with identity $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ that has no zero divisors: $\forall x, y\ (x \cdot y = 0 \implies x = 0 \text{ or } y = 0)$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot, 1\rangle$, the ring of integers with addition, subtraction, zero, and multiplication is an integral domain.

Basic Results

Every finite integral domain is a field.

Properties

Classtype	Universal class
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0$

Subclasses

UFDom: Unique Factorization Domains

Superclasses

CRng_1: Commutative rings with identity

Cont|Po|J|M|L|D|To|B|U|Ind

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54. DivRng: Division rings

Definition

A *division ring* (also called *skew field*) is a ring with identity $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ such that

$\mathbf{R}$ is non-trivial: $0 \neq 1$

every non-zero element has a multiplicative inverse: $x \neq 0 \implies \exists y(x \cdot y = 1)$

Remark: The inverse of x is unique, and is usually denoted by x^{-1}.

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$

$1 \cdot x = x$

$x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

$0 \neq 1$

$x \neq 0 \implies x \cdot x^{-1} = 1$

Examples

Example 1: $\langle \mathcal{Q}, +, -, 0, \cdot, 1\rangle$, the division ring of quaternions with addition, subtraction, zero, multiplication, and one.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = x$ and $x \cdot 0 = 0$.

Every finite division ring is a field (i.e. $\cdot$ is commutative).

Properties

Classtype	first-order
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 3,\ f_5 = 5,\ f_6 = 0,\ f_7 = 7,\ f_8 = 4$

Subclasses

Fld: Fields

Superclasses

Neofld: Neofields

OreDom: Ore domains

RegRng: Regular rings

Cont|Po|J|M|L|D|To|B|U|Ind

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55. Sfld: Semifields

Definition

A *semifield* is a semiring with identity $\mathbf{S} = \langle S, +, \cdot, 1\rangle$ such that

$\langle S^*, \cdot, 1\rangle$ is a group, where $S^* = S - \{0\}$ if S has an absorbtive 0, and $S = S^*$ otherwise.

Basic Results

The only finite semifield that is not a field is the 2-element Boolean semifield: https://arxiv.org/pdf/1709.06923.pdf

Properties

Locally finite	No
Residual size	Unbounded
Congruence distributive	No

Finite Members

$f_1 = 1,\ f_2 = 2,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0$

Subclasses

Fld: Fields

Superclasses

Srng_1: Semirings with identity

Cont|Po|J|M|L|D|To|B|U|Ind

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56. Fld: Fields

Definition

A *field* is a commutative ring with identity $\mathbf{F} = \langle F, +, -, 0, \cdot, 1\rangle$ such that

$\mathbf{F}$ is non-trivial: $0 \neq 1$

every non-zero element has a multiplicative inverse: $x \neq 0 \implies \exists y(x \cdot y = 1)$

Remark: The inverse of x is unique, and is usually denoted by x^{-1}.

Formal Definition

$(x + y) + z = x + (y + z)$

$x + 0 = x$

$-x + x = 0$

$x + y = y + x$

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$x \cdot 1 = x$
$x \cdot y = y \cdot x$
$x \cdot (y + z) = x \cdot y + x \cdot z$
$0 \neq 1$
$x \neq 0 \implies x \cdot x^{-1} = 1$
$0^{-1} = 0$ (needed to avoid multiple isomorphic copies)

Examples

Example 1: $\langle \mathbb{Q}, +, -, 0, \cdot, 1 \rangle$, the field of rational numbers with addition, subtraction, zero, multiplication, and one.

Basic Results

0 is a zero for $\cdot$: $0 \cdot x = x$ and $x \cdot 0 = 0$.

Properties

Classtype	first-order
Locally finite	No
Residual size	Unbounded
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members

$f_1 = 0,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0,\ f_7 = 1,\ f_n = 1 \iff n = p^k$ for some $k > 0$ and prime p, i.e., n is a prime power.

Subclasses

Superclasses

CRegRng: Commutative regular rings

CRng: Commutative rings

DivRng: Division rings

EucDom: Euclidean Domains

NFld: Near-fields

Sfld: Semifields

Cont|Po|J|M|L|D|To|B|U|Ind

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57. CnjMag: Conjugative magmas

Definition

A *conjugative magma* is a magma $\mathbf{A} = \langle A, \cdot \rangle$ such that
$\cdot$ is conjugative: $\exists w,\ x \cdot w = y \iff \exists w,\ w \cdot x = y$.

Properties

Classtype	first-order
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No

Finite Members

$f_1 = 1,\ f_2 = 4,\ f_3 = 215$

Subclasses

CMag: Commutative magmas

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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58. Dtoid: Directoids

Definition

A *directoid* is an algebra $\mathbf{A} = \langle A, \cdot \rangle$, where $\cdot$ is an infix binary operation such that

$\cdot$ is idempotent: $x \cdot x = x$

Formal Definition

$(x \cdot y) \cdot x = x \cdot y$

$y \cdot (x \cdot y) = x \cdot y$

$x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$

$x \cdot x = x$

Basic Results

The relation $x \leq y \iff x \cdot y = x$ is a partial order.

Properties

Classtype	Variety
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence types	semilattice (5)
Equationally def. pr. cong.	No

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 7,\ f_5 = 61$

Subclasses

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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59. UFDom: Unique Factorization Domains

Definition

A *unique factorization domain* is an integral domain D such that

every element is a product of irreducibles: $\forall a \in D \exists p_1, ..., p_r \in D, n_1, ..., n_r \in \mathbb{N}$ such that $a = p_1^{n_1} \cdot_2^{n_2} ... p_r^{n_r}$ and p_i is irreducible for $i = 1, \ldots, r$

the product is unique up to associates: $\forall$ irreducibles p_i, q_j if $a = p_1^{n_1} \cdot p_2^{n_2} ... p_r^{n_r} = q_1^{m_1} \cdot q_2^{m_2} ... q_s^{m_s}$ then $r = s$ and each p_i is an associate of some q_j

Examples

Example 1: $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients.

Properties

Classtype	second-order

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0$

Subclasses

PIDom: Principal Ideal Domain

Superclasses

IntDom: Integral Domain Cont|Po|J|M|L|D|To|B|U|Ind

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60. OreDom: Ore domains

Definition

An *Ore domain* is a ring with identity $\mathbf{A} = \langle A, +, -, 0, \cdot, 1\rangle$ such that
$\cdot$ is *integral*: $xy = 0 \implies x = 0$ or $y = 0$
nonzero common multiples exist: $x \neq 0 \neq y \implies \exists u \exists v(xu = yv \neq 0)$ and $\exists u \exists v(ux = vy \neq 0)$

Properties

Finite Members

Subclasses

DivRng: Division rings

Superclasses

Rng_1: Rings with identity Cont|Po|J|M|L|D|To|B|U|Ind

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61. PIDom: Principal Ideal Domain

Definition

A *principal ideal domain* is an integral domain $\mathbf{R} = \langle R, +, -, 0, \cdot, 1\rangle$ in which
every ideal is principal: $\forall I \in Idl(R)\ \exists a \in R\ (I = aR)$
Ideals are defined for commutative rings

Examples

Example 1: $a + b\theta | a, b \in Z, \theta = \langle 1 + \langle -19\rangle^{1/2}\rangle/2$ is a Principal Ideal Domain that is not an Euclidean domain See Oscar Campoli's "A Principal Ideal Domain That Is Not a Euclidean Domain" in ¡i¿The American Mathematical Monthly¡/i¿ 95 (1988): 868-871

Properties

Classtype	Second-order

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0$

Subclasses

EucDom: Euclidean Domains

Superclasses

UFDom: Unique Factorization Domains Cont|Po|J|M|L|D|To|B|U|Ind

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62. EucDom: Euclidean Domains

Definition

A *Euclidean domain* is an integral domain $\langle D, +, -, 0, \cdot, 1\rangle$ together with a function $d : D \setminus \{0\} \to \mathbf{N}$ such that
$\forall a, b\ (a \neq 0,\ b \neq 0 \implies d(a) \leq d(ab))$
$\forall a, b \exists q, r\ (a = b \cdot q + r,\ (r = 0 \text{ or } d(r) < d(b)))$

Examples

Example 1: $\langle \mathbb{Z}, +, -, 0, \cdot, 1, d\rangle$, the ring of integers with addition, subtraction, zero, and multiplication is a Euclidean domain with $d(a) = |a|$.

Properties

Classtype	first-order

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 1,\ f_5 = 1,\ f_6 = 0$

Subclasses

Fld: Fields

Superclasses

PIDom: Principal Ideal Domain

Cont|Po|J|M|L|D|To|B|U|Ind

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63. Mset: M-sets

Definition

An $\mathbf{M}$-*set* is an algebra $\mathbf{A} = \langle A, f_m(m \in M)\rangle$, where $\mathbf{M} = \langle M, \cdot, 1\rangle$ is a monoid, such that

f_1 is the identity map: $1x = x$ and

the monoid action associates: $(m \cdot n)x = m(nx)$

Remark: $f_m(x) = mx$ is a unary operation called *the monoid action by* m.

Properties

Classtype	Variety

Finite Members

Subclasses

Gset: G-sets

Superclasses

Unar: Unary Algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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64. Gset: G-sets

Definition

A G-*set* is an algebra $\mathbf{A} = \langle A, f_g(g \in G)\rangle$, where $\langle G, \cdot, ^{-1}, 1\rangle$ is a group, such that

f_1 is the identity map: $1x = x$ and

the group action associates: $(g \cdot h)x = g(hx)$

Remark: $f_g(x) = gx$ is a unary operation called *the group action by* g.

If follows from the associativity that $f_{g^{-1}}$ is the inverse function of f_g.

Properties

Finite Members

Subclasses

RMod: Modules over a ring

Superclasses

Mset: M-sets

Cont|Po|J|M|L|D|To|B|U|Ind

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65. RMod: Modules over a ring

Definition

Let $\mathbf{R}$ be a ring with identity. A *module over* $\mathbf{R}$ (or $\mathbf{R}$-module) is an algebra $\mathbf{A} = \langle A, +, -, 0, f_r\ (r \in R)\rangle$ such that

$\langle A, +, -, 0\rangle$ is an abelian group and for all $r, s \in R$

f_r preserves addition: $f_r(x + y) = f_r(x) + f_r(y)$

f_1 is the identity map: $f_1(x) = x$
$f_{r+s}(x)) = f_r(x) + f_s(x)$
$f_{r\cdot s}(x) = f_r(f_s(x))$
Remark: f_r is called *scalar multiplication by* r, and $f_r(x)$ is usually written simply as rx.

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

Subclasses

FVec: Vector spaces over a field

Superclasses

Gset: G-sets

Cont|Po|J|M|L|D|To|B|U|Ind

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66. FVec: Vector spaces over a field

Definition

A *vector space* over a field $\mathbf{F}$ is an algebra $\mathbf{V} = \langle V, +, -, 0, f_a\ (a \in F)\rangle$ such that
$\langle V, +, -, 0\rangle$ is an abelian group
scalar product f_a distributes over vector addition: $a(x + y) = ax + ay$
f_1 is the identity map: $1x = x$
scalar product distributes over scalar addition: $(a + b)x = ax + bx$
scalar product associates: $(a \cdot b)x = a(bx)$
Remark: $f_a(x) = ax$ is called *scalar multiplication by* a.

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	No
Equationally def. pr. cong.	No

Finite Members

Subclasses

NaA: Nonassociative algebras

Superclasses

RMod: Modules over a ring

Cont|Po|J|M|L|D|To|B|U|Ind

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67. JorA: Jordan algebras

Definition

A *Jordan algebra* is a nonassociative algebra $\langle A, +, -, 0, \cdot, s_r\ (r \in F)\rangle$ such that

$\cdot$ is commutative: $x \cdot y = y \cdot x$

the Jordan identity holds: $(xy)x^2 = x(yx^2)$

Properties

Finite Members

Subclasses

Superclasses

NaA: Nonassociative algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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68. LNeofld: Left neofields

Definition

A *left neofield* is an algebra $\mathbf{F} = \langle F, +, \backslash, /, 0, \cdot, 1, ^{-1}\rangle$ of type $\langle 2, 2, 2, 0, 2, 0, 1\rangle$ such that

$\langle F, +, \backslash, /, 0\rangle$ is a loop

$\langle F - \{0\}, \cdot, 1, ^{-1}\rangle$ is a group

$\cdot$ left-distributes over $+$: $x \cdot (y + z) = x \cdot y + x \cdot z$

Properties

Finite Members

Subclasses

Neofld: Neofields

Superclasses

Lp: Loops

Cont|Po|J|M|L|D|To|B|U|Ind

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69. BilinA: Bilinear algebras

Definition

A *bilinear algebra* is an algebra $\mathbf{A} = \langle A, +, -, 0, \cdot, s_r\ (r \in F)\rangle$ of type $\langle 2, 1, 0, 2, 1_r\ (r \in F)\rangle$ such that

$\langle A, +, -, 0, s_r\ (r \in F)\rangle$ is a vector space over a field F

$\cdot$ is *bilinear*: $x(y + z) = xy + xz$, $(x + y)z = xz + yz$, and $s_r(xy) = s_r(x)y = xs_r(y)$

Properties

Classtype	Variety

Finite Members

Subclasses

AAlg: Associative algebras

LieA: Lie algebras

Superclasses

NaA: Nonassociative algebras

Cont|Po|J|M|L|D|To|B|U|Ind

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70. CliffSgrp: Clifford semigroups

Definition

A *Clifford semigroup* is an inverse semigroup $\mathbf{S} = \langle S, \cdot, ^{-1} \rangle$ that is also a completely regular semigroup.

Definition

A *Clifford semigroup* is an algebra $\mathbf{S} = \langle S, \cdot, ^{-1} \rangle$ such that

$\cdot$ is associative: $(xy)z = x(yz)$

$^{-1}$ is an inverse: $xx^{-1}x = x$, $(x^{-1})^{-1} = x$

$xx^{-1} = x^{-1}x$, $xx^{-1}y^{-1}y = y^{-1}yxx^{-1}$, $xx^{-1} = x^{-1}x$

Properties

Classtype	Variety
Locally finite	No
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Congruence extension property	No
Equationally def. pr. cong.	No
Amalgamation property	No
Strong amalgamation property	No
Epimorphisms are surjective	Yes

Finite Members

Subclasses

Grp: Groups

Superclasses

InvSgrp: Inverse semigroups Cont|Po|J|M|L|D|To|B|U|Ind

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71. LieA: Lie algebras

Definition

A *Lie algebra* is a bilinear algebra $\mathbf{A} = \langle A, +, -, 0, \cdot, s_r \ (r \in F)\rangle$ over a field $\mathbf{F}$ such that

$xx = 0$ and

$(xy)z + (yz)x + (zx)y = 0.$

Properties

Classtype	Variety

Finite Members

Subclasses

Superclasses

BilinA: Bilinear algebras Cont|Po|J|M|L|D|To|B|U|Ind

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72. MedMag: Medial magmas

Definition

A *medial magma* is an algebra $\mathbf{G} = \langle G, \cdot \rangle$, where $\cdot$ is an infix binary operation such that

$\cdot$ mediates: $(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot w)$

Formal Definition

$(x \cdot y) \cdot (z \cdot w) = (x \cdot z) \cdot (y \cdot w)$

Examples

Example 1: $\langle S,*\rangle$, where $\langle S,+,\cdot\rangle$ is any commutative semiring, $a,b\in S$, and $x*y=a\cdot x+b\cdot y$.

Properties

Classtype	Variety
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No

Finite Members

$f_1=1,\ f_2=7,\ f_3=75,\ f_4=3969$

Subclasses

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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73. NlGrp: Nilpotent groups

Definition

A *nilpotent group* is a group $\mathbf{G}=\langle G,\cdot,^{-1},1\rangle$ that is

nilpotent: if $Z_0=\{1\}$ and $\forall i(Z_{i+1}=\{x\in G:\forall y\ xyx^{-1}y^{-1}\in Z_i\})$ then $\exists n(Z_n=G)$

Remark: Note that $Z_1=Z(G)$, the center of G. The smallest n for which $Z_n=G$ is the *nilpotence class of* G. E.g. Abelian groups are of nilpotence class 1.

Properties

Classtype	higher-order
Congruence modular	yes
Congruence n-permutable	Yes, $n=2$
Congruence regular	yes
Congruence uniform	yes

Finite Members

Subclasses

AbGrp: Abelian groups

Superclasses

Grp: Groups Cont|Po|J|M|L|D|To|B|U|Ind

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74. NaA: Nonassociative algebras

Definition

A *(nonassociative) algebra* is an algebra $\mathbf{A}=\langle A,+,-,0,\cdot,s_r\ (r\in F)\rangle$ of type $\langle 2,1,0,2,1_r\ (r\in F)\rangle$ such that

$\langle A,+,-,0,s_r\ (r\in F)\rangle$ is a vector space over a field F

$\cdot$ is *bilinear*: $x(y+z)=xy+xz$, $(x+y)z=xz+yz$, and $s_r(xy)=s_r(x)y=xs_r(y)$

Properties

Finite Members

Subclasses

BilinA: Bilinear algebras
JorA: Jordan algebras
Superclasses
FVec: Vector spaces over a field Cont|Po|J|M|L|D|To|B|U|Ind

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75. OrdA: Order algebras

Formal Definition
An *order algebra* Freese et al. [2002] is an algebra $\mathbf{A} = \langle A, \cdot \rangle$, where $\cdot$ is an infix binary operation such that
$\cdot$ is idempotent: $x \cdot x = x$
$(x \cdot y) \cdot x = y \cdot x$
$(x \cdot y) \cdot y = x \cdot y$
$x \cdot ((x \cdot y) \cdot z) = x \cdot (y \cdot z)$
$((x \cdot y) \cdot z) \cdot y = (x \cdot z) \cdot y$
Properties

Classtype	Variety
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence regular	No
Congruence uniform	No
Equationally def. pr. cong.	No

Finite Members
$f_1 = 1,\ f_2 = 2,\ f_3 = 7,\ f_4 = 36,\ f_5 = 251$
Subclasses
Bnd: Bands
Superclasses
Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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76. pGrp: P-groups

Definition
A *p-group* is a group $\mathbf{G} = \langle G, \cdot, ^{-1}, 1 \rangle$ such that
p is a prime number and $\forall x \exists n \in \mathbb{N}(x^{(p^n)} = 1)$
Properties

Classtype	higher-order
Congruence distributive	No
Congruence modular	Yes
Congruence n-permutable	Yes, $n = 2$
Congruence regular	Yes
Congruence uniform	Yes

Finite Members
Subclasses
AbpGrp: Abelian p-groups
Superclasses
Grp: Groups Cont|Po|J|M|L|D|To|B|U|Ind

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77. Qnd: Quandles

Formal Definition

A *quandle* is an algebra $\mathbf{Q} = \langle Q, \triangleright, \triangleleft\rangle$ of type $\langle 2,2\rangle$ such that

$\triangleright$ is *left-selfdistributive*: $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$

$\triangleleft$ is *right-selfdistributive*: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

$(x \triangleright y) \triangleleft x = y$

$x \triangleright (y \triangleleft x) = y$

$\triangleright$ is *idempotent*: $x \triangleright x = x$

Remark: The last identity can equivalently be replaced by $\triangleleft$ is *idempotent*: $x \triangleleft x = x$

Examples

Example 1: If $\langle G, \cdot, ^{-1}, 1\rangle$ is a group and $x \triangleright y = xyx^{-1}$, $x \triangleleft y = x^{-1}yx$ (conjugation) then $\langle G, \triangleright, \triangleleft\rangle$ is a quandle.

Properties

Classtype	Variety
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	Yes, $n = 2$

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 7,\ f_5 = 22,\ f_6 = 73,\ f_7 = 298,\ f_8 = 1581,\ f_9 = 11079$

Subclasses

Superclasses

Mag: Magmas

Cont|Po|J|M|L|D|To|B|U|Ind

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78. qMV: Quasi-MV-algebras

Formal Definition

A *quasi-MV-algebra* Ledda et al. [2006] is a structure $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ such that

$(x \oplus y) \oplus z = x \oplus (y \oplus z)$

$x'' = x$

$x \oplus 1 = 1$

$(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$

$(x \oplus 0)' = x' \oplus 0$

$(x \oplus 0) \oplus 0 = x \oplus 0$

$0' = 1$

Examples

The standard qMV-algebra is $\mathbf{S} = \langle [0,1]^2, \oplus, ', \mathbf{0}, \mathbf{1}\rangle$ where $\langle a,b\rangle \oplus \langle c,d\rangle = \langle \min(1, a+c), \frac{1}{2}\rangle$, $\langle a,b\rangle' = \langle 1-a, 1-b\rangle$, $\mathbf{0} = \langle 0, \frac{1}{2}\rangle$ and $\mathbf{1} = \langle 1, \frac{1}{2}\rangle$.

Basic Results

The variety of qMV-algebras is generated by the standard qMV-algebra.

The operation $\oplus$ is commutative: $x \oplus y = y \oplus x$.

Every qMV-algebra that satisfies $x \oplus 0 = x$ is an MV-algebra.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence e-regular	No
Congruence uniform	No
Congruence extension property	Yes
Equationally def. pr. cong.	No
Amalgamation property	yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 1,\ f_4 = 9,\ f_5 = 9,\ f_6 = 467$

Subclasses

MV: MV-algebras

sqMV: Sqrt-quasi-MV-algebras

Superclasses

CSgrp: Commutative semigroups

Cont|Po|J|M|L|D|To|B|U|Ind

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79. sqMV: Sqrt-quasi-MV-algebras

Definition

A $\sqrt{\prime}$*quasi-MV-algebra*Giuntini et al. [2007] is a structure $\mathbf{A} = \langle A, \oplus, \sqrt{\prime}, ', 0, 1, k\rangle$ such that $\sqrt{\prime}$ is a unary operation,

$\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ is a quasi-MV-algebra,

$x' = \sqrt{\prime}\sqrt{\prime}x$,

$k' = k$, and

$\sqrt{\prime}(x \oplus 0) \oplus 0 = k$.

Examples

The standard $\sqrt{\prime}$qMV-algebra is $\mathbf{S}_r = \langle [0,1]^2, \oplus, \sqrt{\prime}, ', \mathbf{0}, \mathbf{1}, \mathbf{k}\rangle$ where $\langle a, b\rangle \oplus \langle c, d\rangle = \langle \min(1, a + c), \frac{1}{2}\rangle$, $\sqrt{\prime}\langle a, b\rangle' = \langle b, 1 - a\rangle$, $\langle a, b\rangle' = \langle 1 - a, 1 - b\rangle$, $\mathbf{0} = \langle 0, \frac{1}{2}\rangle$, $\mathbf{1} = \langle 1, \frac{1}{2}\rangle$ and $\mathbf{k} = \langle \frac{1}{2}, \frac{1}{2}\rangle$.

Basic Results

The variety of $\sqrt{\prime}$qMV-algebras is generated by the standard $\sqrt{\prime}$qMV-algebra.

The operation $\oplus$ is commutative: $x \oplus y = y \oplus x$.

Only the trivial $\sqrt{\prime}$qMV-algebra is an MV-algebra.

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
Locally finite	No
Residual size	Unbounded
Congruence distributive	No
Congruence modular	No
Congruence n-permutable	No
Congruence e-regular	No
Congruence uniform	No
Congruence extension property	Yes
Equationally def. pr. cong.	No
Amalgamation property	yes

Finite Members

$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 2,\ f_5 = 5,\ f_6 = 5,\ f_7 = 8$

Subclasses

Superclasses

qMV: Quasi-MV-algebras Cont|Po|J|M|L|D|To|B|U|Ind

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80. QtMag: Quasitrivial magmas

Formal Definition

A *quasitrivial magma* is a magma $\mathbf{A} = \langle A, \cdot \rangle$ such that

$\cdot$ is *quasitrivial*: $x \cdot y = x$ or $x \cdot y = y$

Basic Results

Quasitrivial magmas are in 1-1 correspondence with reflexive relations. E.g. a translations is given by $x \cdot y = x$ iff $\langle x, y \rangle \in E$.

Properties

Classtype	Universal

Finite Members

$f_1 = 1,\ f_2 = 3,\ f_3 = 16$

Subclasses

Superclasses

Mag: Magmas Cont|Po|J|M|L|D|To|B|U|Ind

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81. TrivA: Trivial algebras

Definition

A *trivial algebra* is an algebra with exactly one element. We assume that the algebras in this variety have a signature with all possible operation symbols of each finite arity. Hence this category is the unique category at the bottom of the hierarchy.

Formal Definition

$x = y$

Properties

Classtype	Variety
Equational theory	Decidable
Quasiequational theory	Decidable
First-order theory	Decidable
Locally finite	Yes
Residual size	1
Congruence distributive	Yes
Congruence modular	Yes
Congruence n-permutable	Yes
Congruence regular	Yes
Congruence uniform	Yes
Congruence extension property	Yes
Definable principal congruences	Yes
Equationally def. pr. cong.	Yes
Amalgamation property	Yes
Strong amalgamation property	Yes
Epimorphisms are surjective	Yes

Finite Members

$f_1 = 1,\ f_2 = 0,\ f_n = 0$ for all $n > 1$.

Subclasses

Superclasses

AbToGrp: Abelian totally ordered groups

ActLat: Action lattices

BCIInFL: Boolean commutative integral involutive FL-algebras

BGrp: Boolean groups

BRMod: Boolean modules over a relation algebra

BSlat: Boolean semilattices

Bilat: Bilattices

CA_2: Cylindric algebras of dimension 2

CanRL: Cancellative residuated lattices

FRng: Function rings

IMTLChn: Involutive monoidal t-norm logic chains https://www.overleaf.com/project/60bfec78c1e72aa63c5a0e8dhttps://ww

IRA: Integral relation algebras

LLA: Linear logic algebras

MonA: Monadic algebras

TA: Tense algebras

Cont|Po|J|M|L|D|To|B|U|Ind

Appendix

The table below contains an initial segment of the fine spectrum for each of the classes in this survey. The classes are ordered in lexicographically decreasing order of their fine spectrum sequence and, if available, the sequence is followed by a link to the `oeis.org` entry for this sequence.

Name	Fine spectrum	OEIS
PoMag	1, 16, 4051	No
PoImpA	1, 16, 3981	No
PoSgrp	1, 11, 173, 4753, 198838,...	No
Mag	1, 10, 3330, 178981952,...	A001329
Srng	1, 10, 132, 2341	No
CPoSgrp	1, 7, 83, 1468, 37248,...	No
MedMag	1, 7, 75, 3969	No
IdPoSgrp	1, 7, 69, 1035	No
MMag	1, 6, 280	
JImpA	1, 6, 245	
MImpA	1, 6, 220	
JMag	1, 6, 220	
ToMag	1, 6, 175	
ToImpA	1, 6, 175	
MultLat	1, 6, 175	
DLMag	1, 6, 175	
DLImpA	1, 6, 175	
LMag	1, 6, 175	
LImpA	1, 6, 175	
DivPos	1, 6, 123	
LrPoMag	1, 6, 110	
MSgrp	1, 6, 70, 1437	No
JSgrp	1, 6, 61, 866	No
CDivPos	1, 6, 55, 1434	No
DLSgrp	1, 6, 44, 479	No
LSgrp	1, 6, 44, 479	No
ToSgrp	1, 6, 44, 386	A084965
PoUn	1, 6, 43, 452	No
PoNUn	1, 6, 39, 386, 5203	No
BMag	1, 6, 0, 1176, 0, 0, 0	No
BImpA	1, 6, 0, 1176, 0, 0, 0	No
BSgrp	1, 6, 0, 93, 0, 0, 0	No
LrPoSgrp	1, 5, 28, 273, 3788	No
Sgrp	1, 5, 24, 188, 1915, 28634,...	A027851
DivJslat	1, 4, 281	No
DivMslat	1, 4, 216	
DivLat	1, 4, 216	
ToDivLat	1, 4, 216	
DDivLat	1, 4, 216	
CnjMag	1, 4, 215	
CMag	1, 4, 129, 43968, 254429900,...	A001425
CDivJslat	1, 4, 79, 7545	No
CDivMslat	1, 4, 64, 6208	No
CDivLat	1, 4, 64, 6208	No
PoMon	1, 4, 37, 549	No
CMSgrp	1, 4, 32, 432	??
CJSgrp	1, 4, 29, 289	No
IdMSgrp	1, 4, 28, 308, 4694	No
CPoMon	1, 4, 27, 301, 4887	No
IdJSgrp	1, 4, 23, 166, 1379	No
Srng_0	1, 4, 22, 283	No
Srng_1	1, 4, 22, 169, 1819	No
CDLSgrp	1, 4, 20, 149, 1106	No
CLSgrp	1, 4, 20, 149, 1427	No
CToSgrp	1, 4, 20, 114, 710, 4726,...	A346414
IdLSgrp	1, 4, 17, 100, 674	No
DIdLSgrp	1, 4, 17, 100, 576	No
IdToSgrp	1, 4, 17, 82, 422	??
RPoUn	1, 4, 16, 87, 562	No
GalPos	1, 4, 15, 83, 539	No
InPoMag	1, 4, 12, 77, 498	No
CyInPoMag	1, 4, 12, 76, 481	No
CInPoMag	1, 4, 12, 69, 354, 3632	No
InPoSgrp	1, 4, 10, 50, 210, 1721	No
CyInPoSgrp	1, 4, 10, 50, 196, 1397	No
CInPoSgrp	1, 4, 10, 50, 194, 1356	No
BCSgrp	1, 4, 0, 35, 0, 0, 0, 1237, 0	No
BIdSgrp	1, 4, 0, 18, 0, 0, 0, 88, 0, 0	No
LrMMag	1, 3, 52, 4827	No
LrJMag	1, 3, 52, 4827	No
LrLMag	1, 3, 50, 4441	No
DLrLMag	1, 3, 50, 4441	No
LrToMag	1, 3, 50, 4116	??
RtQgrp	1, 3, 44, 14022	??
RPoMag	1, 3, 28, 1200	No
IdPoMon	1, 3, 23, 238, 3356	No
CDDivLat	1, 3, 20, 364	??
CToDivLat	1, 3, 20, 294	No
LrMSgrp	1, 3, 19, 199, 2946	No
LrJSgrp	1, 3, 19, 192	No
CIdPoSgrp	1, 3, 19, 171, 2069	No
LrLSgrp	1, 3, 18, 183, 2500	No
DLrLSgrp	1, 3, 18, 183, 1968	No
LrToSgrp	1, 3, 18, 144, 1370	No
MUn	1, 3, 17, 138, 1555	No
QtMag	1, 3, 16, 218	??
CRPoMag	1, 3, 16, 180, 4761	No
RPoSgrp	1, 3, 16, 154, 2100	No
JUn	1, 3, 16, 104, 822	No
MNUn	1, 3, 15, 113, 1167	No
JNUn	1, 3, 15, 113, 1167	No
CIdPoMon	1, 3, 13, 86, 759	No
CRPoSgrp	1, 3, 12, 76, 670	No
IdLrPoSgrp	1, 3, 12, 71, 524	No
CSgrp	1, 3, 12, 58, 325, 2143, 17291,...	A001426
pPos	1, 3, 11, 47, 243	No

LNUn	1, 3, 10, 56, 457	No
DLNUn	1, 3, 10, 56, 276	No
LUn	1, 3, 10, 50, 313	No
DLUn	1, 3, 10, 50, 226	No
Bnd	1, 3, 10, 46, 251, 1682, 13213	A058112
ToNUn	1, 3, 10, 35, 126, 462	
ToUn	1, 3, 10, 35, 126, 462	
RegSgrp	1, 3, 9, 42, 206, 1352, 10168,...	A001427
NBnd	1, 3, 8, 30, 114, 536	No
InPoMon	1, 3, 5, 20, 39, 179, 500	No
CyInPoMon	1, 3, 5, 20, 39, 176, 493	No
CInPoMon	1, 3, 5, 20, 39, 174, 488	No
InPos	1, 3, 5, 16, 30, 108	No
NRng	1, 3, 5, 35, 10, 99, 24, 3856,...	A305858
BDivLat	1, 3, 0, 325	No
BLrMag	1, 3, 0, 325, 0, 0, 0	No
BCDivLat	1, 3, 0, 70, 0, 0, 0	No
BLrSgrp	1, 3, 0, 39, 0	No
BUn	1, 3, 0, 15, 0, 0, 0, 147, 0	No
BNUn	1, 3, 0, 15, 0, 0, 0, 147, 0	No
Shell	1, 2, 243	
RMMag	1, 2, 20, 1116	No
RJMag	1, 2, 20, 1116	No
RLMag	1, 2, 20, 1116	No
DRLMag	1, 2, 20, 1116	No
RToMag	1, 2, 20, 980	??
MMon	1, 2, 14, 168, 3488	No
RMSgrp	1, 2, 12, 129, 1852	No
RJSgrp	1, 2, 12, 129, 1852	No
RLSgrp	1, 2, 12, 129, 1852	No
IdSrng_0	1, 2, 12, 129, 1852	No
DRLSgrp	1, 2, 12, 129, 1437	No
RToSgrp	1, 2, 12, 101, 1003	No
Sgrp_0	1, 2, 12, 90, 960	No
JMon	1, 2, 11, 73, 703	No
CRMMag	1, 2, 10, 148, 4398	No
CRJMag	1, 2, 10, 148, 4398	No
CRLMag	1, 2, 10, 148, 4398	No
CDRLMag	1, 2, 10, 148, 3554	No
CRToMag	1, 2, 10, 112, 2772	??
CMMon	1, 2, 10, 92, 1322	No
IdMMon	1, 2, 10, 81, 950	No
FL	1, 2, 9, 79, 737	No
FL_e	1, 2, 9, 63, 492	No
CJMon	1, 2, 9, 55, 437	No
GalJslat	1, 2, 9, 52, 361, 2947	No
CRMSgrp	1, 2, 8, 57, 550	No
CRJSgrp	1, 2, 8, 57, 550	No
CRLSgrp	1, 2, 8, 57, 550	No
CRSlSgrp	1, 2, 8, 57, 392	No
CDRLSgrp	1, 2, 8, 57, 392	No
CIdMSgrp	1, 2, 8, 53, 498	No
IdLrMSgrp	1, 2, 8, 46, 345, 3180	No
LMon	1, 2, 8, 45, 347	No
IdLrJSgrp	1, 2, 8, 45, 304	No
DLMon	1, 2, 8, 45, 279	No
CRToSgrp	1, 2, 8, 41, 241	No
ToMon	1, 2, 8, 34, 184, 1218,...	A346413
IdLrLSgrp	1, 2, 7, 40, 273	No
DIdLrLSgrp	1, 2, 7, 40, 213	No
Mon	1, 2, 7, 35, 228, 2237, 31559	A058129
CIdJSgrp	1, 2, 7, 33, 185	
IdLrToSgrp	1, 2, 7, 30, 144, 740	No

IdJMon	1, 2, 7, 29, 136	??
OrdA	1, 2, 7, 36, 251	No
Srng_{01}	1, 2, 6, 40, 295, 3246	No
FL_c	1, 2, 6, 39, 279	No
LrPoMon	1, 2, 6, 32, 234, 2493	No
CIdMMon	1, 2, 6, 31, 228, 2205	No
CLMon	1, 2, 6, 31, 199	No
FL_{ec}	1, 2, 6, 31, 199	No
CDLMon	1, 2, 6, 31, 149	No
GalMslat	1, 2, 6, 30, 184, 1373	No
GalLat	1, 2, 6, 30, 184	No
DGalLat	1, 2, 6, 30, 126	No
IdLMon	1, 2, 6, 22, 93, 439	No
CToMon	1, 2, 6, 22, 92, 426	
DIdLMon	1, 2, 6, 22, 75, 274	No
GalToLat	1, 2, 6, 20, 70, 252, 924	
IdToMon	1, 2, 6, 16, 44, 120	
InLMag	1, 2, 5, 42, 342	No
CyInLMag	1, 2, 5, 42, 328	No
DInLMag	1, 2, 5, 42, 164	No
CyDInLMag	1, 2, 5, 42, 156	No
CInLMag	1, 2, 5, 38, 238, 2722	No
CDInLMag	1, 2, 5, 38, 90, 858	No
InLSgrp	1, 2, 5, 29, 146, 1308	No
CyInLSgrp	1, 2, 5, 29, 132, 1018	No
CInLSgrp	1, 2, 5, 29, 130, 984	No
DInLSgrp	1, 2, 5, 29, 63, 454	No
CyDInLSgrp	1, 2, 5, 29, 55, 353	No
CDInLSgrp	1, 2, 5, 29, 53, 330	No
CInSlSgrp	1, 2, 5, 29, 53, 330	No
RPoMon	1, 2, 5, 28, 186	No
CRPoMon	1, 2, 5, 24, 131, 1001	No
InToMag	1, 2, 5, 22, 142	No
CyInToMag	1, 2, 5, 22, 138	??
BCI	1, 2, 5, 22, 118, 974	No
CIdLSgrp	1, 2, 5, 19, 86, 462	No
CMon	1, 2, 5, 19, 78, 421, 2637	A058131
CDIdLSgrp	1, 2, 5, 19, 68	No
CInToMag	1, 2, 5, 18, 72, 384	No
CIdJMon	1, 2, 5, 17, 66, 288	No
Pos	1, 2, 5, 16, 63, 318, 2045, 16999,...	A000112
pMslat	1, 2, 5, 16, 60, 262, 1315	No
pJslat	1, 2, 5, 16, 60, 262, 1315	No
InvSgrp	1, 2, 5, 16, 52, 208, 911, 4637,...	A001428
CInvSgrp	1, 2, 5, 16, 51, 201,...	A234843
InToSgrp	1, 2, 5, 14, 43, 147, 578	??
CIdToSgrp	1, 2, 5, 14, 42, 132	
CyInToSgrp	1, 2, 5, 14, 39, 119	No
CInToSgrp	1, 2, 5, 14, 37, 107	No
CIdLMon	1, 2, 4, 12, 41, 159	No
CDIdLMon	1, 2, 4, 12, 31, 90, 241	No
CIdToMon	1, 2, 4, 8, 16, 32, 64	
pLat	1, 2, 3, 7, 21, 75, 315	No
pDLat	1, 2, 3, 7, 13, 27, 50	No
pToLat	1, 2, 3, 4, 5, 6,...	A000027
DivRng	1, 2, 3, 3, 5, 0, 7, 4	No
Rng	1, 2, 2, 11, 2, 4	A027623
CRng	1, 2, 2, 9, 2, 4	A037289
LtCanSgrp	1, 2, 2, 4, 2, 5, 2, 9	No
RecBnd	1, 2, 2, 3, 2, 4, 2, 4, 3, 4	

Sfld	1, 2, 1, 1, 1, 0	
BRMag	1, 2, 0, 136, 0	No
BCRMag	1, 2, 0, 36, 0, 0	No
BRSgrp	1, 2, 0, 28, 0, 0	No
BInMag	1, 2, 0, 20, 0	No
BCyInMag	1, 2, 0, 20, 0	No
BCInMag	1, 2, 0, 20, 0	No
BCRSgrp	1, 2, 0, 16, 0, 0	
BInSgrp	1, 2, 0, 15, 0, 0	No
BCyInSgrp	1, 2, 0, 15, 0, 0	No
BCInSgrp	1, 2, 0, 15, 0, 0	No
BMon	1, 2, 0, 11, 0, 0, 0, 383	No
BRUn	1, 2, 0, 10, 0, 0, 0, 104	No
BIdLrSgrp	1, 2, 0, 10, 0, 0	No
BGalLat	1, 2, 0, 10, 0, 0	No
BCMon	1, 2, 0, 9, 0, 0, 0	No
BIdMon	1, 2, 0, 6, 0, 0, 0, 24	No
BCIdSgrp	1, 2, 0, 5, 0, 0, 0, 13	No
BCIdMon	1, 2, 0, 4, 0, 0, 0, 9	No
pBA	1, 2, 0, 3, 0, 0, 0, 1, 0, 0	
Qgrp	1, 1, 5, 35, 1411,...	A057991
MouQgrp	1, 1, 5, 29, 1351	No
LrMMon	1, 1, 4, 24, 195, 2146	No
IdRMSgrp	1, 1, 4, 24, 169, 1404	No
IdRJSgrp	1, 1, 4, 24, 169, 1404	No
IdRPoSgrp	1, 1, 4, 24, 169	No
IdRLSgrp	1, 1, 4, 24, 169	No
DIdRLSgrp	1, 1, 4, 24, 124	No
LrLMon	1, 1, 4, 23, 169, 1635	No
LrJMon	1, 1, 4, 23, 169, 1635	No
DLrLMon	1, 1, 4, 23, 130, 976	No
LrToMon	1, 1, 4, 17, 92, 609	No
IdRToSgrp	1, 1, 4, 17, 82	No
RL	1, 1, 3, 20, 149, 1488, 18554,...	No??
IdSrng_{01}	1, 1, 3, 20, 149, 1488, 18554,...	No
bRL	1, 1, 3, 20, 149, 1488	No
RMMon	1, 1, 3, 20, 149, 1488	No
RJMon	1, 1, 3, 20, 149, 1488	No
KA	1, 1, 3, 20, 149, 1488	No
KLat	1, 1, 3, 16, 149, 1488	No
ActLat	1, 1, 3, 16, 149, 1488	No
DRL	1, 1, 3, 20, 115, 899, 7782,...	No
CRL	1, 1, 3, 16, 100, 794, 7493,...	No
CRMMon	1, 1, 3, 16, 100, 794	No
CRJMon	1, 1, 3, 16, 100, 794	No
CDRL	1, 1, 3, 16, 70, 399	No
RToMon	1, 1, 3, 15, 84, 575	No
BCKJslat	1, 1, 3, 14, 87, 745	No
IdLrPoMon	1, 1, 3, 12, 59, 350	No
IdLrMMon	1, 1, 3, 12, 59, 348, 2372	No
CRSlMon	1, 1, 3, 12, 47, 220	No
IdLrJMon	1, 1, 3, 11, 46, 215, 1114	No
IdLrLMon	1, 1, 3, 11, 46, 215	No
CRToMon	1, 1, 3, 11, 46, 213	
DIdLrLMon	1, 1, 3, 11, 37, 134	No
IdLrToMon	1, 1, 3, 8, 22, 60, 164	??
Qnd	1, 1, 3, 7, 22, 73, 298, 1581,...	A181769
IPoMon	1, 1, 2, 11, 102, 1609	No
IMMon	1, 1, 2, 11, 102, 1569	No
CIPoMon	1, 1, 2, 9, 60, 590	No
CIMMon	1, 1, 2, 9, 60, 572	No
PoIrim	1, 1, 2, 9, 51, 409	No

ILrMMon	1, 1, 2, 9, 51, 408	No
Porim	1, 1, 2, 9, 49, 365	No
IRJMon	1, 1, 2, 9, 49, 364, 3335	No
IRMMon	1, 1, 2, 9, 49, 364	No
IJMon	1, 1, 2, 9, 49, 364	No
IRL	1, 1, 2, 9, 49, 364	No
ILrJMon	1, 1, 2, 9, 49, 364	No
ILrLMon	1, 1, 2, 9, 49, 364	No
ILMon	1, 1, 2, 9, 49, 364	No
DIRL	1, 1, 2, 9, 49, 359	No
DILrLMon	1, 1, 2, 9, 49, 359	No
DILMon	1, 1, 2, 9, 49, 359	No
InFL	1, 1, 2, 9, 21, 101, 284, 1464	No
CyInFL	1, 1, 2, 9, 21, 101, 279, 1433	No
CInFL	1, 1, 2, 9, 21, 100, 276, 1392	No
DInFL	1, 1, 2, 9, 8, 43, 49	No
CyDInFL	1, 1, 2, 9, 8, 43, 48	No
CDInFL	1, 1, 2, 9, 8, 42, 46	No
IToMon	1, 1, 2, 8, 44, 308, 2641,...	A253950
IRToMon	1, 1, 2, 8, 44, 308	
ILrToMon	1, 1, 2, 8, 44, 308	
BCKMslat	1, 1, 2, 8, 38, 265	No
CIdRPoSgrp	1, 1, 2, 8, 36, 203	No
CIdRMSgrp	1, 1, 2, 8, 36, 202	No
CIdRJSgrp	1, 1, 2, 8, 36, 202	No
CIdRLSgrp	1, 1, 2, 8, 36, 202	No
IdRPoMon	1, 1, 2, 8, 32, 148	No
IdRJMon	1, 1, 2, 8, 32, 147, 759	No
IdRMMon	1, 1, 2, 8, 32, 147	No
IdRL	1, 1, 2, 8, 32, 147	No
DIdRL	1, 1, 2, 8, 27, 96	No
CIdRSlSgrp	1, 1, 2, 8, 25, 97	No
CDIdRLSgrp	1, 1, 2, 8, 25, 97	No
RtHp	1, 1, 2, 8, 24, 91	No
Dtoid	1, 1, 2, 7, 61	No
CIRMMon	1, 1, 2, 7, 26, 129, 723	No
CIRL	1, 1, 2, 7, 26, 129, 723	No
CIRJMon	1, 1, 2, 7, 26, 129, 723	No
FL_{ew}	1, 1, 2, 7, 26, 129, 723	No
FL_{w}	1, 1, 2, 7, 26, 129, 723	No
Pocrim	1, 1, 2, 7, 26, 129	No
CIJMon	1, 1, 2, 7, 26, 129	No
CILMon	1, 1, 2, 7, 26, 129	No
BCKLat	1, 1, 2, 7, 26, 129	No
CDIRL	1, 1, 2, 7, 26, 124, 645	No
CDILMon	1, 1, 2, 7, 26, 124, 645	No
CIRSlMon	1, 1, 2, 7, 23, 99, 464	No
CIToMon	1, 1, 2, 6, 22, 94, 451	A030453
CIRToMon	1, 1, 2, 6, 22, 94, 451	Same as above??
CIdRPoMon	1, 1, 2, 6, 20, 78	
CIdRJMon	1, 1, 2, 6, 20, 77, 333	No
CIdRMMon	1, 1, 2, 6, 20, 77	
CIdRL	1, 1, 2, 6, 20, 77	
IdRToMon	1, 1, 2, 6, 16, 44, 120	No
CDIdRL	1, 1, 2, 6, 15, 44, 115	No
Mslat	1, 1, 2, 5, 15, 53, 222, 1078,...	A006966
Jslat	1, 1, 2, 5, 15, 53, 222, 1078,...	A006966
ubJslat	1, 1, 2, 5, 15, 53, 222, 1078,...	A006966
CIdRToSgrp	1, 1, 2, 5, 14, 42	
GBL	1, 1, 2, 5, 10, 23, 49, 111	No
BLA	1, 1, 2, 5, 10, 23, 49, 111	No
Hp	1, 1, 2, 5, 10, 23, 49	No
CIdRSlMon	1, 1, 2, 5, 9, 20, 38	No
CInSlMon	1, 1, 2, 5, 8, 20, 36, 90	No
InToMon	1, 1, 2, 4, 8, 17, 38	??

CyInToMon	1, 1, 2, 4, 8, 17, 38, 91		PIDom	1, 1, 1, 1, 1, 0	
CInToMon	1, 1, 2, 4, 8, 17, 36, 81	No	IntDom	1, 1, 1, 1, 1, 0	
CIdRToMon	1, 1, 2, 4, 8, 16, 32		EucDom	1, 1, 1, 1, 1, 0	
sqMV	1, 1, 2, 2, 5, 5, 8		NRng_1	1, 1, 1, 6, 1, 1, 1, 53, 11, 1	No
qMV	1, 1, 1, 9, 9, 467	No	MouLp	1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1	
Rng_1	1, 1, 1, 4, 1, 1, 1, 11, 4, 1	A037291	LRng	1, 1, 1, 2, 3, 5, 8	
CRng_1	1, 1, 1, 4, 1, 1, 1, 10, 4, 1	A127707	BIdRSgrp	1, 1, 0, 7, 0, 0, 0, 26	No
HilA	1, 1, 1, 3, 8, 27, 113	No	BLrMon	1, 1, 0, 6, 0, 0, 0, 90	??
InLat	1, 1, 1, 3, 5, 14, 27	No	BSlat	1, 1, 0, 5, 0, 0, 0	
InPorim	1, 1, 1, 3, 3, 13, 17, 84	No	BInFL	1, 1, 0, 5, 0, 0, 0, 25	No
IInFL	1, 1, 1, 3, 3, 12, 17, 78	No	BCyInFL	1, 1, 0, 5, 0, 0, 0	(Stopped)
CyInPorim	1, 1, 1, 3, 3, 12, 15, 79	No	BCInFL	1, 1, 0, 5, 0, 0, 0	
CyIInFL	1, 1, 1, 3, 3, 12, 15, 75	No	BRL	1, 1, 0, 5, 0, 0	
InPocrim	1, 1, 1, 3, 3, 12, 15, 73, 116	No	BCRL	1, 1, 0, 5, 0	
CIInFL	1, 1, 1, 3, 3, 12, 15, 70, 112	No	RA	1, 1, 0, 3, 0, 0	
DIInFL	1, 1, 1, 3, 3, 12, 13, 66	No	BIdLrMon	1, 1, 0, 3, 0, 0	
CyDIInFL	1, 1, 1, 3, 3, 12, 12, 65	No	BCIdRSgrp	1, 1, 0, 3, 0, 0	
CDIInFL	1, 1, 1, 3, 3, 12, 12, 60, 73	No	IRA	1, 1, 0, 2, 0, 0, 0, 10, 102, 4412	
MZrd	1, 1, 1, 3, 3, 8, 12, 35	No	BInLat	1, 1, 0, 2, 0, 0	
IMTL	1, 1, 1, 3, 3, 8, 12, 35	No	BIdRL	1, 1, 0, 2, 0, 0	
DInLat	1, 1, 1, 3, 1, 4, 3, 11	No	BCIdRL	1, 1, 0, 2, 0, 0	
DmA	1, 1, 1, 3, 1, 4, 2, 9, 5, 14	No	CplmLat	1, 1, 0, 1, 2	
Lp	1, 1, 1, 2, 6, 109, 23746,...	A057771	CdMLat	1, 1, 0, 1, 1	
Lat	1, 1, 1, 2, 5, 15, 53, 222, 1078,...	A006966	OLat	1, 1, 0, 1, 0, 2, 0, 5, 0, 15	
lbJslat	1, 1, 1, 2, 5, 15, 53	A006966	OMLat	1, 1, 0, 1, 0, 1, 0, 2	
bLat	1, 1, 1, 2, 5, 15, 53	A006966	BA	1, 1, 0, 1, 0, 0, 0, 1, 0, 1	
MsdLat	1, 1, 1, 2, 4, 9, 23, 65, 197, 636	No	BCIInFL	1, 1, 0, 1, 0, 0, 0, 1, 0	
JsdLat	1, 1, 1, 2, 4, 9, 23, 65, 197, 636	No	BIInFL	1, 1, 0, 1, 0, 0, 0, 1	
SdLat	1, 1, 1, 2, 4, 9, 22, 60, 174, 534	A292790	BGrp	1, 1, 0, 1, 0, 0, 0, 1	
ModLat	1, 1, 1, 2, 4, 8, 16, 34, 72, 157	A006981	BCyIInFL	1, 1, 0, 1, 0, 0, 0, 1	
AdLat	1, 1, 1, 2, 4		GBA	1, 1, 0, 1, 0, 0	
IInToMon	1, 1, 1, 2, 3, 7, 12, 35		BIRL	1, 1, 0, 1, 0, 0	
CyIInToMon	1, 1, 1, 2, 3, 7, 12, 35		BCIRL	1, 1, 0, 1, 0, 0	
IMTLChn	1, 1, 1, 2, 3, 7, 12, 31, 59	A034786	BCIMon	1, 1, 0, 1, 0, 0	
HA	1, 1, 1, 2, 3, 5, 8, 15, 26, 47	A006982	BIMon	1, 1, 0, 1, 0	
DLat	1, 1, 1, 2, 3, 5, 8, 15, 26, 47	A006982	BILrMon	1, 1, 0, 1, 0	
BrSlat	1, 1, 1, 2, 3, 5, 8, 15, 26, 47	A006982	Bilat	1, 0, 0, 1, 3, 32, 284	
BrA	1, 1, 1, 2, 3, 5, 8, 15, 26, 47	A006982	RepLGrp	1, 0, 0, 0, 0, 0	
bDLat	1, 1, 1, 2, 3, 5, 8, 15, 26, 47	A006982	AbLGrp	1, 0, 0, 0, 0, 0	
StAlg	1, 1, 1, 2, 2, 4, 5, 10, 16, 28	No	LGrp	1, 0, 0, 0, 0, 0	
CIdInFL	1, 1, 1, 2, 2, 4, 4, 9, 10, 21	No	TrivA	1, 0, 0	
KLA	1, 1, 1, 2, 1, 3, 2, 6, 4, 10	No	ToGrp	1, 0, 0	
PoGrp	1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1	A000001	CanRL	1, 0, 0	
Grp	1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1	A000001	AbToGrp	1, 0, 0	
CanSgrp	1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1	A000001	Fld	0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1	A069513
psMV	1, 1, 1, 2, 1, 2, 1, 3, 2, 2		pcDLat		
GödA	1, 1, 1, 2, 1, 2, 1, 3, 1, 2		pGrp		
MV	1, 1, 1, 2, 1, 2, 1, 3		WaHp		
CanMon	1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1	A000001	Unar		
AbPoGrp	1, 1, 1, 2, 1, 1, 1, 3, 2, 1	A000688	ToRng		
AbGrp	1, 1, 1, 2, 1, 1, 1, 3, 2, 1	A000688	ToFld		
CanCSgrp	1, 1, 1, 2, 1, 1, 1	A000688	TA		
CanCMon	1, 1, 1, 2, 1, 1, 1	A000688	SkLat		
InToLat	1, 1, 1, 1, 1, 1, 1, 1, 1, 1		SeqA		
ToLat	1, 1, 1, 1, 1, 1, 1, 1, 1, 1	A000012	RegRng		
Set	1, 1, 1, 1, 1, 1, 1, 1, 1, 1	A000012	RMod		
UFDom	1, 1, 1, 1, 1, 0		OreDom		
			OckA		
			NlGrp		
			Neofld		
			NdLat		
			NaA		

NVLGrp		
NFld		
NA		
Mset		
MonA		
MTLA		
ModOLat		
MALLA		
MA		
LieA		
LNeofld		
LLA		
LA_n		
JorA		
ImpLat		
ILLA		
Gset		
GMV		
FVec		
FRng		
DunnMon		
DpAlg		
DdpAlg		
DblStAlg		
DmMon		
DDblpAlg		
CliffSgrp		
CToRng		
CRegRng		
CA_2		
CLRng		
BoolLat		
BilinA		
BRMod		
BCK		
AbpGrp		
AAlg		

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Index