Distributive ℓ-Pregroups

R. Ball, N. Galatos, and P. Jipsen

5 August 2013

Definition

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  A *lattice-ordered pregroup*, or just *ℓ-pregroup*, is a structure of the form $\langle L, \cdot, 1, ^l, ^r, \vee, \wedge, \rangle$, where
    - $\langle L, \cdot, 1\rangle$ is a monoid,
    - $\langle L, \vee, \wedge\rangle$ is a lattice,
    - multiplication on either side preserves order,
    - and $x^l x \leq 1 \leq xx^l$ and $xx^r \leq 1 \leq x^r x$.

- Alternatively, L is a residuated lattice such that $x^{lr} = x = x^{rl}$ and $(xy)^l = y^l x^l$.
- An ℓ-pregroup is distributive if it is distributive as a lattice.
- The variety of distributive ℓ-pregroups has the variety of ℓ-groups as an important subvariety. It is picked out by the equation $x^l = x^r$.
- The elements which satisfy the foregoing equation form an ℓ-group inside any ℓ-pregroup.

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The fat question: are all ℓ-pregroups distributive?

- A modular ℓ-pregroup is distributive. This fact first came to light as the result of a two-month run on an automated theorem prover. Peter has reduced this proof to a single page. Nevertheless, the proof remains opaque.
- Is an ℓ-pregroup modular?

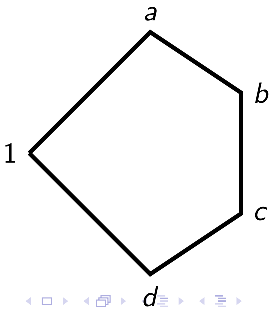
- Theorem

  If a pregroup contains a pentagon then the pivot element cannot be invertible.

- Proof.

It suffices to prove this for pivot element 1.

- $da = (1 \wedge b)a = a \wedge ba \geq b$
- $da = d(1 \vee c) = d \vee dc \leq c$



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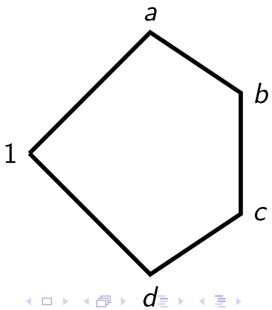

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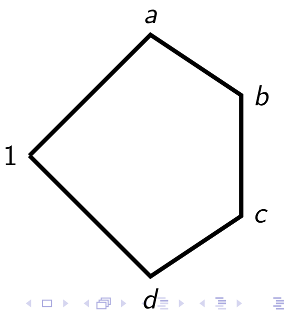

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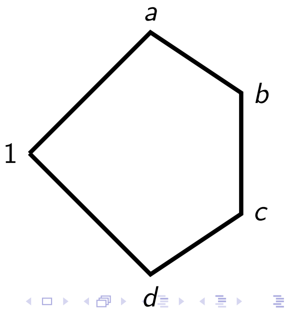

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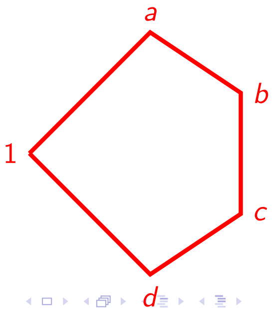

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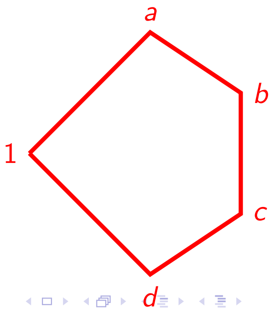

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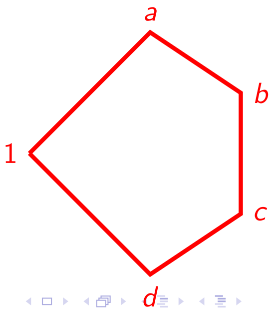

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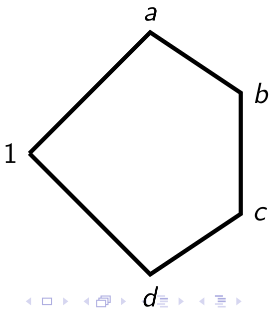

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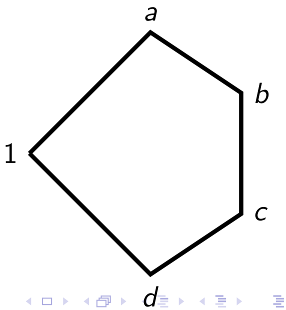

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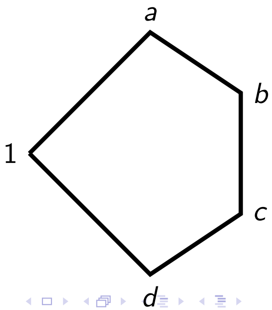

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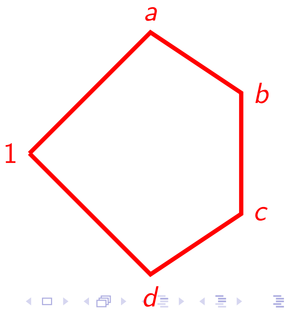

A beautiful theorem of Anderson and Edwards

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  An ℓ-semigroup wih right identity is distributive iff it can be embedding into $\mathrm{End}(\Omega)$, the ℓ-monoid of order-preserving endomorphisms of some chain Ω.
  - The question becomes which $f \in \mathrm{End}(\Omega)$ have residuals f^l and f^r? Which have residuals of all orders?
  - Note that f and f^l form a Galois pair, as do f and f^r. It follows that if both f^l and f^r exist then f must preserve all existing joins and meets in Ω.

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Which endomorphisms have residuals?

- Theorem

  An endomorphism $f \in \mathrm{End}(\Omega)$ has a left residual f^l iff, for each $\alpha \in \Omega$, $\{\beta : \beta f \leq \alpha\}$ contains a greatest element. And in that case

$$\alpha f^l = \bigvee_{\beta f \leq \alpha} \beta$$

  And dually.

- Proof.
  - We claim that $\beta f \leq \alpha$ iff $\beta \leq \alpha f^l$.
  - Recall that $f^l f \leq 1 \leq f f^l$. Therefore
  - $\beta f \leq \alpha$ implies (apply f^l to both sides)
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  - The argument for the converse is similar.
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  An endomorphism $f \in \mathrm{End}(\Omega)$ has a left residual f^l iff, for each $\alpha \in \Omega$, $\{\beta : \beta f \leq \alpha\}$ contains a greatest element. And in that case

$$\alpha f^l = \bigvee_{\beta f \leq \alpha} \beta$$

  And dually.

- Proof.
  - We claim that $\beta f \leq \alpha$ iff $\beta \leq \alpha f^l$.
  - Recall that $f^l f \leq 1 \leq f f^l$. Therefore
  - $\beta f \leq \alpha$ implies (apply f^l to both sides)
  - $\beta = \beta 1 \leq \beta f f^l \leq \alpha f^l$.
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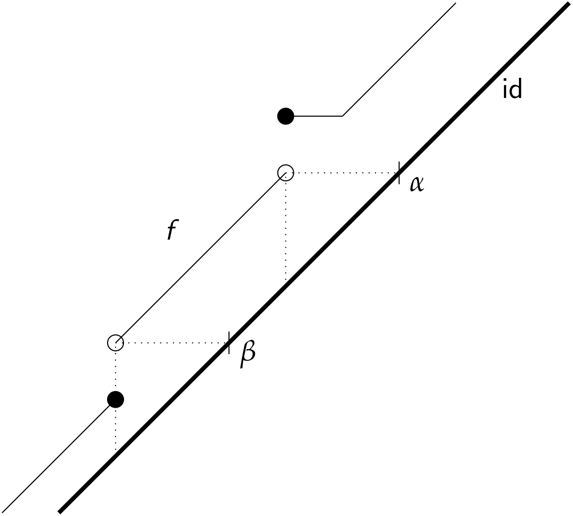

▸ To intervals of constancy ▸ To lacunas in the range

Endomorphisms with residuals must have coterminal range

- In order for an endomorphism $f \in \mathrm{End}(\Omega)$ to have a left residual f^l, its range $[\Omega]f$ must be co-initial in Ω, i.e., for all $\alpha \in \Omega$ there must be some $\beta \in \Omega$ such that $\beta f \leq \alpha$.
- In order for f^r to exist, the range of f must be cofinal in Ω, i.e., for all $\alpha \in \Omega$ there must be some $\beta \in \Omega$ such that $\beta f \geq \alpha$.
- We say that the range of f is *coterminal in* Ω if it is both co-initial and cofinal.

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  An *interval of constancy* of an endomorphism f is a convex subset $\Lambda \subseteq \Omega$ of cardinality at least 2 such that $\alpha f = \beta f$ for all $\alpha, \beta \in \Lambda$. Such an interval is said to be maximal if it is contained in no strictly larger interval of constancy.

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What else can we say about intervals of constancy?

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  Supppose f is an endomorphism whose second order residuals exist. Suppose also that $[\alpha, \beta]$ is a maximal interval of constancy of f. Then beta is covered and α is a cover.

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What else can we say about lacunas in the range?

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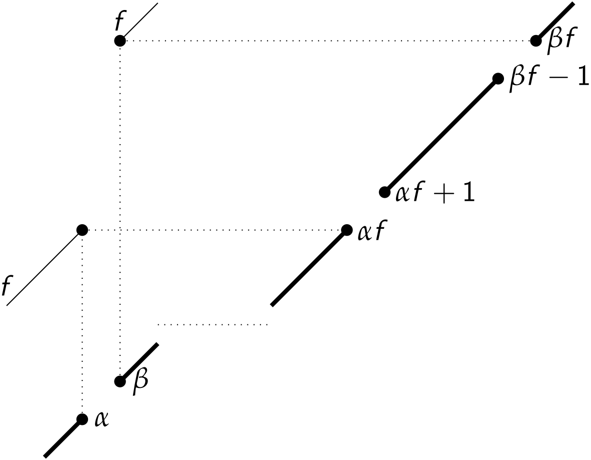

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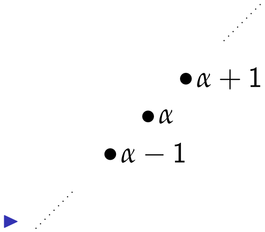

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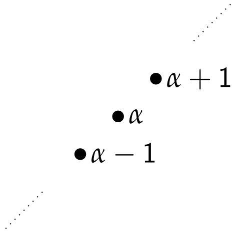

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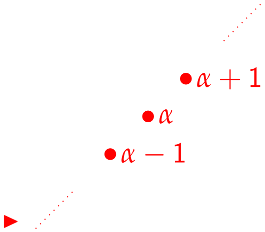

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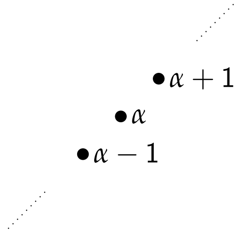
- 

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Which properties suffice?

- Theorem
  An endomorphism $f \in \mathrm{End}(\Omega)$ has residuals of all orders iff it has these properties.
  - *The range of f is coterminal in Ω.*
  - *For each $\alpha \in \Omega$, the set $\{\beta : \beta f \leq \alpha\}$ has a greatest element, and dually.*
  - *Each maximal interval of constancy of f has the form $[\alpha, \beta]$, where α and β are integral points.*
  - *Each maximal lacuna in the range of f has the form (α, β) for integral points α and β.*
- Theorem
  The family of endomorphisms which satisfy these conditions, call it $E(\Omega)$, forms a distributive ℓ-pregroup. It is the unique largest ℓ-pregroup contained in $\mathrm{End}(\Omega)$.

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  - *The range of f is coterminal in Ω.*
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- Theorem
  Every ℓ-pregroup is isomorphic to a sub-ℓ-pregroup of $E(\Omega)$ for some chain Ω.
  - If Ω has no covering pairs then $\mathrm{End}(\Omega) = \mathrm{Aut}(\Omega)$. In fact, if If Ω has no integral points then $\mathrm{End}(\Omega) = \mathrm{Aut}(\Omega)$.
  - Every automorphism of $E(\Omega)$ must take integeral points to integral points.
  - A sub-ℓ-pregroup $G \subseteq E(\Omega)$ is called *quasitransitive* if it has a point $\alpha_0 \in \Omega$, called the source, such that for all $\beta \in \Omega$ there is some $g \in G$ for which $\alpha_0 g = \beta$.
  - The quasitransitive sub-ℓ-pregroups of $E(\Omega)$ are the building blocks of a structure theory.
  - The theory of ℓ-permutation groups is well-developed and deep. The theory fof ℓ-pregroups which are not ℓ-groups should be simpler.

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An example

$$\Omega \equiv \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$$

$$(m, n)\, f \equiv \begin{cases} (2n, n) & \text{if} \quad n \geq 1 \\ (2n-1, n) & \text{if} \quad n \leq 0 \end{cases}$$

$$(k, l)\, f^{l} \equiv \begin{cases} \left(\frac{k}{2}, l\right) & \text{if} \quad k \text{ is even and } l \geq 1 \\ \left(\frac{k}{2}, 0\right) & \text{if} \quad k \text{ is even and } l \leq 0 \\ \left(\frac{k+1}{2}, l\right) & \text{if} \quad k \text{ is odd and } l \leq 0 \\ \left(\frac{k+1}{2}, 0\right) & \text{if} \quad k \text{ is odd and } l \geq 1 \end{cases}$$

f has no intervals of constancy but infinitely many lacunas in its range.
f^{l} has infinitely many intervals of constancy and no lacunas in its range.

Thank you!