The finite embeddability property for some noncommutative knotted extensions of RL.

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Preliminaries

Finite embeddability property

A class of algebras $\mathcal{K}$ has the finite embeddability property (FEP) if for every $\mathbf{A} \in \mathcal{K}$, every finite *partial subalgebra* $\mathbf{B}$ of $\mathbf{A}$ can be embedded in a finite $\mathbf{D} \in \mathcal{K}$.

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A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

$(L, \wedge, \vee)$ is a lattice,

$(L, \cdot, 1)$ is a monoid and

for all $a, b, c \in L$, $ab \leq c \iff b \leq a\backslash c \iff a \leq c/b$.

RL denotes the variety of residuated lattices.

Knotted axioms

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Some known examples of these include

contraction $x \leq x^2$,

mingle $x^2 \leq x$, and

integrality $x \leq 1$.

Some Results

Theorem (Van Alten)

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Theorem

The variety of residuated lattices axiomatized by $xyx = x^2y$ *and a knotted axiom* $x^m \leq x^n$ *has the FEP.*

Generalization

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The previous equalities can be represented by

$$xyx = x^{a_0} y x^{a_1},$$

where $a_0 + a_1 = 2$ and $a_0 a_1 = 0$.

Generalization

We consider the generalization

$$xy_1xy_2x\cdots xy_rx = x^{a_0}y_1x^{a_1}y_2x^{a_2}\cdots x^{a_{r-1}}y_rx^{a_r}, \tag{1}$$

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Theorem

For $n > m \geq 1, r \geq 1$, the variety $\mathcal{V}_r$ of residuated lattices axiomatized by (1) and a knotted axiom $x^m \leq x^n$ has the FEP.

Residuated frames

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We define S_W to be the set of *unary linear polynomial* (sections) of $(W, \circ, 1)$. Elements of S_W are of the form $u(\_) = y \circ \_ \circ w$ for $y, w \in W$. Let $W' = S_W \times B$, and define

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We define $y \setminus\!\!\setminus (u, b) = \{(u(y \circ \_), b)\}$ and $(u, b) /\!\!/ y = \{(u(\_ \circ y), b)\}$. The relation N is a nuclear relation, because it satisfies the condition

$$(x \circ y)Nz \Leftrightarrow yN(x \setminus\!\!\setminus z) \Leftrightarrow xN(z /\!\!/ y)$$

Then $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \circ, \setminus\!\!\setminus, /\!\!/, \{1\})$ is a unital residuated frame.

Galois algebra

For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^{\triangleright} = \{b \in W' : xNb, \text{ for all } x \in X\}$$
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The *Galois algebra* of $\mathbf{W_{A,B}}$ is

$$\mathbf{W_{A,B}}^{+} = (\gamma_N[\wp(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{1\})),$$

which is a complete residuated lattice.

The embedding

The map $b \mapsto \{(id, b)\}^{\lhd}$ is an embedding of the partial subalgebra **B** of **A** into $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ [Galatos, Jipsen].

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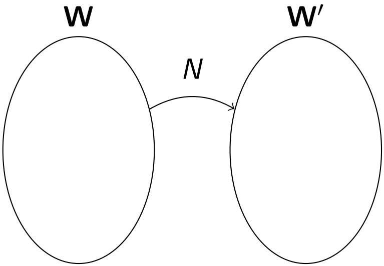
Furthermore, $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ and **A** belong to $\mathcal{V}_k$ and the closed sets $\{(u, b)\}^{\lhd}$ for $u \in S_W, b \in B$ form a basis for $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$.

The setting



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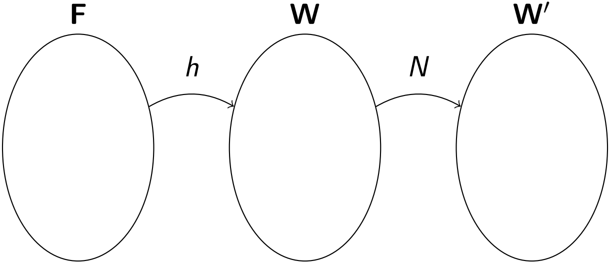

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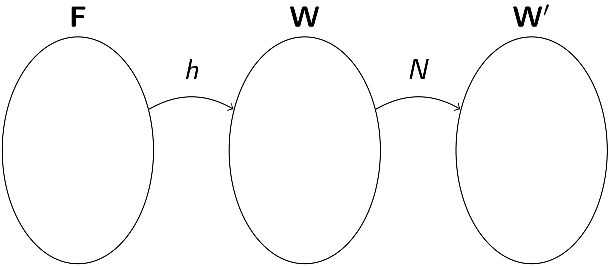

F is a pomonoid and h is an order preserving homomorphism.
Furthermore, h is surjective and **F** is a well partially ordered set.

Well partially ordered sets

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If $\langle P, \leq \rangle$ is well partially ordered, then it is known that for each $k \in \mathbb{N}$, P^k is well partially ordered under the direct product ordering. Furthermore, homomorphic images, finite disjoint unions, and subposets of well partially ordered sets are well partially ordered.

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Consider the poset $\langle P, \leq \rangle$. An infinite sequence $p_1, p_2, \ldots$ of elements of P is called *bad* when $i < j$ implies that $p_i \not\leq p_j$. Note that an infinitely descending chain or antichain would be a bad sequence. A poset is well partially ordered if and only if it has no bad sequences.

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$$y_1 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_1 \leq^{\mathbf{F}} y_2 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_2 \text{ and } h(y_1) \circ h(x) \circ h(w_1) \leq h(y_2) \circ h(x) \circ h(w_2).$$

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Now if $z \in \{(h(y_2) \circ \_ \circ h(w_2), b)\}^{\lhd}$, then $h(y_2) \circ h(x) \circ h(w_2) \leq b$. Hence $h(y_1) \circ h(x) \circ h(w_1) \leq b$ and $z = h(x) \in \{(h(y_1) \circ \_ \circ h(w_1), b)\}^{\lhd}$.

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C_b is finite for every $b \in B$. Thus, there are finitely many sets $\{(u, b)\}^{\lhd}$.

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For instance, in the monoid on generators $\{z_1, z_2, z_3, z_4, z_5\}$, $z_5^3 z_1^4 z_3^2$ will be represented by $((4, 0, 2, 0, 3), 513)$

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For that purpose we need to look at the defining equation and obtain information out of it.

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Further work

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Thank you for your attention.